

POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR NTH ORDER Q-DIFFERENTIAL EQUATIONS

Moustafa El-Shahed

Department of Mathematics, College of Education, Qassim University P.O. Box 3771 Unizah, Qassim, Saudi Arabia Tel: 00966501219508

E-mail: elshahedm@yahoo.com

and

Maryam Al-Yami*

Department of Mathematics, College of Education, P.O.Box 102290 Jeddah, King Abdulaziz University Saudi Arabia

E-mail: mohad20020@hotmail.com

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ABSTRACT

In this paper, we investigate the problem of existence of positive solutions for the nonlinear q-boundary value problem or quantum boundary value problem:

$$D_q^n u(t) + \lambda a(t)f(u(t)) = 0, \quad 0 < t < 1,$$

satisfying three kinds of q-different boundary value conditions. Our analysis relies on Krasnoselskii's fixed point theorem of cone.

Keywords: Q-difference equations; Fixed point theorem; Boundary value problem; Positive solution

1. INTRODUCTION:

There is currently a great deal of interest in positive solutions for several types of boundary value problems. A large part of the literature on positive solutions to boundary value problems seems to be traced back to Krasnoselskii's work on nonlinear operator equations [15], especially the part dealing with the theory of cones in Banach spaces. In 1994, Erbe and Wang [6] applied Krasnoselskii's work to eigenvalue problems to establish intervals of the parameter λ for which there is at least one positive solution. In 1995, Elloe and Henderson [2] obtained the solutions that are positive to a cone for the boundary value problem

$$\begin{aligned} u^{(n)}(t) + a(t)f(u) &= 0, \quad 0 < t < 1, \\ u^{(i)}(0) = u^{(n-2)}(1) &= 0, \quad 0 \leq i \leq n-2. \end{aligned}$$

Since this pioneering works, a lot research has been done in this area [3, 6, 11, 16, 19, 20]. In 2008, EL-Shahed [4] obtained the existence of positive solutions to nonlinear nth order boundary value problems

$$\begin{aligned} u^{(n)}(t) + \lambda a(t)f(u(t)) &= 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = \dots = u^{(n-1)}(0) &= 0, \quad u'(1) = 0, \\ u(0) = u'(0) = u''(0) = \dots = u^{(n-2)}(0) &= 0, \quad u'(1) = 0, \\ u(0) = u'(0) = u''(0) = \dots = u^{(n-2)}(0) &= 0, \quad u''(1) = 0 \end{aligned}$$

El-Shahed and Hassan [5] studied the existence of positive solutions of the q-difference boundary value problem:

***Corresponding author: Maryam Al-Yami* *E-mail: mohad20020@hotmail.com**

$$\begin{aligned}
 -D_q^2 u(t) &= a(t)f(u(t)), \quad 0 \leq t \leq 1, \\
 \alpha u(0) - \beta D_q u(0) &= 0, \\
 \gamma u(1) + \delta D_q u(1) &= 0.
 \end{aligned}$$

The purpose of this paper is to establish the existence of positive solutions to nonlinear nth order q`boundary value problems:

$$D_q^n u(t) + \lambda a(t)f(u(t)) = 0, \quad 0 < t < 1, \tag{1}$$

$$u(0) = D_q^2 u(0) = D_q^3 u(0) = \dots = D_q^{n-1} u(0) = 0, D_q u(1) = 0, \tag{2}$$

$$u(0) = D_q u(0) = D_q^2 u(0) = \dots = D_q^{n-2} u(0) = 0, D_q u(1) = 0, \tag{3}$$

$$u(0) = D_q u(0) = D_q^2 u(0) = \dots = D_q^{n-2} u(0) = 0, D_q^2 u(1) = 0, \tag{4}$$

where λ is a positive parameter. Throughout the paper, we assume that

C1: $f : [0, \infty) \rightarrow [0, \infty)$ is continuous

C2: $a : (0, 1) \rightarrow [0, \infty)$ is continuous function such that $\int_0^1 a(t) d_q t > 0$.

2. PRELIMINARIES:

For the convenience of the reader, we present here some notations and lemmas that will be used in the proof our main results.

