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A COMMON FIXED POINT THEOREM FOR FOUR SELF MAPS ON A MENGER SPACE WITH HADZIC TYPE t-NORM

K.P.R. Sastry [1], Ch. Srinivasa Rao [2],
*R. Venkata Bhaskar [3], G.Srirama Murthy [4] and D.Narayana Rao [5]

[1] *8-28-8/1, Tamil Street, Chinna Waltair, Visakhapatnam -530 017, India*
E-mail: kprsastry@hotmail.com
[2] *Department of Mathematics, Mrs. A.V.N. College, Visakhapatnam -530 001, India*
E-mail: drcsr41@yahoo.com
[3] *Department of Mathematics, Raghu Engineering College, Visakhapatnam -531 162, India*
E-mail: rvenkatab@gmail.com
[4] *Department of Mathematics, M.R.P.G.College, Vizianagaram -535 002, India.*
E-mail: sriramamurthy.gedela@yahoo.in
[5] *Department of Mathematics, Andhra University, Visakhapatnam – 530 003, India*
E-mail: narayanaramanajp@gmail.com

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ABSTRACT

In this paper, we proved a common fixed point theorem for four self maps on a complete Menger space and obtain a result of Geeta Modi and S. S. Khare [1] as a corollary.

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***Key words**: Menger space, fixed point theorem, Hadzic type t-norm, R-weakly commuting mappings.*

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1. INTRODUCTION:

There have been a number of generalizations of a metric space one among them is Menger space introduced in 1942 by Menger [3] who used distribution functions instead of non-negative real numbers as values of the metric. Schweizer and Sklar [8] studied this concept and established some fundamental results on this space. In 1978, Hadzic [2] introduced a class $\mathcal{H}$ of t – norms. Throughout this paper, $\mathbb{R}$ represents the real line, $\mathbb{R}^+ = [0, \infty)$ and N is set of positive integers.

Definition: 1.1

A mapping $F: \mathbb{R} \longrightarrow \mathbb{R}^+$ is said to be a distribution function if

(i) F is non- decreasing

(ii) F is left continuous

(iii) $\inf_{x\in\mathbb{R}} F(x) = 0$ and $\sup_{x\in\mathbb{R}} F(x) = 1$.

We denote the set of all distribution functions by $\mathcal{D}$.

Define $H: \mathbb{R} \longrightarrow \mathbb{R}^+$ by $H(t) = \begin{cases} 0 \; if \; t \le 0 \\ 1 \; if \; t > 0 \end{cases}$, then H is called the Heaviside function. Clearly H is a distribution function.

Definition: 1.2 (B. Schweizer and A. Sklar, [6]):

A Probabilistic Metric Space is an ordered pair (X, F), where X is non-empty set and F is a function defined on $X \times X$ to $\mathcal{D}$ which satisfies the following conditions:

Corresponding author: R Venkata Bhaskar[3]*, *E-mail: rvenkatab@gmail.com
[3] *Department of Mathematics, Raghu Engineering College, Visakhapatnam -531 162, India*

For $x, y, z \in X$

(i) $F_{x,y}(0) = 0$

(ii) $F_{x,y}(u) = 1$ for all $u > 0$ if and only if $x = y$

(iii) $F_{x,y}(u) = F_{y,x}(u)$

(iv) $F_{x,y}(u) = 1$ and $F_{y,z}(v) = 1 \Longrightarrow F_{x,z}(u+v) = 1$.

Definition: 1.3 (B. Schweizer and A. Sklar, [8]):

A function $t: [0,1] \times [0,1] \to [0,1]$ is said to be a t-norm if it satisfies the following conditions: For $a, b, c, d \in [0,1]$

(i) $t(a, 1) = a$

(ii) $t(a, b) = t(b, a)$

(iii) $t(t(a, b), c) = t\big(a, t(b, c)\big)$

(iv) $t(c, d) \geq t(a, b)$ if $c \geq a$ and $d \geq b$.

For $a, b \in [0,1]$, if we define $t(a, b) = min\{a, b\}$, then t is a t-norm.

Definition: 1.4 (K.Menger, [3]):

Let X be a non-empty set, t a t-norm and F is a function defined on $X \times X$ to $\mathcal{D}$ satisfy:

(i) $F_{x,y}(0) = 0 \ \forall \ x, y \in X$

(ii) $F_{x,y}(u) = 1$ for all $u > 0$ if and only if $x = y$

(iii) $F_{x,y}(u) = F_{y,x}(u) \ \forall \ x, y \in X$

(iv) $F_{x,y}(u+v) \geq t\left(F_{x,z}(u), F_{z,y}(v)\right) \forall \ u, v \geq 0$ and $x, y, z \in X$.

Then the triple (X, F, t) is called Menger space.

