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BEHAVIOR OF SOLUTIONS FOR FREE BOUNDARY PROBLEMS

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ABSTRACT

In this paper we shows that the free boundary is Lipschitz Continues in any Dimensions

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INTRODUCTION:

In the earlier works [8], [10], the boundary behavior of solutions of non-parametric least area, or area-type problems were studied. The main strategy was to reduce the problem to a free boundary problem so that the regularity theory for free boundaries applies.

In [8], only the two-dimensional case was considered since the free boundary was well understood in this case, see for example [3]. Here we adapt an idea from [1] to show that the free boundary is Lipschitz continuous in any dimension.

1. NOTATION AND MAIN RESULTS:

Let M be a $C^{2,1}$ bounded domain in R^n, and let $\Phi \in C^0(\partial M)$. We are here interested in the behavior of the trace of the unique solution U of the non-para-metric least problem (1.1) :

$\min\{I[V]: V \in BV(M)\}$,

(1.1) $I[V] \equiv \int_M (1+(DV)^2)^{\frac{1}{2}}\,dx + \int_{\partial M} |V-\phi| dX^{n-1}$

Let $T = T_+ \cup T_-$, where

(1.2) $T_+ = \{x \in \partial M = \phi(x) > U(x) and\, H_{\partial M}(x) < 0\}$,

$T_- = \{x \in \partial M = \phi(x) < U(x) and\, H_{\partial M}(x) < 0\}$.

Up to X^{n-1} *measure zero, sets* $T_\pm$ *and* T *are well defined. Suppose*

$x_0 \in T_+\ (T_-)$, *and* $\partial M \cap B_r(x_0) \sim T_+$, $(\partial M \cap B_r(x_0) \sim T_-$, *respectively)*

is of X^{n-1} *, measure zero, for some* $r > 0$*. Then we have the following result.*

THEOREM 1.*The restriction of U to* $\partial M \cap B_{\frac{r}{2}}(x_0)$ *is a* $C^{1,\alpha}$ *function,*

for any $\alpha \in (0,1)$, *and* $U \in C^{\frac{1}{2}}\left(\overline{M} \cap B_{\frac{r}{2}}(x_0)\right)$.

Next, let us consider the equation for surfaces with prescribed mean curvature:

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$$(1.3)\quad \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\frac{\partial U}{\partial x_i}/(1+(DU)^2)^{\frac{1}{2}}\right] = H(x) in\, M.$$

Here we are interested in the behavior of the trace of the unique solution U (up to an additive constant) to the extremal problem, i.e., the mean curvature function H satisfies

$$(1.4)\quad |\int_A H dx| < X^{n-1}\,(\partial A)$$

for every Caccioppoli set A $\subset$M, A$\neq$ $\phi, M\ and$

$$(1.5)\quad |\int_M H dx = X^{n-1}\,(\partial M),$$

(cf. [4]).

THEOREM 2: Let M be a $C^{2,1}$ bounded domain in R^n, and let H be a Lipschitz continuous function on $\overline{M}$, satisfies (1.4), 1.5). Suppose that

$$(1.6).\quad H_{\partial M}(x) \leqq H(x) - C_0\ for\ x \in \partial M \cap B_r(x_0),$$

for a positive constant C_0. Then the restriction of U to $\partial M \cap B_{\frac{r}{2}}(x_0)$ is a $C^{1,\alpha}$ function, for any $\alpha \in (0,1), and\ U \in C^{1/2}\ (\overline{M} \cap B_{\frac{r}{2}}(x_0))$.

As a consequence of Theorem 1, Theorem 2 and [5], [6] (or [7]), we have –

COROLLARY: In Theorem 1, if $\partial M\ is\ C^{k,\alpha}$ (analytic),then the restriction of U to $\partial M \cap B_{\frac{r}{2}}(x_0)$ is $C^{k-1,\alpha}$(analytic), for k=3,4,.......... In Theorem 2, if ∂M is $C^{k,\alpha}$(analytic) and H is $C^{k-2,\alpha}$ (analytic), then the restriction of U to $\partial M \cap B_{\frac{r}{2}}(x_0)$ is $C^{k-1,\alpha}$(analytic), for k=3,4,...

1. Proofs:

We consider first the minimal surface case. As in [8], Theorem 1 follows from the $C^{1,\alpha}$ regularity of the free boundary of the following variational inequality.

