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THE APPROXIMATE SOLUTIONS OF COUPLED MATRIX RICCATI CONVOLUTION DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, we present the common approximate solutions of coupled matrix Riccati convolution differential equations by using the successive approximation method and Kronecker convolution products. The maximum error of $n-$approximation by using this method is also considered. Furthermore, an illustrative example is given.

Keywords: *Convolution Product, Kronecker Convolution Product, Coupled Matrix Riccati Convolution Differential Equations, 2-Convolution Norm.*

1. INTRODUCTION AND PRELIMINARY RESULTS:

In the field of matrix convolution algebra and system identification; there has been interest in convolution and Kronecker convolution products of matrices which are very useful in applications. In fact, these products are very important role in control system analysis, semi-Markov system, statistics, stability theory of differential equations, communication systems, perturbation analysis of matrix differential equations and other fields of pure and applied mathematics [e.g.,1-5,8-11]. For example, Nikolaos [11] established some inequalities involving convolution product of matrices and presented a new method to obtain closed form solutions of transition probabilities and dependability measures and then solved the renewal matrix equation by using the convolution product of matrices, Sumita [12] established the matrix Laguerre transformation to calculate matrix convolutions and evaluated a matrix renewal function, Boshnakov [3] showed that the entries of the autocovariances matrix function can be expressed in terms of the Kronecker convolution product and Kilicman and Al-Zhour [8] presented the iterative solution of such coupled matrix equations based on the Kronecker convolution structures.

One family of matrix problems is the coupled matrix Riccati convolution differential equations; depending on the problem considered, different terms may appear. However, in this case, the system is difficult to find the exact solution and it is often not necessary to compute exact solutions, approximate solutions are sufficient because sometimes computational efforts rapidly increase with the size of matrix functions. In this paper, we find the common numerical solutions of the coupled matrix Riccati convolution differential equations by using the successive approximation method and Kronecker convolution products, the way exists which transform the coupled matrix differential equations into forms for which solutions may be readily computed. An illustrative example is also considered. The solution procedure presented here may be considered as a continuation of the method proposed in [6].

2. BASIC DEFINITIONS AND RESULTS:

We begin this section by recalling the successive approximation method, matrix convolution products of matrices (namely, convolution and Kronecker convolution products), and study some basic results related to these products that will be used in our investigation to the common solution of coupled matrix Riccati convolution differential equations. Before starting, throughout we consider matrices over the field of real numbers R. The set of m-by-n absolutely integrable real matrices for $t \geq 0$ is denoted by $M^{I}_{m,n}(R)$ or $R^{m\times n}$. For simplicity we write $M^{I}_{m,n}$ instead of $M^{I}_{m,n}(R)$, and when $m=n$ we write M^{I}_{n} instead of $M^{I}_{n,n}$.

Definition: 2.1. Let

$$u^{'} = g(t,u) \ , \ u(t_0) = u_0 \tag{2.1}$$

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be differential equation such that g is real continuous function on $D=[a,b]\times[c,d]$. Then $u(t)$ is a solution of the differential equation in (2.1) on the interval $[a,b]$ if the following conditions hold

(*i*) $(\mathrm{t},\mathrm{u}(\mathrm{t}))\in \mathrm{D}$ for all $t\in[a,b]$, (*ii*) $u'(t)=g(t,u(t))$ for all $t\in[a,b]$

(*iii*) $[a,b]$ in t_0 for any initial value $u(t_0)=u_0$

Definition: 2.2. The function $g(t,u)$ is said to be satisfy Lipschtiz condition with a variable u on D if there exists a constant K such that

$$|g(t,u_1)-g(t,u_2)| \;\le\; K|u_1-u_2| \text{ for all } (t,u_1),(t,u_2)\in D. \tag{2.2}$$

Lemma: 2.3. Let $g(t,u)$ be continuous function and satisfy Lipschtiz condition with a variable u on D. Then the differential equation:

$$u' = g(t,u)\ ,\ u(t_0)=u_0,\ a\le t\le b, \text{ for any } t_0 \text{ in} [a,b]$$

has a unique solution $u(t)$ on the interval $[a,b]$.

The differential equation defined in (2.1) can be solved numerically by using the *Successive Approximation Method* as:

$$u_n(t)=u_0+\int_{t_0}^{t} g(s,u_{n-1}(s))ds \qquad , \qquad n=1,2,... \tag{2.3}$$

This method generates the following sequence of functions:

$$\{u_n(t)\}=\{u_0(t),\ u_1(t),\ ...,\ u_n(t);...\}, \tag{2.4}$$

and each function of this sequence satisfy the initial condition $u(t_0)=u_0$, but in general not satisfy the differential equation $u'=g(t,u)$. If there exists positive integer n such that $u_{k+1}(t)=u_k(t)$ for all $n\le k$, then $u_n(t)$ is a solution of the integral equation defined in (2.3) and also a solution of the differential equation defined in (2.1).