Definition: 1.5 (B. Schweizer and A. Sklar, [7]):

A sequence $\{x_n\}$ in Menger space (X, F, t) is a Cauchy sequence if for any $\varepsilon, \lambda > 0 \ \exists \ N(\varepsilon, \lambda)$ such that $F_{x_n, x_m}(\varepsilon) > 1 - \lambda$ for $n, m > N$.

Definition: 1.6 (B. Schweizer and A. Sklar, [7]):

A sequence $\{x_n\}$ in Menger space (X, F, t) is said to converge to x if for any $\varepsilon, \lambda > 0 \ \exists \ N(\varepsilon, \lambda)$ such that $F_{x_n, x}(\varepsilon) > 1 - \lambda$ for $n > N$.

Definition: 1.7 (B. Schweizer and A. Sklar, [7]):

A Menger space (X, F, t) is said to be complete if every Cauchy sequence in (X, F, t) is convergent.

Note: This notion of converge of sequences in X gives raise to a topology which is Hausdorff if t is continuous.

Definition: 1.8 (R. P. Panth, [4]):

Two mappings f, g of a Menger space (X, F, t) into itself are said to be R-weakly commuting provided there exist some positive real number R such that $F_{fgx, gfx}(u) \geq F_{fx, gx}\left(\frac{u}{R}\right)$ for every $x \in X$.

Definition: 1.9:
Let (X, F, t) be a Menger space such that t is continuous and f, g be mappings from X into itself. Then f and g are said to be compatible if $\lim_{n\to\infty} F_{fgx_n, gfx_n}(u) = 1 \; \forall u > 0$ when ever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Definition: 1.10:

Two mappings f, g of a Menger space (X, F, t) into itself are said to be reciprocally continuous if $\lim_{n\to\infty} fgx_n = fp$ and $\lim_{n\to\infty} gfx_n = gp$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p$ for some $p \in X$.

Definition: 1.11 (O.Hadzic , [2]):

Let t be a t- norm. For any $x \in [0,1]$ write $t^0(x) = 1$ and $t^1(x) = t(t^0(x), x) = t(1, x) = x$. In general recursively define $t^{n+1}(x) = t(t^n(x), x)$, for $n = 0,1,2$ ...

Suppose that given ε in $(0,1)$ $\exists\, \delta \in (0,1)$ $\ni x > 1 - \delta \Rightarrow t^n(x) > 1 - \varepsilon \;\; \forall\, n \in N$

Then the sequence $\{t^n\}$ is said to be equicontinuous at 1. If $\{t^n\}$ is equicontinuous at 1, then we say that t is a Hadzic type t- norm. Define t_{min} by $t_{min}(a, b) = min\{a, b\}$ for $a, b \in [0,1]$, then we observe that t_{min} is a continuous t-norm of Hadzic type.

The following Lemma is proved in Sastry, Babu and Sandhya [5].

Lemma: 1.12 (K. P. R. Sastry, G. V. R. Babu and M. L. Sandhya , [5]):

Let (X, F, t) be a Menger space with continuous Hadzic-type t- norm t and $0 < a < 1$. Suppose $\{x_n\}$ is a sequence in X such that for any $u > 0$, $F_{x_n, x_{n+1}}(u) \geq F_{x_0, x_1}\left(\frac{u}{a^n}\right)$. Then $\{x_n\}$ is a Cauchy sequence.

Geeta Modi and S. S. Khare [1] proved the following theorem.

Theorem: 1.13 (Geeta Modi and S.S.Khare, [1]):

Let (X, F, t) be a complete Menger space where t is defined as $t(a, b) = min\{a, b\}, a, b \in [0,1]$. A, B, S and T be mappings from X to itself such that

$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ (1.13.1)

the pair (A, S) or (B, T) are compatible pair of reciprocally continuous mappings (1.13.2)

(A, S) , (B, T) are point wise R-weakly commuting pair of mappings (1.13.3)

for all $x, y \in X, k \in (0,1), u > 0$ (1.13.4)

$$F^3_{Ax,By}(ku) \geq max\{F^3_{Sx,Ty}(u), F^3_{Ax,Sx}(u), F^3_{By,Ty}(u), F_{Ax,Ty}(2u), F_{By,Sx}(2u), F^2_{By,Ty}(u)\}$$

for all $x, y \in X, \lim_{u\to\infty} F_{x,y}(u) = 1$ (1.13.5)

Then A, B, S and T have a unique common fixed point in X.

2. MAIN RESULT:

In this paper, we show that Theorem 1.13 [1] is not in general valid, but valid if 0 < R< 1. Further, when 0 < R< 1, we improve Theorem 1.13 significantly by

(i) replacing the minimum t norm by Hadzic type t – norm

(ii) do away with condition (1.13.2) and

(iii) relax condition (1.13.4)

Also we conclude that under the given conditions, $A = B =$ constant.

