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A STUDY OF LINEAR MEASURES OF INEQUALITY

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ABSTRACT

Mehran 1976 introduced some linear class of measures and gave some desirable properties of such class measures $[I_k, k \geq 2]$ *and* $[J_k, k \geq 1]$. *In this paper we obtain particular score functions for these measures. Test of significance for one sample and simulation work is carried out to find the power of these tests.*

Keywords: *Gini Index, Relation Mean Deviation, Simulation.*

INTRODUCTION:

In the class of linear measures of inequality we show that the general class of measures $[I_k, k \geq 2]$ and $[J_k, k \geq 1]$ (Ref. Klefsjo (1984)) can be visualized as linear measures of income inequality with suitable score functions (Ref. Mehran (1976)). Infact, we observe that Lorenz family of inequality measures given as J_k or $D_k(F)$ is a sub-family of Linear measures given by Mehran (1976), whereas correspond to the Extended Gini family, which itself is a sub-family of Lorenz family of inequality measures. Some conditions (in terms of score functions) are derived under which these measures satisfy some desirable properties of inequality indices viz. Pigou-Dalton Transfer Principle and Diminishing Transfer Principle.

1. LINEAR INEQUALITY MEASURES:

Mehran (1976) defined the class of Linear measures of inequality as

$$I = \frac{1}{\mu}\int_0^1 \left[F^{-1}(p) - \mu\right] w(p)\,dp \tag{1}$$

where $w(p)$ is a score function. It is assumed that $\int_0^1 w(p)\,dp = 0$ and each score function defines a particular linear inequality measure.

Integrating (1) by parts and using $L_F(0)=0$ and $L_F(1)=1$, we obtain

$$I = \int_0^1 \left[p - L_F(p)\right] dw(p).$$

Hence each linear inequality index is the weighted area between the Lorenz Curve and the line of perfect equality. Mehran (1976) pointed out that famous inequality measures like Gini Index and Relative Mean Deviation are particular cases of linear measures of income inequality. In particular, if $w(p) = 2p - 1$ for $0 \leq p \leq 1$, then I corresponds to the Gini Index.

In the results given below, we verify that members of the Extended Gini family measures I_k and Lorenz family measures J_k, given by

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(A) $$I_k = \int_0^1 k(k-1)(1-p)^{k-2}\left[p - L_F(p)\right]dp$$
$$= 1 - k(k-1)\int_0^1 (1-p)^{k-2} L_F(p)\,dp, \quad k \geq 2,$$

(B) $$J_k = (k+1)\int_0^1 p^{k-1}\left[p - L_F(p)\right]dp, \quad k \geq 1$$ (Ref. Klefsjo (1984)).

can be considered as linear measures of inequality with different score functions. We derive the score functions in each case and check whether these measures satisfy the two Transfer Principles so that comparison of these inequality measures can be carried out by looking at their properties.

RESULT: 1

(a) The inequality measure I_2 satisfies Pigou-Dalton Transfer Principle but fails to satisfy the stronger Diminishing Transfer Principle.

(b) The general inequality measure $I_k, k \geq 3$ satisfies both the Transfer Principles i.e., Pigou–Dalton Principle and Diminishing Transfer Principle.

Proof: **(a)** By definition, the Gini Index is

$$I_2 = 2\int_0^1 \left(p - L_F(p)\right)dp,$$

which can be looked upon as a linear measure with $dw(p) = 2dp$.
The restriction $\int_0^1 w(p)\,dp = 0$ yields the score function

$$w(p) = 2p - 1.$$

We observe that

$$w'(p) > 0 \text{ and } w''(p) = 0$$

So, I_2 satisfies Pigou-Dalton Transfer Principle (as its score function is strictly increasing) but fails to satisfy the stronger Principle of Diminishing Transfer because the score function does not have strictly decreasing derivative.

(b) In general

$$I_k = \int_0^1 k(k-1)(1-p)^{k-2}\left[p - L_F(p)\right]dp,$$

is a linear measure of inequality with $dw(p) = k(k-1)(1-p)^{k-2}\,dp$.

Again the condition $\int_0^1 w(p)\,dp = 0$ leads to the score function

$$w(p) = \left\lfloor 1 - k(1-p)^{k-1} \right\rfloor 0 \leq p \leq 1.$$

Since $w'(p) > 0 \text{ and } w''(p) < 0 \text{ for } k \geq 3$,

hence I_k satisfies both the Transfer Principles for $k \geq 3$.

