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A RESULT ON COMMON FIXED POINT THEOREM IN COMPLETE L-FUZZY METRIC SPACES

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ABSTRACT

The notion of fuzzy sets was introduced by Zadeh. Many authors have studied fixed point theory in fuzzy metric spaces. In the sequel, we shall adopt the usual terminology, notation and conventions of L-fuzzy metric spaces introduced by Saadati [13]. Which are a generalization of fuzzy metric spaces and intuitionistic fuzzy metric spaces. In this paper we prove a common fixed point theorem in L Fuzzy Metric Space which is generalization of result in Adibi et al. [1]

KEY WORDS: *Fuzzy Metric Spaces, Intuitionistic Fuzzy metrics spaces, L – Fuzzy metric spaces.*

Definition: 1 Let $L = (L, \_ \leq_L)$ be a complete lattice, and U a non-empty set called a universe. An L-fuzzy set A on U is defined as a mapping $A: U \to L$. For each u in U, A (u) represents the degree (in L) to which u satisfies A.

Definition: 2 A triangular norm (t-norm) on L is a mapping $T: L^2 \to L$ satisfying the following conditions:

(1) $(\forall x \in L)(T(x, 1_L) = x)$, (boundary condition)
(2) $(\forall (x, y) \in L^2)(T(x, y) = T(y, x))$, (commutativity)
(3) $(\forall (x, y, z) \in L^3)(T(x, T(y, z)) = T(T(x, y), z))$, (associativity)
(4) $(\forall (x, x_0, y, y_0) \in L^4)$ $(x \leq_L x_0$ and $y \leq_L y_0)$ $T(x, y) \leq_L T(x_0, y_0))$. (monotonicity)

Definition: 3 A t–norm T on L is said to be continuous if for any $x, y \in L$ and any sequences$\{x_n\}$ and $\{y_n\}$ which converge to x and y we have

$$\lim_{n \to \infty} T(x_n, y_n) = T(x, y)$$

Definition: 4. The 3-tuple (X,M,T) is said to be an L-fuzzy metric space if X is an arbitrary (non-empty) set, T is a continuous t–norm on L and M is an L-fuzzy set on $X^2 \times\,]0,+\infty[$ satisfying the following conditions for every x, y, z in X and t, s in $]0,+\infty[$

(a) $M(x, y, t) >_L 0_L$,
(b) $M(x, y, t) = 1_L$ for all $t > 0$ if and only if $x = y$,
(c) $M(x, y, t) = M(y, x, t)$,
(d) $T(M(x, y, t), M(y, z, s)) \leq_L M(x, z, t + s)$,
(e) $M(x, y, \cdot): [0, \infty] \to L$ is continuous and $\lim_{t \to \infty} M(x, y, t) = 1_L$.

Definition: 5 A sequence $\{x_n\}$ $n \in N$ in an L-fuzzy metric space (X,M, T) is called a Cauchy sequence, if for each $\varepsilon \in L \setminus \{0_L\}$ and $t > 0$, there exists $n_0 \in N$ such that for all $m \geq n \geq n_0$ $(n \geq m \geq n_0)$, $M(x_m, x_n, t) \geq_L N(\varepsilon)$.

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Definition: 6 The sequence $\{x_n\}$ $n\in N$ is said to be convergent to $x \in X$ in the L-fuzzy metric space (X, M, T) $M(x_n, x, t) = M(x, x_n, t) \to 1_L$ whenever $n \to +\infty$ for every $t > 0$.

Definition: 7 A L-fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent.

Definition: 8 Let A and S be mappings from an L-fuzzy metric space (X,M, T)into itself. Then the mappings are said to be compatible if $\lim_{n \to\infty} (ASx_n, SAx_n, t) = 1_L$, $\forall t > 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$.