We observe that (1.13.5) is unnecessary, since it is a part of the definition of a distribution function.

Now we state our main result.

Theorem: 2.1
Let (X, F, t) be a complete Menger space where t is Hadzic type t - norm. Suppose A, B, S and T are mappings from X to itself such that

$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ (2.1.1)

(A, S) , $(B, T)$are R-weakly commuting pair of mappings. (2.1.2)

There exist $k \in (0,1)$ such that for all $x, y \in X$ and $u > 0$ (2.1.3)

$$F_{Ax,By}(ku) \geq max\{F_{Ax,Sx}(u), F_{By,Ty}(u)\}$$

Then $A = B$ is a constant function, Further, if 0 < R< 1, then A, B, S and T have a unique common fixed point in X.

Proof: Let$x_0 \in X$. By (2.1.1), there exist $x_1 \in X$ such that $Ax_0 = Tx_1 = y_1$(say). Inductively, construct a sequence $\{y_n\}$ in X such that $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n = 1,2,3$ ...

We have

$$F_{y_{2n+1},y_{2n+2}}(ku) = F_{Ax_{2n},Bx_{2n+1}}(ku)$$

$$\geq max\{F_{Ax_{2n},Sx_{2n}}(u), F_{Bx_{2n+1},Tx_{2n+1}}(u)\} \quad \text{from (2.1.3)}$$

$$= max\{F_{y_{2n+1},y_{2n}}(u), F_{y_{2n+2},y_{2n+1}}(u)\}$$

$$= F_{y_{2n+1},y_{2n}}(u)$$

$$\therefore F_{y_{2n+1},y_{2n+2}}(ku) \geq F_{y_{2n+1},y_{2n}}(u) \quad (2.1.4)$$

Also

$$F_{y_{2n},y_{2n+1}}(ku) = F_{Bx_{2n-1},Ax_{2n}}(ku)$$

$$= F_{Ax_{2n},Bx_{2n-1}}(ku)$$

$$\geq max\{F_{Ax_{2n},Sx_{2n}}(u), F_{Bx_{2n-1},Tx_{2n-1}}(u)\}$$

$$= max\{F_{y_{2n+1},y_{2n}}(u), F_{y_{2n},y_{2n-1}}(u)\}$$

$$= F_{y_{2n},y_{2n-1}}(u)$$

$$\therefore F_{y_{2n},y_{2n+1}}(ku) \geq F_{y_{2n-1},y_{2n}}(u) \quad (2.1.5)$$

From (2.1.4) and (2.1.5), we have

$$F_{y_n,y_{n+1}}(ku) \geq F_{y_{n-1},y_n}(u) \quad (2.1.6)$$

From (2.1.6), we have

$$F_{y_n,y_{n+1}}(u) \geq F_{y_{n-1},y_n}\left(\frac{u}{k}\right) \geq F_{y_{n-2},y_{n-1}}\left(\frac{u}{k^2}\right) \geq F_{y_{n-3},y_{n-2}}\left(\frac{u}{k^3}\right) \dots \geq F_{y_0,y_1}\left(\frac{u}{k^n}\right)$$

$$\therefore \quad F_{y_n,y_{n+1}}(u) \geq F_{y_0,y_1}\left(\frac{u}{k^n}\right)$$

Since t is of Hadzic type t – norm, from Lemma 1.12, $\{y_n\}$ is a Cauchy sequence.

Since (X, F, t) is complete, there exist $z \in X$ such that $\lim_{n\to\infty} y_n = z$

Then $\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} Tx_{2n-1} = \lim_{n\to\infty} Ax_{2n-2} = z$ and
$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Bx_{2n-1} = z$

Put $x = x_{2n}$ and $y = z$ in (2.1.3), we get

$$F_{Ax_{2n},Bz}(ku) \geq max\{F_{Ax_{2n},Sx_{2n}}(u), F_{Bz,Tz}(u)\}$$

On letting $n \to \infty$

$$F_{z,Bz}(ku) \geq max\{F_{z,z}(u), F_{Bz,Tz}(u)\}$$

$$= max\{1, F_{Bz,Tz}(u)\} = 1$$

$\therefore\ F_{z,Bz}(ku) \geq 1$

$\therefore Bz = z.$

Put $x = z$ and $y = x_{2n}$ in (2.1.3), we get

$$F_{Az,Bx_{2n}}(ku) \geq max\{F_{Az,Sz}(u), F_{Bx_{2n},Tx_{2n}}(u)\}$$

On letting $n \to \infty$

$$F_{Az,z}(ku) \geq max\{F_{Az,Sz}(u), F_{z,z}(u)\}$$

$$= max\{F_{Az,Sz}(u), 1\} = 1$$

$\therefore\ F_{Az,z}(ku) \geq 1$

$\therefore$ A$z = z.$

Thus A$z = Bz = z$

$\therefore\ A = B$ is a constant function.