Let $q \in (0, 1)$ and defined [14]

$$[a]_q = \frac{q^a - 1}{q - 1} = q^{a-1} + \dots + 1, \quad a \in \mathbb{R}.$$

The q-analogue of the power function $(a - b)^n$ with $n \in \mathbb{N}$ is

$$(a - b)^0 = 1, \quad (a - b)^n = \prod_{k=0}^{n-1} (a - bq^k), \quad a, b \in \mathbb{R}, \quad n \in \mathbb{N}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a - b)^{(\alpha)} = a^\alpha \prod_{i=0}^{\infty} \frac{(a - bq^i)}{(a - bq^{\alpha+i})}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1 - q)^{(x-1)}}{(1 - q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \dots\}, \quad 0 < q < 1,$$

and satisfies $\Gamma_q(x + 1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function f is here defined by

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q - 1)x},$$

and q-derivatives of higher order by

$$D_q^n f(x) = \begin{cases} f(x) & \text{if } n = 0, \\ D_q D_q^{n-1} f(x) & \text{if } n \in \mathbb{N}. \end{cases}$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$\int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, \quad 0 \leq |q| < 1, \quad x \in [0, b].$$

If $a \in [0, b]$ and f defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Similarly as done for derivatives, it can be defined an operator I_q^n , namely,

$$(I_q^0 f)(x) = f(x) \quad \text{and} \quad (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q , i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [14]. We now point out three formulas that will be used later (${}^i D_q$ denotes the derivative with respect to variable i) [8]

$$[a(t-s)]^{(\alpha)} = a^\alpha (t-s)^{(\alpha)}, \quad (5)$$

$${}_i D_q (t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)}, \quad (6)$$

$$\left({}_x D_q \int_0^x f(x,t) d_q t \right) (x) = \int_0^x {}_x D_q f(x,t) d_q t + f(qx, x). \quad (7)$$

Remark: 2.1. We note that if $\alpha > 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$ [8].

Definition: 2.1. Let $\alpha \geq 0$ and f be a function defined on $[0, 1]$. The fractional q-integral of the Riemann–Liouville type is $({}_{RL} I_q^\alpha f)(x) = f(x)$ and

$$({}_{RL} I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_a^x (x-qt)^{(\alpha-1)} f(t) d_q t, \quad \alpha \in \mathbb{N}^+, x \in [0, 1].$$

Definition: 2.2. [18] The fractional q-derivative of the Riemann–Liouville type of order $\alpha \geq 0$ is defined by $({}_{RL} D_q^0 f)(x) = f(x)$ and $({}_{RL} D_q^\alpha f)(x) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f)(x)$, $\alpha > 0$, where $[\alpha]$ is the smallest integer greater than or equal to α .

Definition: 2.3. [18] The fractional q -derivative of the Caputo type of order $\alpha \geq 0$ is defined by

$$({}_C D_q^\alpha f)(x) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f)(x), \quad \alpha > 0,$$

where $[\alpha]$ is the smallest integer greater than or equal to α .

Lemma: 2.1. Let $\alpha, \beta \geq 0$ and f be a function defined on $[0, 1]$. Then, the next formulas hold:

1. $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x),$
2. $(D_q^\alpha I_q^\alpha f)(x) = f(x).$

The next result is important in the sequel. It was proved in a recent work by the author [8].

Theorem: 2.1. Let $\alpha > 0$ and p be a positive integer. Then, the following equality holds:

$$({}_{RL} I_q^\alpha {}_{RL} D_q^p f)(x) = (D_q^p I_q^\alpha f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^k f)(0).$$

Theorem: 2.2. [18] Let $x > 0$ and $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Then, the following equality holds:

$$(I_q^\alpha {}_C D_q^\alpha f)(x) = f(x) - \sum_{k=0}^{[\alpha]-1} \frac{x^k}{\Gamma_q(k+1)} (D_q^k f)(0).$$

Definition: 2.4. Let X be a real Banach space. A nonempty closed convex set $P \subset X$ is called cone of X if it satisfies the following conditions

1. $x \in P, \sigma \geq 0$ Implies $\sigma x \in P$;
2. $x \in P, -x \in P$ Implies $x = 0$

Definition: 2.5. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Theorem: 2.3. [10,15] Let X be a Banach space and $P \subset X$ is a cone in X . Assume that Ω_1 and Ω_2 are open subsets in X of with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P$ be completely continuous operator. In addition suppose either:

H1: $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_2$ or

H2: $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_2$ and $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_1,$

holds. Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3. GREEN FUNCTIONS AND THEIR PROPERTIES:

Lemma: 3.1. Let $y \in C[0,1]$, then the boundary value problem

$$D_q^n u_2(t) + y(t) = 0, \quad 0 < t < 1,$$

$$u_2(0) = D_q^2 u_2(0) = D_q^3 u_2(0) = \dots = D_q^{n-1} u_2(0) = 0, D_q u_2(1) = 0,$$

has a unique solution

$$u_2(t) = \int_0^1 G_2(t, qs) y(s) d_q s,$$

where

$$G_2(t, s) = \begin{cases} \frac{t(1-s)^{n-2}}{\Gamma_q(n-1)} - \frac{(t-s)^{n-1}}{\Gamma_q(n)}, & 0 \leq s \leq t \leq 1, \\ \frac{t(1-s)^{n-2}}{\Gamma_q(n-1)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Proof: We may apply Lemma 2.1 and Theorem 2.2, we see that

$$u_2(t) = u_2(0) + \frac{D_q u_2(0)}{\Gamma_q(2)} t + \frac{D_q^2 u_2(0)}{\Gamma_q(3)} t^2 + \frac{D_q^3 u_2(0)}{\Gamma_q(4)} t^3 + \dots + \frac{D_q^{n-1} u_2(0)}{\Gamma_q(n)} t^{n-1} - I_q^\alpha y(t).$$

By using the boundary conditions $u_2(0) = D_q^2 u_2(0) = D_q^3 u_2(0) = \dots = D_q^{n-1} u_2(0) = 0$, we get

$$u_2(t) = B_2 t - \int_0^t \frac{(t-qs)^{n-1}}{\Gamma_q(n)} y(s) d_q s. \tag{8}$$

Differentiating both sides of (8) one obtain, with the help (6) and (7),

$$(D_q u_2)(t) = B_2 - \int_0^t \frac{[n-1]_q (t-qs)^{n-2}}{\Gamma_q(n)} y(s) d_q s,$$

then by the condition $D_q u_2(1) = 0$, we have

$$B_2 = \int_0^1 \frac{(1-qs)^{n-2}}{\Gamma_q(n-1)} y(s) d_q s,$$

the proof is complete.

Lemma: 3.2. Function G_2 defined above satisfies the following conditions:

$$G_2(t, qs) \geq 0 \text{ and } G_2(t, qs) \leq G_2(1, qs), \quad 0 \leq t, s \leq 1, \tag{9}$$

$$G_2(t, qs) \geq \eta_2(t) G_2(1, qs), \quad 0 \leq t, s \leq 1 \text{ with } \eta_2(t) = t. \tag{10}$$

Proof: We start by defining two functions

$$g_1(t, s) = \frac{t(1-s)^{n-2}}{\Gamma_q(n-1)} - \frac{(t-s)^{n-1}}{\Gamma_q(n)}, \quad 0 \leq s \leq t \leq 1,$$

and

$$g_2(t, s) = \frac{t(1-s)^{n-2}}{\Gamma_q(n-1)}, \quad 0 \leq t \leq s \leq 1.$$

It is clear that $g_2(t, qs) \geq 0$. Now, $g_1(0, qs) = 0$ and, in view of Remark 2.1, for $t \neq 0$

$$\begin{aligned} g_1(t, qs) &= \frac{t(1-qs)^{n-2}}{\Gamma_q(n-1)} - \frac{(t-qs)^{n-1}}{\Gamma_q(n)} \\ &\geq \frac{t(1-qs)^{n-2}}{\Gamma_q(n-1)} - \frac{t(1-qs)^{n-1}}{\Gamma_q(n)} \\ &= \frac{t}{\Gamma_q(n)} \left[[n-1]_q (1-qs)^{n-2} - (1-qs)^{n-1} \right] \geq 0, \end{aligned}$$

therefore, $G_2(t, qs) \geq 0$. Moreover, for fixed $s \in [0, 1]$,

$$\begin{aligned} {}_t D_q g_1(t, qs) &= \frac{(1-qs)^{n-2}}{\Gamma_q(n-1)} - \frac{[n-1]_q (t-qs)^{n-2}}{\Gamma_q(n)} \\ &= \frac{[n-1]_q [(1-qs)^{n-2} - (t-qs)^{n-2}]}{\Gamma_q(n)} \geq 0, \end{aligned}$$

i.e., $g_1(t, qs)$ is an increasing function of t . Obviously, $g_2(t, qs)$ is increasing in t , therefore $G_2(t, qs)$ is an increasing function of t for fixed $s \in [0, 1]$. This concludes the proof of (9).