Lemma: 1 [13,14] Consider the set L* and the operation $\leq_L$ defined by: $L^* = \{ (x_1, x_2) : (x_1, x_2) \in [0, 1]^2$ and $x_1 + x_2 \leq 1\}$, $(x_1, x_2) \leq_{L^*} (y_1, y_2)$ $x_1\leq y_1$ and $x_2\leq y_2$, for every $(x_1, x_2), (y_1, y_2) \in L^*$ Then $(L^*,\leq_{L^*})$ is a complete lattice .

Lemma; 2 **[15]** Let (X, M, T) be an L-fuzzy metric space. Then, M(x, y, t) is Non decreasing with respect to t, for all x, y in X.

Lemma: 3 [7] Let (X, M, T) be an L-fuzzy metric space. Then M is a continuous function on $X \times X\times [0,\infty]$.

Lemma: 4 [13] Let (X, M, T) be an L-fuzzy metric space. Define $E_{\lambda,M} : X^2 \to R^+ \cup \{0\}$ by $E_{\lambda,M}(x, y) = \inf\{t > 0 : M(x, y, t) >_L N(\lambda)\}$ for each $\lambda \in L \setminus \{0_L, 1_L\}$ and $x, y \in X$. Then we have

(i) For any $\mu \in L \setminus \{0_L, 1_L\}$ there exists $\lambda \in L \setminus \{0_L, 1_L\}$ such that $E_{\mu,M}(x_1, x_n) \leq E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n)$ for any $x_1, \ldots, x\in X$,

(ii) The sequence $\{x_n\}$ $n\in N$ is convergent to x w.r.t. L-fuzzy metric M if and only if $E_{\lambda,M} : (x_n, x) \to 0$.Also the sequence $\{x_n\}$ $n\in N$ is Cauchy w.r.t. L-fuzzy metric M if and only if it is Cauchy with $E_{\lambda,M}$.

Lemma: 5 [13] Let (X,M, T) be an L-fuzzy metric space. If $M(x_n, x_{n+1}, t) \geq_L M(x_0, x_1, k^n t)$ for some $k > 1$ and $n \in N$. Then $\{x_n\}$ is a Cauchy sequence.

MAIN RESULT:

Theorem .Let A, B, S and T be self-mappings of a complete L-fuzzy metric space (X, M, T), which has property (C), satisfying:

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and T(X), S(X) are two closed subsets of X, the pairs (A, S) and (B, T) are weak compatible,

$$M(Ax, By, t) \geq_L [\{M(Sx, Ty, kt), M(Ax, Sx, kt), M(Ay, Tx, kt), M(Ax, Ty, kt)\}]$$

for every x, y in X and some $k > 1$.Then A,B, S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be an arbitrary point in X. By (i), there exists $x_1, x_2 \in X$ such that $y_0 = Ax_0 = Tx_1$, $y_1 = Bx_1 = Sx_2$. Inductively, we can construct sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$,for $n = 0, 1, 2, \cdots$. Now, we prove that $\{y_n\}$ is a Cauchy sequence.

Let

$$d_m(t) = M(y_m, y_{m+1}, t), \quad t > 0.$$

Then, we have

$$d_{2n(t)} = M(y_{2n}, y_{2n+1}, t)$$

$$= M(Ax_{2n}, Bx_{2n+1}, t) \geq_L$$

$$\{M(Sx_{2n}, Tx_{2n+1}, kt), M(Ax_{2n}, Sx_{2n}, kt), M(Ax_{2n+1}, Tx_{2n+1}, kt), M(Ax_{2n}, Tx_{2n+1}, kt)\}$$

$=M(y_{2n-1}, y_{2n}, kt) = d_{2n-1}(kt)$.

Thus $d_{2n}(t) \geq_L d_{2n-1}(kt)$ for every $m = 2n \in N$ and $\forall t > 0$. Similarly for an odd integer $m = 2n + 1$,

we have $d_{2n+1}(t) \geq_L d_{2n}(kt)$. Hence, for every $n \in N$,

we have $d_n(t) \geq_L d_{n-1}(kt)$.