Remarks: 2.4. (***i***) If $g(t,u)$ is a continuous function on D, then $g(t,u)$ is bounded, that is

$$|g(t,u)|\le M \text{ , for all } (t,u)\in D, \tag{2.5}$$

where M is a constant. The sequence $\{u_n(t)\}$ exist if the following condition holds

$$|t-t_0|\le h\ ,\ |u-u_0|\le b,,\ h=\min\left\{a,\frac{b}{M}\right\}. \tag{2.6}$$

(***ii***) The infinite series

$$u_0(t)+\sum_{k=0}^{\infty}[u_{k+1}(t)-u_k(t)] \tag{2.7}$$

is convergent if and only if

$$u_0(t)+\sum_{k=0}^{\infty}[u_{k+1}(t)-u_k(t)] \tag{2.8}$$

is convergent. This occurs when the infinite sequence $\{u_n(t)\}_{n=0}^{\infty}$ is convergent.

(***iii***) If $\lim_{n\to\infty} u_n(t) = u(t)$, the we obtain the exact solution as follows:

$$u(t) = \lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} \left[u_0 + \int_{t_0}^{t} g(s, u_{n-1}(s)) ds \right]$$

$$= u_0 + \lim_{n\to\infty} \int_{t_0}^{t} g(s, u_{n-1}(s)) ds = u_0 + \int_{t_0}^{t} g\left(s, \lim_{n\to\infty} u_{n-1}(s) \right) ds$$

$$= u_0 + \int_{t_0}^{t} g(s, u(s)) ds . \tag{2.9}$$

(***iv***) The maximum error of $n-$ approximation by using successive approximation method is given by

$$R_n(t) = |u(t) - u_n(t)| \le M \sum_{i=1}^{n} K^{i-1} \frac{h^i}{i!}, \tag{2.10}$$

where K , M and h are defined in (2.2), (2.5) and (2.6), respectively.

Definition: 2.5. Let $A(t) = [f_{ij}(t)] \in M_{m,n}^{I}$ and $B(t) = [g_{jr}(t)] \in M_{n,p}^{I}$. The convolution and Kronecker convolution products of $A(t)$ and $B(t)$ are matrix functions defined for $t \ge 0$ by (see, e.g.,[2,3,8,9,11]):

(***i***) *Convolution Product*

$$A(t) * B(t) = [h_{ir}(t)] \text{ with } h_{ir}(t) = \sum_{k=1}^{n} \int_{0}^{t} f_{ik}(t-x) g_{kr}(x) dx = \sum_{k=1}^{n} f_{ik}(t) * g_{kr}(t) . \tag{2.11}$$

(***ii***) *Kronecker Convolution Product*

$$A(t) \overset{c}{\otimes} B(t) = [f_{ij}(t) * B(t)]_{ij} . \tag{2.12}$$

where $f_{ij}(t) * B(t)$ is the ij -th submatrix of order $n \times p$, $A(t) \overset{c}{\otimes} B(t)$ is of order $mn \times np$ and $A(t) * B(t)$ is of order $m \times p$.

Lemma: 2.6 Let $A(t)$, $B(t)$, $C(t) \in M_n^I$ and $D_n(t) = \delta(t) I_n$ (where $I_n \in M_n$ is scalar identity matrix) . Then for any constants α and β (see [2, 8, 11])

(***i***) $(\alpha A(t) + \beta B(t)) * C(t) = \alpha (A(t) * C(t)) + \beta (B(t) * C(t))$;

(***ii***) $A(t) * (B(t) * C(t)) = (A(t) * B(t)) * C(t)$;

(***iii***) $A(t) * D_n(t) = D_n(t) * A(t) = A(t)$;

(***iv***) $(A(t) * B(t))^T = B^T(t) * A^T(t)$.

(**v**) $\|A(t) * B(t)\| \le \|A(t)\| . \|B(t)\|$ for any matrix norm $\|.\|$.

Lemma: 2.7 Let $A(t) \in M_{m,n}^{I}$, $B(t) \in M_{p,q}^{I}$, $C(t) \in M_{n,r}^{I}$ and $D(t) \in M_{q,s}^{I}$. Then (see [8, 11])

$$\left(A(t) \overset{c}{\otimes} B(t) \right) * \left(C(t) \overset{c}{\otimes} D(t) \right) = (A(t) * C(t)) \overset{c}{\otimes} (B(t) * D(t)) . \tag{2.13}$$

But

$$\left(A(t)\overset{c}{\otimes}B(t)\right)\left(C(t)\overset{c}{\otimes}D(t)\right)\neq\left(A(t)C(t)\right)\overset{c}{\otimes}\left(B(t)D(t)\right). \tag{2.14}$$

Corollary: 2.8 Let $A(t)\in M_n^I$, $B(t)\in M_m^I$ and let $D_n(t)=\delta(t)I_n$ be Dirac identity matrix .Then (see [8, 11])

(*i*) $D_n(t)\overset{c}{\otimes}A(t)=diag\left(A(t),A(t),\cdots,A(t)\right)$;

(*ii*) $\left(A(t)\overset{c}{\otimes}D_m(t)\right)*\left(D_n(t)\overset{c}{\otimes}B(t)\right)=\left(D_n(t)\overset{c}{\otimes}B(t)\right)*\left(A(t)\overset{c}{\otimes}D_m(t)\right)=A(t)\overset{c}{\otimes}B(t)$;

(iii) $tr(A\overset{c}{\otimes}B)=tr(A)*tr(B)$.