Suppose now that $t \geq qs$, Then

$$\begin{aligned} \frac{G_2(t, qs)}{G_2(1, qs)} &= \frac{[n-1]_q t(1-qs)^{n-2} - (t-qs)^{n-1}}{[n-1]_q (1-qs)^{n-2} - (1-qs)^{n-1}} \\ &\geq \frac{[n-1]_q t(1-qs)^{n-2} - t(1-qs)^{n-1}}{[n-1]_q (1-qs)^{n-2} - (1-qs)^{n-1}} = t. \end{aligned}$$

If $t \leq qs$. Then

$$\begin{aligned} \frac{G_2(t, qs)}{G_2(1, qs)} &= \frac{t(1-qs)^{n-2}/\Gamma_q(n-1)}{G_2(1, qs)} \\ &> \frac{t(1-qs)^{n-2}/\Gamma_q(n-1)}{(1-qs)^{n-2}/\Gamma_q(n-1)} = t, \end{aligned}$$

and this finishes the proof of (10).

Lemma: 3.3. Let $y \in C[0, 1]$, then the q -boundary value problem

$$D_q^n u_3(t) + y(t) = 0, \quad 0 < t < 1,$$

$$u_3(0) = D_q u_3(0) = D_q^2 u_3(0) = \dots = D_q^{n-2} u_3(0) = 0, D_q u_3(1) = 0,$$

has a unique solution

$$u_3(t) = \int_0^1 G_3(t, qs) y(s) d_q s,$$

where

$$G_3(t, s) = \begin{cases} \frac{t^{n-1}(1-s)^{n-2}}{\Gamma_q(n)} - \frac{(t-s)^{n-1}}{\Gamma_q(n)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{n-1}(1-s)^{n-2}}{\Gamma_q(n)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Proof: We may apply Lemma 2.1 and Theorem 2.2, we see that

$$u_3(t) = u_3(0) + \frac{D_q u_3(0)}{\Gamma_q(2)} t + \frac{D_q^2 u_3(0)}{\Gamma_q(3)} t^2 + \dots + \frac{D_q^{n-1} u_3(0)}{\Gamma_q(n)} t^{n-1} - I_q^\alpha y(t).$$

By using the boundary conditions $u_3(0) = D_q u_3(0) = D_q^2 u_3(0) = \dots = D_q^{(n-2)} u_3(0) = 0$, we get

$$u_3(t) = B_3 t^{n-1} - \int_0^t \frac{(t-qs)^{n-1}}{\Gamma_q(n)} y(s) d_q s. \tag{11}$$

Differentiating both sides of (11) one obtain, with the help (6) and (7),

$$(D_q u_3)(t) = B_3 [n-1]_q t^{n-2} - \int_0^t \frac{[n-1]_q (t-qs)^{n-2}}{\Gamma_q(n)} y(s) d_q s,$$

then by the condition $D_q u_3(1) = 0$, we have

$$B_3 = \int_0^1 \frac{(1-qs)^{n-2}}{\Gamma_q(n)} y(s) d_q s,$$

the proof is complete.

Lemma: 3.4. Function G_3 defined above satisfies the following conditions:

$$G_3(t, qs) \geq 0 \text{ and } G_3(t, qs) \leq G_3(1, qs), \quad 0 \leq t, s \leq 1, \tag{12}$$

$$G_3(t, qs) \geq \eta_3(t) G_3(1, qs), \quad 0 \leq t, s \leq 1 \text{ with } \eta_3(t) = t^{n-1}. \tag{13}$$