That is, $M(y_n, y_{n+1}, t) \geq_L M(y_{n-1}, y_n, kt) \geq_L \ldots$

$\geq_L M(y_0, y_1, k^n t)$.

So, by Lemma 5, $\{y_n\}$ is Cauchy and the completeness of X implies $\{y_n\}$ converges to y in X. That is, $\lim_{n\to\infty} y_n = y$

$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1}$

$= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y$.

As $B(X) \subseteq S(X)$, there exists $u \in X$ such that $Su = y$. By (iii), we have

$M(Au,Bx_{2n+1}, t) \geq_L [M(Su, Tx_{2n+1}, kt)), M(Ax, Sx, kt), M(Ay, Tx, kt), M(Ax, Ty, kt)\}] \vee M(Au,Su, kt)$

Since M is continuous, we get (whenever $n \to \infty$ in the above inequality),

$M(Au, y, t) \geq_L M(y, y, kt) = 1_L$.

Thus $M(Au, y, t) = 1_L$, i.e. $Au = y$. Therefore, $Au = Su = y$.

Since $A(X) \subseteq T(X)$, there exists $v \in X$, such that $Tv = y$. Thus,

$M(y,Bv, t) = M(Au,Bv,t) \geq_L [M(Su, Tv, kt)), M(Ax, Sx, kt), M(Ay, Tx, kt), M(Ax, Ty, kt)\}] \vee M(Au,Su,kt)] = 1_L$.

Hence $Tv = Bv = Au = Su = y$. Since (A, S) is weak compatible, we conclude that $ASu = SAu$, that is $Ay = Sy$. Also, since (B, T) is weak compatible then, $TBv = BTv$, that is $Ty = By$.

We now prove that $Ay = y$. By (iii), we have

$M(Ay, y, t) = M(Ay,Bv,t) \geq_L [M(Sy, Tv, kt)), M(Ax, Sx, kt), M(Ay, Tx, kt), M(Ax, Ty, kt)\}] \vee M(Ay,Bv, t)$

$= M(Ay, y, kt) \ldots \geq_L M(Ay, y, k^n t)$.

On the other hand, from Lemma 2, we have that

$M(Ay, y, t) \geq_L M(Ay, y, k_n t)$. Hence, $M(Ay, y, t) = C$ for all $t > 0$. Since (X, M, T) has property (C), it follows that $C = 1_L$, i.e., $Ay = y$,

therefore $Ay = Sy = y$. Similarly we prove that $By = y$. By (iii), we have

$M(y,By, t) = M(Ay,By, t) \geq L [M(Sy, Ty, kt) \vee M(Ay,Sy, kt) = M(y,By, kt) .. \geq_L M(y,By, k_n t)$.

On the other hand, from Lemma 2, we have that

$M(y,By, t) \geq_L M(y,By, k^n t)$. Hence, $M(y,By, t) = C$

for all $t > 0$. Since (X, M, T) has property (C), it follows that $C = 1_L$, i.e., $By = y$. Therefore, $Ay = By = Sy = Ty = y$, that is, y is a common fixed of A,B, S and T. For uniqueness, let x be another common fixed point of A,B, S and T i.e., $x = Ax = Bx = Sx = Tx$. Hence

$M(y, x, t) = M(Ay,Bx, t) \geq_L [M(Sy, Tx, kt)), M(Ax, Sx, kt) , M(Ay, Tx, kt), M(Ax, Ty, kt) \}] \vee M(Ay,Sy,kt)$

$= M(y, x, kt).. \geq_L M(y, x, k^n t)$

On the other hand, from Lemma 2 ,we have that

$$M(y, x, t) \geq_L M(y, x, k^n t).$$

Hence, $M(y, x, t) = C$ for all $t > 0$. Since (X,M, T) has property (C), it follows that $C = 1_L$, i.e., $y = x$. Therefore, y is the unique common fixed point of self-maps A, B, S and T.

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