Lemma: 2.9 Let $A(t)\in M_{m,n}^I$, $B(t)\in M_{p,q}^I$ and $X(t)\in M_{n,p}^I$. Then (see, [8])

$$Vec\left(A(t)*X(t)*B(t)\right)=\left(B^T(t)\overset{c}{\otimes}A(t)\right)*VecX(t). \tag{2.15}$$

3. MAIN PROBLEM AND SOLUTION PROCEDURE:

In this section, we will present the approximate solutions of the following coupled matrix Riccati convolution differential equations:

$$\begin{aligned}X_1'(t)=\{&Q_1(t)+B_1(t)*X_1(t)+X_1(t)*A_1(t)+X_1(t)*S_{11}(t)*X_1(t)\\&+X_1(t)*S_{22}(t)*X_2(t)+X_2(t)*S_{22}(t)*X_1(t)+X_2(t)*S_{12}(t)*X_2(t)\},\end{aligned} \tag{3.1}$$

$$\begin{aligned}X_1'(t)=\{&Q_2(t)+B_2(t)*X_2(t)+X_2(t)*A_2(t)+X_2(t)*S_{22}(t)*X_2(t)\\&+X_2(t)*S_{11}(t)*X_1(t)+X_1(t)*S_{11}(t)*X_2(t)+X_1(t)*S_{21}(t)*X_1(t)\}.\end{aligned} \tag{3.2}$$

Subject to:

$$X_1(t_f)=X_{1f},\quad X_2(t_f)=X_{2f}, \tag{3.3}$$

where X_{1f} and X_{2f} are constant matrices, and

$$Q_1(t),B_1(t),A_1(t),S_{11}(t),S_{22}(t),S_{12}(t),\ \ Q_2(t),\ \ B_2(t),A_2(t),S_{21}(t)\in M_n^I \tag{3.4}$$

are real continuous matrix functions on the interval $z=[t_0,t_f]$ and called "*Time-Varying Matrix convolution Functions*"

By using the Vec -notation in Lemma 2.9 of (3.1)-(3.3), we obtain:

$$\begin{aligned}VecX_1'(t)=VecQ_1(t)&+\left\{A_1^T(t)\overset{c}{\otimes}D_n(t)+D_n(t)\overset{c}{\otimes}B_1(t)\right\}*VecX_1(t)\\&+\left\{X_2^T(t)\overset{c}{\otimes}X_1(t)+X_1^T(t)\overset{c}{\otimes}X_2(t)\right\}*VecS_{22}(t)\\&+\left\{X_2^T(t)\overset{c}{\otimes}X_2(t)\right\}*VecS_{12}(t)+\left\{X_1^T(t)\overset{c}{\otimes}X_1(t)\right\}*VecS_{11}(t),\end{aligned} \tag{3.5}$$

$$VecX_2'(t)=VecQ_2(t)+\left\{A_2^T(t)\overset{c}{\otimes}D_n(t)+D_n(t)\overset{c}{\otimes}B_2(t)\right\}*VecX_2(t)$$

$$+\left\{X_1^T(t)\overset{c}{\otimes}X_2(t)+X_2^T(t)\overset{c}{\otimes}X_1(t)\right\}*VecS_{11}(t)$$
$$+\left\{X_1^T(t)\overset{c}{\otimes}X_1(t)\right\}*VecS_{21}(t)+\left\{X_2^T(t)\overset{c}{\otimes}X_2(t)\right\}*VecS_{22}(t). \tag{3.6}$$

Subject to

$$Vec\big(X_1(t_f)\big)=Vec\big(X_{1f}\big)\ ,\ \ Vec\big(X_2(t_f)\big)=Vec\big(X_{2f}\big). \tag{3.7}$$

Define the 2-convolution norm of $A(t)\in M_n^I$ for all $t\geq 0$ as follows:

$$\|A(t)\|_2^2=tr\big(A^T(t)*A(t)\big). \tag{3.8}$$

Thus, it is easy to show that:

(i) for $A_{ij}(t)\in M_n^I$ $(1\leq i,j\leq 2)$,

$$\left\|\begin{bmatrix}A_{11}(t) & A_{12}(t)\\ A_{21}(t) & A_{22}(t)\end{bmatrix}\right\|_2\leq 4\max_{1\leq i,j\leq 2}\|A_{ij}(t)\|_2, \tag{3.9}$$

(ii) for $A_i(t)\in M_n^I$ $(i=1,2)$,

$$\left\|A_1(t)\overset{c}{\otimes}A_2(t)\right\|_2=\|A_1(t)\|_2*\|A_2(t)\|_2\ . \tag{3.10}$$

Lemma 3-1- Let $A_i(t), B_i(t)$ and $S_{ij}(t)$ $(i,j=1,2)$ be matrix functions determined by (3.4). Also let

$$X_i(t),\ \ Y_i t)\in R^{n\times n}\ ,\ L_i(t)=\left\{A_i^T(t)\overset{c}{\otimes}D_n(t)+D_n(t)\overset{c}{\otimes}B_i(t)\right\}\ (i,j=1,2), \tag{3.11}$$

$$X(t)=\big[X_1(t),X_2(t)\big]\in M^I{}_{n,2n},\ \ Y(t)=\big[Y_1(t),Y_2(t)\big]\in M_{n,2n}^I\ . \tag{3.12}$$

Then

$$VecX(t)=Vec\big[X_1(t),X_2(t)\big]=\begin{bmatrix}VecX_1(t)\\ VecX_2(t)\end{bmatrix}\in M_{2n^2,1}^I\ . \tag{3.13}$$