Proof: We start by defining two functions

$$g_3(t, s) = \frac{t^{n-1}(1-s)^{n-2}}{\Gamma_q(n)} - \frac{(t-s)^{n-1}}{\Gamma_q(n)}, \quad 0 \leq s \leq t \leq 1,$$

and

$$g_4(t, s) = \frac{t^{n-1}(1-s)^{n-2}}{\Gamma_q(n)}, \quad 0 \leq t \leq s \leq 1.$$

It is clear that $g_4(t, qs) \geq 0$. Now, $g_3(0, qs) = 0$ and, in view of Remark 2.1, for $t \neq 0$

$$\begin{aligned} g_3(t, qs) &= \frac{t^{n-1}(1-qs)^{n-2}}{\Gamma_q(n)} - \frac{(t-qs)^{n-1}}{\Gamma_q(n)} \\ &\geq \frac{t^{n-1}(1-qs)^{n-2}}{\Gamma_q(n)} - \frac{t^{n-1}(1-qs)^{n-1}}{\Gamma_q(n)} \\ &= \frac{t^{n-1}}{\Gamma_q(n)} \left[(1-qs)^{n-2} - (1-qs)^{n-1} \right] \geq 0, \end{aligned}$$

therefore, $G_3(t, qs) \geq 0$. Moreover, for fixed $s \in [0, 1]$,

$$\begin{aligned} {}_t D_q g_3(t, qs) &= \frac{[n-1]_q t^{n-2} (1-qs)^{n-2} - [n-1]_q (t-qs)^{n-2}}{\Gamma_q(n)} \\ &\geq \frac{[n-1]_q t^{n-2} (1-qs)^{n-2} - [n-1]_q t^{n-2} (1-qs)^{n-2}}{\Gamma_q(n)} = 0, \end{aligned}$$

i.e., $g_3(t, qs)$ is an increasing function of t . Obviously, $g_4(t, qs)$ is increasing in t , therefore $G_3(t, qs)$ is an increasing function of t for fixed $s \in [0, 1]$. This concludes the proof of (12).

Suppose now that $t \geq qs$, Then

$$\begin{aligned} \frac{G_3(t, qs)}{G_3(1, qs)} &= \frac{t^{n-1} (1-qs)^{n-2} - (t-qs)^{n-1}}{(1-qs)^{n-2} - (1-qs)^{n-1}} \\ &\geq \frac{t^{n-1} (1-qs)^{n-2} - t^{n-1} (1-qs)^{n-1}}{(1-qs)^{n-2} - (1-qs)^{n-1}} = t^{n-1}. \end{aligned}$$

If $t \leq qs$. Then

$$\begin{aligned} \frac{G_3(t, qs)}{G_3(1, qs)} &= \frac{t^{n-1} (1-qs)^{n-2} / \Gamma_q(n)}{G_3(1, qs)} \\ &> \frac{t^{n-1} (1-qs)^{n-2} / \Gamma_q(n)}{(1-qs)^{n-2} / \Gamma_q(n)} = t^{n-1}, \end{aligned}$$

and this finishes the proof of (13).

Lemma: 3.5. Let $y \in C[0, 1]$, then the q -boundary value problem

$$D_q^n u_4(t) + y(t) = 0, \quad 0 < t < 1,$$

$$u_4(0) = D_q u_4(0) = D_q^2 u_4(0) = \dots = D_q^{n-2} u_4(0) = 0, D_q^2 u_4(1) = 0,$$

has a unique solution

$$u_4(t) = \int_0^1 G_4(t, qs) y(s) d_q s,$$

where

$$G_4(t, s) = \begin{cases} \frac{t^{n-1} (1-s)^{n-3}}{\Gamma_q(n)} - \frac{(t-s)^{n-1}}{\Gamma_q(n)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{n-1} (1-s)^{n-3}}{\Gamma_q(n)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

The proof of Lemma 3.5 is very similar to that of Lemma 3.3 and therefore omitted.