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RESULT: 2 The inequality measure $J_k, k \geq 1$ satisfies Pigou-Dalton Transfer Principle but fails to satisfy the Diminishing Transfer Principle.

Proof: We have

$$J_k = \int_0^1 (k+1) p^{k-1} \left[p - L_F(p) \right] dp, \quad k \geq 1,$$

which can be put as a linear measure of inequality with $dw(p) = (k+1)p^{k-1}dp$.

Using $\int_0^1 w(p)\,dp = 0$, the score function is

$$w(p) = \tfrac{1}{k}\left[(1+k)p^k - 1 \right].$$

Since $w'(p) > 0$ and $w''(p) > 0$, hence J_k satisfies Pigou–Dalton Transfer Principle but fails to satisfy the stronger Principle of Diminishing Transfer.

REMARK: 1 By representing $\{I_k, k \geq 2\}$ and $\{J_k, k \geq 1\}$ as linear measures of inequality, we observe that $\{I_k, k \geq 3\}$ represents a better set of inequality measures that $\{J_k, k \geq 1\}$ in terms of Transfer Principles.

2. ASYMPTOTIC TEST FOR SIGNIFICANCE OF AN INEQUALITY MEASURE:

Let I_F be the linear inequality measure *I* corresponding to a population with distribution function *F*. Let $X_1, \ldots, X_n$ be a random sample form distribution *F* with T_F as sample linear inequality measure and consider the problem of testing

$$H_0 : I_F = I_F^{(0)}$$

vs. $H_A : I_F \neq I_F^{(0)}$

where $I_F^{(0)}$ is some specified value of I_F such that $0 < I_F^{(0)} < 1$.

Under some regularity conditions, using Theorem, we have

$$\frac{\sqrt{n}\left(T_F - \mu(T_F)\right)}{\sigma(T_F)} \xrightarrow{D} N(0,1)$$

where $E(T_F) \to \mu(T_F) = \dfrac{1}{\mu}\displaystyle\int_0^1 w(u) F^{-1}(u)\,du$

and $\quad nV(T_F) \to \sigma^2(T_F) = \dfrac{2}{\mu^2}\displaystyle\int_0^1 \int_0^y w(x) w(y) x(1-y)\, dF^{-1}(x)\, dF^{-1}(y)$, for large *n*.

Under H_0, the critical region is of the form

$$\frac{\sqrt{n}\left(T_F - I_F^{(0)}\right)}{\hat{\sigma}(T_F)} > \Phi^{-1}(1-\alpha), \tag{2}$$

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where $\hat{\sigma}^2(T_F)$ is a consistent estimator of $\sigma^2(T_F)$ and is given by

$$\hat{\sigma}^2(T_F) = \frac{1}{n^2\mu^2}\sum_{i=1}^{n-1}\sum_{k=1}^{i} a_{ik} w\left(\frac{i}{n+1}\right) w\left(\frac{k}{n+1}\right) k(n-i)\left[X_{(i+1)} - X_{(i)} \mathbf{I} X_{(k+1)} - X_{(k)}\right] \text{where}$$

$a_{ik} = 2$ for $i \neq k$ and $a_{ii} = 1$ (Ref. Rojo and Wang (1994).

REMARK: 2 It can be observed that the null hypothesis, $I_F = 0$ or $I_F = 1$ corresponds to a population with no or perfect inequality and from an applied perspective, this seems to be an unrealistic hypothesis that will be rejected in most of the practical situations.

SIMULATION STUDY:

To investigate the behaviour of the test defined by (2) for finite sample sizes in terms of its power function, a simulation study was carried out with

$$F = 1 - e^{-x},$$
$$G_1 = 1 - e^{-2x},$$
$$G_2 = 1 - e^{-\left(x + \frac{x^2}{2}\right)},$$

and $$G_3 = \frac{1}{1 + e^{-x}}.$$

Let $I_{k,F}$ be the inequality measure I_k corresponding to F for $k \geq 2$. The experiment is conducted 10,000 times with $\alpha = 0.05$. The results of the simulation for the power of suggested test in one sample case are summarized in the following tables and figures.