Furthermore, if the following functions: $\psi_i\ :\ [t_0,t_f]\times R^{2n^2\times 1}\to R^{2n^2\times 1}$, $i=1,2,3$ are defined by

(*a*)
$$\psi_1\big(t,VecX(t)\big)=\begin{bmatrix}L_1(t)*\big[VecX_1(t)\big]\\ L_2(t)*\big[VecX_2(t)\big]\end{bmatrix}; \tag{3.14}$$

(*b*)
$$\psi_2\big(t,VecX(t)\big)=\begin{bmatrix}X_1^T(t)\overset{c}{\otimes}X_1(t) & . & X_2^T(t)\overset{c}{\otimes}X_1(t)+X_1^T(t)\overset{c}{\otimes}X_2(t)\\ & . & \\ X_1^T(t)\overset{c}{\otimes}X_2(t)+X_2^T\overset{c}{\otimes}X_1(t) & . & X_2^T(t)\overset{c}{\otimes}X_2(t)\end{bmatrix}*\begin{bmatrix}VecS_{11}(t)\\ VecS_{22}(t)\end{bmatrix}; \tag{3.15}$$

(*c*)
$$\psi_3\big(t,VecX(t)\big)=\begin{bmatrix}\left(X_2^T\overset{c}{\otimes}X_2(t)\right)*VecS_{12}(t)\\ \left(X_1^T\overset{c}{\otimes}X_1(t)\right)*VecS_{21}(t)\end{bmatrix}. \tag{3.16}$$

Then the following inequalities are true:

(*i*) $$\left\|\psi_1(t,VecX(t))-\psi_1(t,VecY(t))\right\|_2 \le 4n^{\frac{1}{2}} * \alpha(t) * \left\|VecX(t)-VecY(t)\right\|_2; \qquad (3.17)$$

where

$$\alpha(t)=\max_{t\in z}\left\{\left\|A_i(t)\right\|_2, \left\|B_i(t)\right\|_2, 1\le i\le 2\right\}. \qquad (3.18)$$

(*ii*) $$\left\|\psi_2(t,VecX(t))-\psi_2(t,VecY(t))\right\|_2 \le 8\gamma(t)*\left(\left\|X(t)\right\|_2+\left\|Y(t)\right\|_2\right)*\left\|VecX(t)\text{-}VecY(t)\right\|_2; \qquad (3.19)$$

where

$$\gamma(t)=\max_{t\in z}\left\{\left\|S_{ii}(t)\right\|_2, i=1,2\right\}. \qquad (3.20)$$

(*iii*) $$\left\|\psi_3(t,VecX(t))-\psi_3(t,VecY(t))\right\|_2 \le \eta(t)*\left(\left\|X(t)\right\|_2+\left\|Y(t)\right\|_2\right)*\left\|VecX(t)\text{-}VecY(t)\right\|_2; \qquad (3.21)$$

where

$$\eta(t)=\max_{t\in z}\left\{\left\|S_{ij}(t)\right\|_2 : 1\le i,j\le 2,\ \ i\ne j\right\}. \qquad (3.22)$$

Proof: (*i*) From (3.9) and the definition of ψ_1 we have

$$\psi_1(t,VecX(t))-\psi_1(t,VecY(t))=\begin{bmatrix} L_1(t)*\left[VecX_1(t)-VecY_1(t)\right] \\ L_2(t)*\left[VecX_2(t)-VecY_2(t)\right]\end{bmatrix}.$$

Now

$$\begin{aligned}\left\|\psi_1(t,VecX(t))-\psi_1(t,VecY(t))\right\|_2 &\le 2\max_{t\in z}\left\{\left\|L_i(t)\right\|_2, i=1,2\right\}*\left\|\begin{bmatrix} VecX_1(t)-VecY_1(t) \\ VecX_2(t)-VecY_2(t)\end{bmatrix}\right\|_2 \\ &= 2\max_{t\in z}\left\{\left\|L_i(t)\right\|_2, i=1,2\right\}*\left\|\begin{bmatrix} VecX_1(t) \\ VecX_2(t)\end{bmatrix}-\begin{bmatrix} VecY_1(t) \\ VecY_2(t)\end{bmatrix}\right\|_2 \\ &= 2\max_{t\in z}\left\{\left\|L_i(t)\right\|_2, i=1,2\right\}*\left\|VecX(t)-VecY(t)\right\|_2. \qquad (3.23)\end{aligned}$$

Also, from (3.10) we have

$$\begin{aligned}\left\|L_i(t)\right\|_2 &= \left\|A_i^T(t)\overset{c}{\otimes}D_n(t)+D_n(t)\overset{c}{\otimes}B_i(t)\right\|_2 \quad ,(i,j=1,2) \\ &\le \left\|D_n(t)\overset{c}{\otimes}B_i(t)\right\|_2+\left\|A_i^T(t)\overset{c}{\otimes}D_n(t)\right\|_2 \\ &= \left\|D_n(t)\right\|_2*\left\|B_i(t)\right\|_2+\left\|A_i^T(t)\right\|_2*\left\|D_n(t)\right\|_2 \\ &= n^{\frac{1}{2}}*\left\{\left\|B_i(t)\right\|_2+\left\|A_i^T(t)\right\|_2\right\}. \qquad (3.24)\end{aligned}$$

Thus from (3.23) and (3.24) we have

$$\begin{aligned}\left\|\psi_1(t,VecX(t))-\psi_1(t,VecY(t))\right\|_2 &\le 2\max_{t\in z}\ n^{\frac{1}{2}}*\left[\left\|B_i(t)\right\|_2+\left\|A_i^T(t)\right\|_2\right]*\left\|VecX(t)-VecY(t)\right\|_2 \\ &= 4n^{\frac{1}{2}}*\max_{t\in z}\left\{\left\|B_i(t)\right\|_2, \left\|A_i(t)\right\|_2\right\}*\left\|VecX(t)-VecY(t)\right\|_2 \\ &= 4n^{\frac{1}{2}}*\alpha(t)*\left\|VecX(t)-VecY(t)\right\|_2.\end{aligned}$$

Similarly we can prove (*ii*) and (*iii*).