Lemma: 3.6. Function G_4 defined above satisfies the following conditions:

$$G_4(t, qs) \geq 0 \text{ and } G_4(t, qs) \leq G_4(1, qs), \quad 0 \leq t, s \leq 1, \quad (14)$$

$$G_4(t, qs) \geq \eta_4(t)G_4(1, qs), \quad 0 \leq t, s \leq 1 \text{ with } \eta_4(t) = t^{n-1}. \quad (15)$$

Proof: We start by defining two functions

$$g_5(t, s) = \frac{t^{n-1}(1-s)^{n-3}}{\Gamma_q(n)} - \frac{(t-s)^{n-1}}{\Gamma_q(n)}, \quad 0 \leq s \leq t \leq 1,$$

and

$$g_6(t, s) = \frac{t^{n-1}(1-s)^{n-3}}{\Gamma_q(n)}, \quad 0 \leq t \leq s \leq 1.$$

It is clear that $g_6(t, qs) \geq 0$. Now, $g_5(0, qs) = 0$ and, in view of Remark 2.1, for $t \neq 0$

$$\begin{aligned} g_5(t, qs) &= \frac{t^{n-1}(1-qs)^{n-3}}{\Gamma_q(n)} - \frac{(t-qs)^{n-1}}{\Gamma_q(n)} \\ &\geq \frac{t^{n-1}(1-qs)^{n-3}}{\Gamma_q(n)} - \frac{t^{n-1}(1-qs)^{n-1}}{\Gamma_q(n)} \\ &= \frac{t^{n-1}}{\Gamma_q(n)} \left[(1-qs)^{n-3} - (1-qs)^{n-1} \right] \geq 0, \end{aligned}$$

therefore, $G_4(t, qs) \geq 0$. Moreover, for fixed $s \in [0, 1]$,

$$\begin{aligned} {}_t D_q g_5(t, qs) &= \frac{[n-1]_q t^{n-2} (1-qs)^{n-3} - [n-1]_q (t-qs)^{n-2}}{\Gamma_q(n)} \\ &\geq \frac{t^{n-2} (1-qs)^{n-3} - t^{n-2} (1-qs)^{n-2}}{\Gamma_q(n-1)} \\ &= \frac{t^{n-2}}{\Gamma_q(n-1)} \left[(1-qs)^{n-3} - (1-qs)^{n-2} \right] \geq 0, \end{aligned}$$

i.e., $g_5(t, qs)$ is an increasing function of t . Obviously, $g_6(t, qs)$ is increasing in t , therefore $G_4(t, qs)$ is an increasing function of t for fixed $s \in [0, 1]$. This concludes the proof of (14).

Suppose now that $t \geq qs$, Then

$$\begin{aligned} \frac{G_4(t, qs)}{G_4(1, qs)} &= \frac{t^{n-1}(1-qs)^{n-3} - (t-qs)^{n-1}}{(1-qs)^{n-3} - (1-qs)^{n-1}} \\ &\geq \frac{t^{n-1}(1-qs)^{n-3} - t^{n-1}(1-qs)^{n-1}}{(1-qs)^{n-3} - (1-qs)^{n-1}} = t^{n-1}. \end{aligned}$$

If $t \leq qs$. Then

$$\frac{G_4(t, qs)}{G_4(1, qs)} = \frac{t^{n-1}(1-qs)^{n-3}/\Gamma_q(n)}{G_4(1, qs)} > \frac{t^{n-1}(1-qs)^{n-3}/\Gamma_q(n)}{(1-qs)^{n-3}/\Gamma_q(n)} = t^{n-1},$$

and this finishes the proof of (15).

4. Main results :

In this section, we will apply Krasnoselskii’s fixed point theorem to the eigenvalue problem (1), (i) (i=2,3,4).

Remark: 3.1 : If we let $0 < \tau < 1$, then

$$\min_{t \in [\tau, 1]} G_i(t, qs) \geq \eta_i(\tau)G_i(1, qs), \text{ for } s \in [0, 1]. \tag{16}$$

Let $X = C[0, 1]$ be the Banach space endowed with norm $\|u_i\| = \max_{t \in [\tau, 1]} |u_i(t)|$. Let $\tau = q^n$ [9] for a given $n \in \mathbb{N}$

and define the cone $P \subset X$ by

$$P = \left\{ u_i \in X : u_i(t) \geq 0, \min_{t \in [\tau, 1]} u_i(t) \geq \eta_i(\tau) \|u_i\| \right\}.$$

Remark: 3.2: It follows from the non-negativeness and continuity of G_i, a and f that the operator $T : P \rightarrow X$ defined by

$$Tu_i(t) = \lambda \int_0^1 G_i(t, qs) a(s) f(u_i(s)) d_q s,$$

is completely continuous. Moreover, for $u_i \in P, (Tu_i)(t) \geq 0$ on $[0, 1]$ and

$$\begin{aligned} \min_{t \in [\tau, 1]} (Tu_i)(t) &= \min_{t \in [\tau, 1]} \lambda \int_0^1 G_i(t, qs) a(s) f(u_i(s)) d_q s \\ &\geq \eta_i(\tau) \int_0^1 G_i(1, qs) a(s) f(u_i(s)) d_q s \\ &= \eta_i(\tau) \|Tu_i\|, \end{aligned}$$

that is, $T(P) \subset P$.