Table 1

Power of the test for I_2 (Gini Index) with $\alpha = 0.05$ and score function $w(p) = 2p - 1$

	$n = 50$	$n = 80$	$n = 100$	$n = 200$
$I_{2,F}$	0.5213	0.5416	0.8128	0.9230
I_{2,G_1}	0.4681	0.5191	0.7115	0.9015
I_{2,G_2}	0.4918	0.5232	0.7242	0.9141
I_{2,G_3}	0.5903	0.6280	0.8316	0.9356

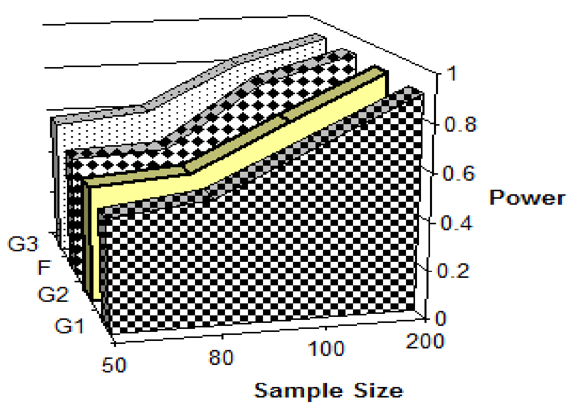

Fig. 1 Power vs. sample size for I_2 (Gini Index) with score function $w(p) = 2p - 1$

Table 2

Power of the test for I_3 with $\alpha = 0.05$ and score function $w(p) = 1 - 3(1-p)^2$

	$n = 50$	$n = 80$	$n = 100$	$n = 200$
$I_{3,F}$	0.5343	0.5821	0.8199	0.9299
I_{3,G_1}	0.4891	0.5211	0.7203	0.9111
I_{3,G_2}	0.5124	0.5340	0.7312	0.9215
I_{3,G_3}	0.6123	0.6349	0.8499	0.9416

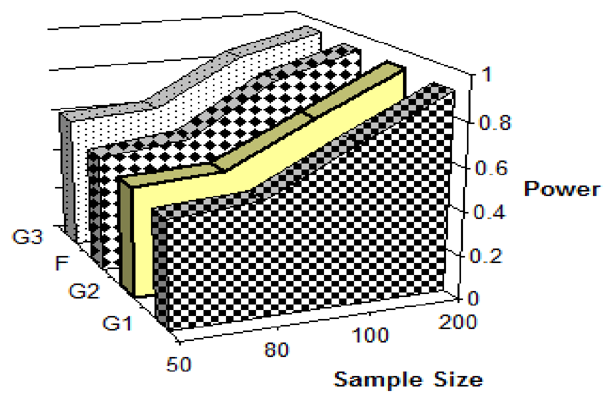

Fig. 2 Power vs. sample size for I_3 with score function $w(p) = 1 - 3(1-p)^2$

Table 3

Power of the test for J_2 (Piesch Measure) with $\alpha = 0.05$ and score function $w(p) = \frac{1}{2}(3p^2 - 1)$

	$n = 50$	$n = 80$	$n = 100$	$n = 200$
$J_{2,F}$	0.5124	0.5291	0.7914	0.9196
J_{2,G_1}	0.4613	0.5018	0.7018	0.8981
J_{2,G_2}	0.4898	0.5113	0.7187	0.9012
I_{2,G_3}	0.5792	0.6041	0.8192	0.9276

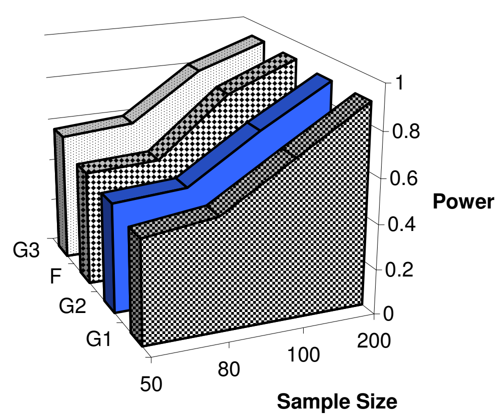

Fig. 3 Power vs. sample size for J_2 with score function $w(p) = \frac{1}{2}(3p^2 - 1)$

Proof: we have

$$J_k = \int_0^1 (k+1) p^{k-1} \left[p - L_F(p) \right] dp, \qquad k \geq 1,$$

which can be put as a linear measure of inequality with $dw(p) = (k+1) p^{k-1} dp$. Using $\int_0^1 w(p) dp = 0$.

CONCLUSION:

From the results in the above tables, it is observed that

- as we increase the sample size, power of the test improves,
- power of the test is highest corresponding to G_3, the Logistic distribution and
- power of the test is more for I_3 than for I_2. This leads us to believe that the test will be more powerful for higher values of k.

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