If we let

$$\left.\begin{array}{l} Q(t) = \begin{bmatrix} Q_1(t) & Q_2(t) \end{bmatrix} \\ X(t) = \begin{bmatrix} X_1(t) & X_2(t) \end{bmatrix} \\ X(t_f) = \begin{bmatrix} X_{1f} & X_{2f} \end{bmatrix} \end{array}\right\} \in R^{n\times 2n} \quad \text{and} \quad \left.\begin{array}{l} \alpha = \max\{\alpha(t) \;:\; t_0 \le t \le t_f\} \\ \gamma = \max\{\gamma(t) \;:\; t_0 \le t \le t_f\} \\ \eta = \max\{\eta(t) \;:\; t_0 \le t \le t_f\} \end{array}\right\}. \tag{3.25}$$

Then by Lemma 3.1, the coupled matrix Riccati convolution differential equations defined by (3.5)-(3.7) can be rewrited as follows:

$$VecX'(t) = F(t, VecX(t)) \;,\; VecX(t_f) = Vec[X_{1f}, X_{2f}], \tag{3.26}$$

where $F : [t_0, t_f] \times R^{2n^2\times 1} \to R^{2n^2\times 1}$ is a function defined by

$$F(t, VecX(t)) = VecQ(t) + \psi_1(t, vecX(t)) + \psi_2(t, VecX(t)) + \psi_3(t, VecX(t)). \tag{3.27}$$

Corollary: 3.2. Let $\{A_i(t), B_i(t), Q_i(t), S_{ij}(t) : 1 \le i, j \le 2\}$ be continuous matrix functions on $z = [t_0, t_f] = [0,1]$, and let $\delta > 0$,

$$X_f = [X_{1f}, X_{2f}] \in R^{n\times 2n},\; \varepsilon = \|X_f\| + \delta, \tag{3.28}$$

$$X(t) = [X_1(t), X_2(t)] \in R^{n\times 2n} \;,\quad Y(t) = [Y_1(t), Y_2(t)] \in R^{n\times 2n}, \tag{3.29}$$

$$\|X(t) - X_f\| \le \delta \quad,\quad \|Y(t) - Y_f\| \le \delta. \tag{3.30}$$

Then the function $F(t, VecX(t))$ determined by (3.27) satisfy the following Lipschtiz condition on $[0,1]$:

$$\|F(t, VecX(t)) - F(t, VecY(t))\|_2 \le K * \|VecX(t) - VecY(t)\|_2, \tag{3.31}$$

where

$$K = 2\left(2n^{\frac{1}{2}}\alpha + 8\gamma\varepsilon + \eta\varepsilon\right). \tag{3.32}$$

Proof: By assumptions and from (3.17), (3.19) and (3.21) we have

$$\|F(t, VecXt)) - F(t, VecY(t))\|_2$$

$$= \|VecQ(t) + \psi_1(t, VecX(t)) + \psi_2(t, VecX(t)) + \psi_3(t, VecX(t)) \\ - VecQ(t) - \psi_1(t, VecY(t)) - \psi_2(t, VecY(t)) - \psi_3(t, VecY(t))\|_2$$

$$= \|\psi_1(t, VecX(t)) - \psi_1(t, VecY(t)) + \psi_2(t, VecX(t)) - \psi_2(t, VecY(t)) + \psi_3(t, VecX(t)) - \psi_3(t, VecY(t))\|_2$$

$$\le \|\psi_1(t, VecX(t)) - \psi_1(tVecY(t))\|_2 + \|\psi_2(t, VecX(t)) - \psi_2(t, VecY(t))\|_2 \\ + \|\psi_3(t, VecX(t)) - \psi_3(t, VecY(t))\|_2 + \eta(t) * \{\|X(t)\|_2 + \|Y(t)\|_2\} * \|VecX(t) - VecY(t)\|_2.$$

$$= \left(4n^{\frac{1}{2}} * \alpha(t) + 8\,\gamma(t) * \{\|X(t)\|_2 + \|Y(t)\|_2\} + \eta(t)\{\|X(t)\|_2 + \|Y(t)\|_2\}\right) * \{\|VecX(t) - VecY(t)\|_2\}$$

By using assumptions: $\varepsilon = \|X_f\|_2 + \delta$, $\|X(t) - X_f\|_2 \le \delta$ implies

$$\left\|X(t)\right\|_2 - \left\|X_f\right\|_2 \le \delta\,,\ \left\|X(t)\right\|_2 \le \left\|X_f\right\|_2 + \delta = \varepsilon\,. \tag{3.33}$$