Let B= {y $\in$ R^n : | y | $\leqq 1\}, and\ let\ U \in C^{1,1}(B)$ be the solution of (2.1) below :

$$(2.1)\quad \int_B \alpha_j(DU)(V-U)_{y_j}\ dy \geqq 0\ for\ all\ V \in K,$$

where

$$\alpha_j(DU) = \frac{\partial U}{\partial y_j}/\,(1+(DU)^2)^{\frac{1}{2}},$$

$$K = \{V \in C^{0,1}(B): V \geqq W\ in\ B\ and\ V = U\ on\ \partial B\},$$

W(y) = W(y_2, y_3, …………, y_n) is a $C^{2,1}$ function with

$$(2.2)\quad \sum_{i=2}^{n} \frac{\partial}{\partial y_i}\left[W_{yi}/\,(1+|DW|^2)^{\frac{1}{2}}\right] \leqq -C_0 < 0, W(0) = |DW(0)| = 0.$$

Moreover, we have –

$$U_{y1} < 0\ in\ M = \{y \in B: h(y) = U(y) - W(y) > 0\}.$$

Set F=$\partial M, and\ M = \sup\{|D^2U(y)| + |D^3W(y)|: y \in B\}$.

Let Δ_g be the Laplace-Beltrami operator on the graph(u), and $|A_g|^2$ the length square of the second fundamental form of the graph (u). Define 2ρ>0 to be a positive constant such that the first eigenvalue of the operator $-\Delta_g - |A_g|^2$ on any $B_{2\rho} \subset$B is positive.

LEMMA: Let $y_0 \in F$ and $B_{2\rho}(y_0) \subset B$. Then there is a cone $\Lambda \subset R^n_+ = \{X \in R^n: x_1 > 0\}$ such that

$$Dh(y) \cdot \xi \leqq 0 \; for \; y \in B_\rho(y_0),$$

$$Dh(y) \cdot \xi < 0 \; for \; y \in M \cap B_\rho(y_0),$$

whenever $\xi \in \Lambda \cap S^{n-1}$.

A geometric consequence of the lemma is that, for $|y_0| + \rho$ sufficiently small,

$$A_+(y_0) = \left\{y \in B_\rho(y_0): \frac{y - y_0}{|y - y_0|} \in \Lambda\right\} \subseteq B_\rho(y_0) \sim M$$

and

$$A_-(y_0) = \left\{y \in B_\rho(y_0): \frac{y_0 \quad y}{|y - y_0|} \in \Lambda\right\} \subseteq \bar{M}$$

hence

$$F = \left(f(y_2, \dots, y_n), y_2, \dots, y_n\right) for \; |(y_2, \dots y_n)| \leqq \rho$$

and f is a uniformly Lipschitz continuous function. The $C^{1,\alpha}, 0 < \alpha < 1$, regularity of F follows from [2] and [7].

For the surfaces with prescribed mean curvature, the proof is similar. In this case, we use the same test function V .

Consider the operator L=$\Delta_g + (|A_g|^2 - H^2 + (DH \cdot \nu))$. Then

$Lv = DH = (H_y, H_u) in \; M \cap B$, where H(y, U) = H($y_2$, …., y_n, U). Thus

$$LV = \sum_{j=2}^{n} \xi_j H_{yj} - \sum_{j=2}^{n} \xi_j L\left(\frac{W_{yj}}{(1 + (DU)^2)^{\frac{1}{2}}}\right) + L(U - W) - \varepsilon LR$$

$in \; M \cap B_{2\rho}(y_0)$.

By choosing $\varepsilon, \sum_{j=2}^{n} |\xi_j|, |y_0| + \rho$, sufficiently small, and by using (1.6), one has

$$LV \geqq \frac{1}{2} C_0 > 0 \; in \; M \cap B_{2\rho}(y_0).$$

1. FINAL REMARKS:

1. Similar results hold for a general class of quasi-linear equations which result from a nonparametric variational integral associated to an elliptic parametric integral, see [8].

2. Consider the problem (1.1) with $\phi \in C^{1,\alpha}(\partial M)$, then the lemma is true even when $y_0 \in$ graph $(\phi) \cap F$. This follows from boundary regularity for obstacle problems. See for example [9]. As a consequence, F is a Lipschitz graph over ∂M near y_0 where $H_{\partial M}(y_0) < 0$.

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