Since the pair (A, S) is R- weakly commuting there exist a positive real number R such that

$$F_{ASz,SAz}(u) \geq F_{Az,Sz}\left(\frac{u}{R}\right)$$

$$\Longrightarrow\ F_{z,Sz}(u) \geq F_{z,Sz}\left(\frac{u}{R}\right)$$

If 0 < R< 1, then $z = Sz$.

Also since, the pair (B, T) is R- weakly commuting, if 0 < R< 1, then we can show that $z = Tz$.

$\therefore Az = Bz = Sz = Tz = z$

Thus z is a common fixed point of A, B, S and T.

Let w be another fixed point of A, B, S and T.

Put $x = z$ and $y = w$ in (2.1.3), we get

$$F_{Az,Bw}(ku) \geq max\{F_{Az,Sz}(u), F_{Bw,Tw}(u)\}$$

$\Longrightarrow\ F_{z,w}(ku) \geq max\{F_{z,z}(u), F_{w,w}(u)\}$

$\Longrightarrow\ F_{z,w}(ku) \geq max\{1,1\} = 1$

$\Longrightarrow\ F_{z,w}(ku) \geq 1$

Therefore $z = w$

Hence z is a unique common fixed point of A, B, S and T.

Under the condition of Theorem 2.1, the following example shows that A, B, S and T may not have a common fixed point if R ≥ 1, even in a metric space.

Example: 2.2:
Let $X = \{1,2,3, \dots\}$. For any $m, n \in X$ and $t \in \mathbb{R}$, define $F_{m,n}(t) = H(t - |m - n|)$.

Define t_{min} by $t_{min}(a, b) = min\{a, b\}$. Then t is a Hadzic type t-norm. Then clearly (X, F, t) is a complete Menger space.

Now define A, B, S and T on X as follows

$An = 3 = Bn, Sn = n + 1 = Tn$, for $n = 1,2,3 \dots$

Then A, B, S and T satisfy the hypothesis of Theorem 2.1 with R ≥ 1. Further A, B, S and T do not have a common fixed point.

Corollary: 2.3: Theorem 1.13 with 0 < R< 1

Let (X, F, t) be a complete Menger space where t is defined as $t(a, b) = min\{a, b\}, a, b \in [0,1]$. A, B, S and T be mappings from X to itself such that

$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ (1.13.1)

the pair (A, S) or (B, T) are compatible pair of reciprocally continuous mappings (1.13.2)

(A, S) , (B, T) are point wise R-weakly commuting pair of mappings with 0 < R< 1 (1.13.3)

for all $x, y \in X, k \in (0,1), u > 0$ (1.13.4)

$$F^3_{Ax,By}(ku) \geq max\{F^3_{Sx,Ty}(u), F^3_{Ax,Sx}(u), F^3_{By,Ty}(u), F_{Ax,Ty}(2u), F_{By,Sx}(2u), F^2_{By,Ty}(u)\}$$

for all $x, y \in X, \lim_{u\to\infty} F_{x,y}(u) = 1$ (1.13.5)

Then . A, B, S and T have a unique common fixed point in X.

Proof: Since (1.13.4) ⇒ (2.1.3), the result follows.

REFERENCES:

[1] Geeta Modi and S.S.Khare : Common fixed points of four mappings of Menger space, The Mathematics Education, Vol.XLI(4), 2007, 289-292.(issued in 2010).

[2] O.Hadzic: A generalization of the contraction principle in probabilistic metric spaces, Univ.u.Nvom Sadu Zb. Road, Prirod-Mat, Fak., 10(1980), 13-21 (1981).

[3] K.Menger: Statistical metrics, Proc. Of the National Academy of sciences of United States of America, 28 (1942), 535-537.

[4] R.P.Panth: Common fixed points of non-commuting mapping, J.Math.Anal.Apply., 188, (1994), 436-440.

[5] K.P.R.Sastry, G.V.R.Babu and M.L.Sandhya: Weak contractions in Menger spaces-II, Journal of Adv.Research in pure mathematics, vol.2.issue 1,2010, 65-73.

[6] B.Schweizer and A. Sklar: Probabilistic metric spaces, North Hollond series in probability and applied mathematics, 1983, MR0790314 .

[7] B.Schweizer and A. Sklar: Statistical metric spaces, North Holland Amsterdam (1983).

[8] B.Schweizer and A. Sklar: Statistical metric spaces, Pacific.j.of mathematics, 10 (1960), 313-334.