Now by (3.33), (3.25) and since $t \in [0,1]$ we have

$$\begin{aligned}\left\|F(t,VecX(t)) - F(t,VecY(t)\right\|_2 &\le \left(4n^{\frac{1}{2}} * \alpha + 16\gamma * \varepsilon + 2\eta * \varepsilon\right) * \left\{\left\|VecX(t) - VecY(t)\ \right\|_2\right\} \\ &= 2\left(2n^{\frac{1}{2}} * \alpha + 8\gamma * \varepsilon + \eta * \varepsilon\right) * \left\{\left\|VecX(t) - VecY(t)\right\|_2\right\} \\ &= 2t\left(2n^{\frac{1}{2}}\alpha + 8\gamma\varepsilon + \eta\varepsilon\right) * \left\{\left\|VecX(t) - VecY(t)\right\|_2\right\} \\ &\le 2\left(2n^{\frac{1}{2}}\alpha + 8\gamma\varepsilon + \eta\varepsilon\right) * \left\{\left\|VecX(t) - VecY(t)\right\|_2\right\} \\ &= K * \left\|VecX(t) - VecY(t)\right\|_2 .\end{aligned}$$

Corollary: 3.3. Let $\delta > 0$, $t \in [t_0, t_f] = [0,1]$, $\left\|X(t) - X_f\right\|_2 \le \delta$, $\varepsilon = \left\|X_f\right\|_2 + \delta$, $X_f = [X_{1f}, X_{2f}]$, and let

$$q = \max\left\{\left\|\ [Q_1(t), Q_2(t)]\ \right\|_2 \quad : \quad t_0 \le t \le t_f\right\}, \tag{3.34}$$

$X(t) = \left[X_1(t), X_2(t)\right]$ satisfy $\left\|X(t) - X_f\right\|_2 \le \delta$. Then

$$\sup\left\{\left\|F(t, vecX)\right\|_2 : t_0 \le t \le t_f\right\} \le M\ , \tag{3.35}$$

where

$$M = q + 2n^{\frac{1}{2}}\alpha\varepsilon + \left(4\gamma + \frac{1}{2}\eta\right)\varepsilon^2, \tag{3.36}$$

and α, γ, η are defined in (3.25).

Proof: For any $t \in [t_0, t_f] = [0,1]$ and $\left\|X(t) - X_f\right\|_2 \le \delta$, we have

$$\begin{aligned}\left\|F(t, VecX(t))\right\|_2 &= \left\|Q(t) + \psi_1(t, VecX(t)) + \psi_2(t, VecX(t)) + \psi_3(t, VecX(t))\right\|_2 \\ &\le \left\|Q(t)\right\|_2 + \left\|\psi_1(t, VecX(t))\right\|_2 + \left\|\psi_2(t, VecX(t))\right\|_2 + \left\|\psi_3(t, VecX(t))\right\|_2\end{aligned}$$

From (3.17)-(3.22) of Lemma 3.1 and (3.25), we have

$$\left\|\psi_1(t, VecX(t)) - \psi_1(t, VecY(t))\right\|_2 \le 4n^{\frac{1}{2}}\alpha\ t * \left\|VecX(t) - VecY(t)\right\|_2 . \tag{3.37}$$

When $Y(t) = 0$, we obtain

$$\left\|\psi_1(t, VecX(t))\right\|_2 \le 4n^{\frac{1}{2}}\alpha\ t * \left\|VecX(t)\right\|_2 \le 4n^{\frac{1}{2}}\alpha t * \varepsilon = 2n^{\frac{1}{2}}\alpha\varepsilon t^2\ . \tag{3.38}$$

$$\left\|\psi_2(t, VecX(t)) - \psi_2(t, VecY(t))\right\|_2 \le 8\gamma * \left(\left\|X(t)\right\|_2 + \left\|Y(t)\right\|_2\right) * \left\|VecX(t) - VecY(t)\right\|_2 . \tag{3.39}$$

When $Y(t) = 0$, we obtain

$$\left\|\psi_2(t, VecX(t))\right\|_2 \le 8\gamma * \left(\left\|X(t)\right\|_2\right) * \left(\left\|VecX(t)\right\|_2\right) \le 8\gamma * \varepsilon^2 t = 4\gamma\varepsilon^2 t^2\,. \tag{3.40}$$

$$\left\|\psi_3(t,VecX(t))-\psi_3(t,VecY(t))\right\|_2 \leq \eta * \left(\left\|X(t)\right\|_2+\left\|Y(t)\right\|_2\right) * \left\|VecX(t)-VecY(t)\right\|_2 . \tag{3.41}$$

When $Y(t)=0$, we obtain

$$\left\|\psi_3(t,VecX(t))\right\|_2 \leq \eta * \varepsilon^2 t = \frac{1}{2}\eta\varepsilon^2 t^2 . \tag{3.42}$$

Since $t \in [0,1]$, then

$$\begin{aligned}\left\|F(t,VecX(t))\right\|_2 &\leq \left\|Q(t)\right\|_2 + \left\|\psi_1(t,VecX(t)\right\|_2 + \left\|\psi_2(t,VecX(t)\right\| + \left\|\psi_3(t,VecX(t)\right\|_2 \\ &\leq q + 2n^{\frac{1}{2}}\alpha\varepsilon t^2 + 4\gamma\varepsilon^2 t^2 + \frac{1}{2}\eta\varepsilon^2 t^2 \\ &= q + \left\{2n^{\frac{1}{2}}\alpha\varepsilon + \left(4\gamma+\frac{1}{2}\eta\right)\varepsilon^2\right\}t^2 . \\ &\leq q + 2n^{\frac{1}{2}}\alpha\varepsilon + \left(4\gamma+\frac{1}{2}\eta\right)\varepsilon^2 = M .\end{aligned}$$

This completes the proof of Corollary 3.3.