For our purposes, let us define two constants

$$\gamma = \left(\lambda \int_0^1 G_i(1, qs) a(s) d_q s \right)^{-1} \text{ and } \beta = \left(\eta_i(\tau) \lambda \int_\tau^1 G_i(1, qs) a(s) d_q s \right)^{-1}.$$

Our existence result is now presented.

Theorem: 3.1. Let $\tau = q^n$ with $n \in \mathbb{N}$. Suppose that $f(u_i)$ is a nonnegative continuous function on $[0, 1] \times [0, \infty)$. If there exist two positive constants $R > r > 0$ such that

$$\max_{(s, u_i) \in [0, 1] \times [0, r]} f(u_i(t)) \leq \gamma u_i, \tag{17}$$

$$\min_{(s, u_i) \in [\tau, 1] \times [\eta_i(\tau)R, R]} f(u_i(t)) \geq \beta u_i, \tag{18}$$

then problem (1)–(4) has a solution u_i satisfying $u_i(t) > 0$ for $t \in (0, 1]$.

Proof: Since the operator $T : P \rightarrow X$ is completely continuous we only have to show that the operator equation $u_i = Tu_i$ has a solution satisfying $u_i(t) > 0$ for $t \in (0, 1]$.

Let $\Omega_1 = \{u_i \in X : \|u_i\| < r\}$. For $u_i \in P \cap \partial\Omega_1$, we have $0 \leq u_i(t) \leq r$ on $[0, 1]$. Using (9),(12),(14) and (17) we obtain,

$$\begin{aligned} \|Tu_i\| &= \max_{t \in [0,1]} \lambda \int_0^1 G_i(t, qs) a(s) f(u_i(s)) d_q s \\ &\leq \lambda \int_0^1 G_i(1, qs) a(s) \max_{(s, u_i) \in [0,1] \times [0,r]} f(u_i(s)) d_q s \\ &\leq \gamma r \lambda \int_0^1 G_i(1, qs) a(s) d_q s \\ &= r = \|u_i\|. \end{aligned}$$

Let $\Omega_2 = \{u_i \in X : \|u_i\| < R\}$. For $u_i \in P \cap \partial\Omega_2$, we have $\eta_i(\tau)R_2 \leq u(t) \leq R$ on $[\tau, 1]$. Using (16) and (18), and the fact that $\tau = q^n$ [9], we obtain

$$\begin{aligned} \|Tu_i\| &= \max_{t \in [0,1]} \lambda \int_0^1 G_i(t, qs) a(s) f(u_i(s)) d_q s \\ &\geq \lambda \int_{\tau}^1 G_i(1, qs) a(s) \min_{(s, u_i) \in [0, \tau] \times [\eta_i(\tau)R, R]} f(u_i(s)) d_q s \\ &\geq \beta \eta_i(\tau) R \lambda \int_{\tau}^1 G_i(1, qs) a(s) d_q s \\ &= R = \|u_i\|. \end{aligned}$$

Now, Theorem 3.1 assures the existence of a fixed point u_i of T such that $r \leq \|u_i\| \leq R$. To finish the proof, note that by (10),(13) and (15)

$$\begin{aligned} u_i(t) &= \lambda \int_0^1 G_i(t, qs) a(s) f(u_i(s)) d_q s \\ &\geq \eta_i(t) \lambda \int_0^1 G_i(1, qs) a(s) f(u_i(s)) d_q s \\ &= \eta_i(t) \|u_i\|, \end{aligned}$$

which implies that $u_i(t) \geq \eta_i(t) r$. Therefore, $u_i(t) > 0$ for $t \in (0, 1]$ and the proof is done.

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