Theorem: 3.4. Let $\{A_i(t),B_i(t),Q_i(t),S_{ij}(t):1\leq i,j\leq 2\}$ be continuous matrix functions on $z=[t_0,t_f]=[0,1]$. Let $\delta>0$, $\varepsilon=\left\|X_f\right\|_2+\delta$ and M determined by (3.36). Suppose that

$$h=\min\left\{1,\frac{\delta}{M}\right\}. \tag{3.43}$$

Then the coupled matrix Riccati convolution differential equations defined in (3.1)-(3.4) has a unique solution

$$X(t)=\left[X_1(t),X_2(t)\right]$$

on the interval $[1-h,1]$ such that $X(t)$ is the limit of 2-convolution norm of the following successive approximation sequence:

$$\left\{X^{(p)}(t)\right\}_{p\geq 0} \tag{3.44}$$

such that

$$X^{(p)}(t)=\begin{Bmatrix}X_1^{(p)}(t)\\ X_2^{(p)}(t)\end{Bmatrix}, \tag{3.45}$$

$$X_i^{(0)}(t)=X_{if} \quad , \quad i=1,2 , \tag{3.46}$$

and

$$\begin{aligned}X_1^{(p+1)}(t)=X_{1f}+\int_1^t &\left[Q_1(u)+B_1(u)*X_1^{(p)}(u)+X_1^{(p)}(u)*A_1(u)\right. \\ &\left.+X_1^{(p)}(u)*S_{11}(u)*X_1^{(p)}(u)\right]du+\int_1^t\left[X_1^{(p)}(u)*S_{22}(u)*X_2^{(p)}(u)\right. \\ &\left.+X_2^{(p)}(u)*S_{22}(u)*X_1^{(p)}(u)+X_2^{(p)}(u)*S_{12}(u)*X_2^{(p)}(u)\right]du .\end{aligned} \tag{3.47}$$

$$\begin{aligned}X_2^{(p+1)}(t)=X_{2f}+\int_1^t &\left[Q_2(u)+B_2(u)*X_2^{(p)}(u)+X_2^{(p)}(u)*A_2(u)\right. \\ &\left.+X_2^{(p)}(u)*S_{22}(u)*X_2^{(p)}(u)\right]du+\int_1^t\left[X_2^{(p)}(u)*S_{11}(u)*X_1^{(p)}(u)\right.\end{aligned}$$

$$+ X_1^{(p)}(u) * S_{11}(u) * X_1^{(p)}(u) + X_1^{(p)}(u) * S_{21}(u) * X_1^{(p)}(u)\Big]du\,. \tag{3.48}$$

Furthermore, the maximum error is given by

$$R_p(t) = \left\| VecX(t) - VecX^{(p)}(t) \right\|_2 \leq M \sum_{r=1}^{p} K^{r-1} \frac{h^r}{r!}\,. \tag{3.49}$$

where M, K and h are defined in (3.36), (3.32) and (3.43), respectively.

Proof: The function $F(t, VecX(t))$ determined by (3.27) satisfies Lipschtiz condition by Corollary 3.2 and bounded with constant M by Corollary 3.3. Now applying Lemma 2.3, then the unique solution of the following equivalent coupled matrix Riccati convolution differential equations determined by (3.26):

$$VecX^{'}(t) = F\big(t, VecX(t)\big) \quad , \quad VecX(t_f) = Vec\big[X_{1f}, X_{2f}\big]$$

on the interval $[1-h, 1]$ is $VecX(t)$ which is given by the sequence: $\left\{X^{(p)}(u)\right\}_{p\geq 0}$, where $X^0(t) = VecX_f$. Also

$$VecX^{(p+1)}(t) = VecX_f + \int_1^t F\big(u, VecX^{(p)}(u)\big)du$$

$$= VecX_f + \int_1^t \begin{bmatrix} Q_1(u) + B_1(u) * X_1^{(p)}(u) + X_1^{(p)}(u) * A_1(u) + X_1^{(p)}(u) * S_{11}(u) * X_1^{(p)}(u) \\ Q_2(u) + B_2(u) * X_2^{(p)}(u) + X_2^{(p)}(u) * A_2(u) + X_2^{(p)}(u) * S_{22}(u) * X_2^{(p)}(u) \end{bmatrix} du$$

$$+ \int_1^t \begin{bmatrix} X_1^{(p)}(u) * S_{22}(u) * X_2^{(p)}(u) + X_2^{(p)}(u) * S_{22}(u) * X_1^{(p)}(u) + X_2^{(p)}(u) * S_{12}(u) * X_2^{(p)}(u) \\ X_2^{(p)}(u) * S_{11}(u) * X_1^{(p)}(u) + X_1^{(p)}(u) * S_{11}(u) * X_1^{(p)}(u) + X_1^{(p)}(u) * S_{21}(u) * X_1^{(p)}(u) \end{bmatrix} du \tag{3.50}$$

This implies:

$$VecX^{(p+1)}(t) = \begin{bmatrix} X_1^{(p+1)}(t) \\ X_2^{(p+1)}(t) \end{bmatrix} \equiv Vec\begin{bmatrix} X_{1f} \\ X_{2f} \end{bmatrix} + Vec\begin{bmatrix} \int_1^t W_1^{(p)}(u)du \\ \int_1^t W_2^{(p)}(u)du \end{bmatrix}, \tag{3.51}$$

where

$$W_1^{(p)}(u) = \{Q_1(u) + B_1(u) * X_1^{(p)}(u) + X_1^{(p)}(u) * A(u) + X_1^{(p)}(u) * S_{11}(u) * X_1^{(p)}(u) + X_1^{(p)}(u) * S_{22}(u) * X_2^{(p)}(u) + X_2^{(p)}(u) * S_{22}(u) * X_1^{(p)}(u) + X_2^{(p)} * S_{12}(u) * X_2^{(p)}(u)\ \}, \tag{3.52}$$

$$W_2^{(p)}(u) = \{Q_2(u) + B_2(u) * X_2^{(p)}(u) + X_2^{(p)}(u) * A_2(u) + X_2^{(p)}(u) * S_{22}(u) * X_2^{(p)}(u) + X_2^{(p)}(u) * S_{11}(u) * X_1^{(p)}(u) + X_1^{(p)}(u) * S_{11}(u) * X_1^{(p)}(u) + X_1^{(p)}(u) * S_{21}(u) * X_1^{(p)}(u)\ \}. \tag{3.53}$$

Thus from (2.10) of Remark 2.4, the maximum error due to the above truncation is given by (3.49).

Remark: 3.5. Observe that the convergence interval is $[1-h, 1]$ and the radius of convergence is $h = \min\left\{1, \frac{\delta}{M}\right\}$, $\delta > 0$.

Since the convergence interval depends on ratio $\frac{\delta}{M}$, then the ratio $\frac{\delta}{M}$ must be maximum value. Since M also depend on δ, we need to find δ such that $\frac{\delta}{M}$ is maximum value. Now we have $h = \frac{\delta}{M}$ implies $h^{-1} = \delta^{-1}M$, that is

$$h^{-1}(\delta) = \delta^{-1}M = \delta^{-1}\left\{q + 2n^{\frac{1}{2}}\alpha\varepsilon + \left(4\gamma + \frac{1}{2}\eta\right)\varepsilon^2\right\}, \tag{3.54}$$

where $\varepsilon = \|X_f\|_2 + \delta$.

Example: 3.6 Consider the coupled matrix Riccati convolution differential equations on $t \in [0,1]$

$$X_1'(t) = Q_1(t) + B_1(t) * X_1(t),\ X_2'(t) = Q_2(t) + B_2(t) * X_2(t),$$

where

$$Q_1(t) = \begin{bmatrix} t & t \\ t & t \end{bmatrix},\ Q_2(t) = \begin{bmatrix} 5t & 3t \\ t & t \end{bmatrix},\ B_1(t) = \begin{bmatrix} t & 2t \\ 2t & 4t \end{bmatrix},\ B_2(t) = \begin{bmatrix} 2t & t \\ 0 & 2t \end{bmatrix},$$

$$X_1(1) = X_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ X_2(1) = X_{21} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Here, the matrix functions $Q_1(t)$, $Q_2(t)$, $B_1(t)$ and $B_2(t)$ are continuous on $[0,1]$. According to (3.27), we construct the function F as follows:

$$F(t, VecX(t)) = VecQ(t) + \psi_1(t, VecX(t)),$$

where

$$X(t) = [X_1(t), X_2(t)],\ X(t) = [Q_1(t), Q_2(t)],$$

$$\psi_1(t, vecX(t)) = \begin{bmatrix} L_1(t)[VecX_1(t)] \\ L_2(t)[VecX_2(t)] \end{bmatrix} :\ L_1(t) = D_2(t) \overset{c}{\otimes} B_1(t),\quad L_2(t) = D_2(t) \overset{c}{\otimes} B_2(t)$$

It is easy to verify that this function is bounded and satisfies the Lipschtiz condition. Furthermore, computation shows that:

$$\|Q_1(t)\|_2 = \sqrt{\frac{2}{3}t^3},\ \|Q_2(t)\|_2 = \sqrt{6t^3},\ \|B_1(t)\|_2 = \sqrt{\frac{25}{6}t^3},\ \|B_2(t)\|_2 = \sqrt{\frac{9}{6}t^3},$$

$$\alpha(t) = \max\{\|B_1(t)\|_2, \|B_2(t)\|_2\} \Rightarrow \alpha = \max_{t\in[0,1]}\left\{\sqrt{\frac{25}{6}t^3},\ \sqrt{\frac{9}{6}t^3}\right\} = \frac{5}{\sqrt{6}} \approx 2.04124,$$

$$q(t) = \max\{\|Q_1(t)\|_2, \|Q_2(t)\|_2\} \Rightarrow q = \max_{t\in[0,1]}\left\{\sqrt{\frac{2}{3}t^3},\ \sqrt{6t^3}\right\} = \sqrt{6} \approx 2.44949,$$

$$\gamma(t) = 0,\quad \eta(t) = 0 \quad \Rightarrow \gamma = 0,\quad \eta = 0.$$

If we choose $\delta = 0.1$, then computation also shows that:

$$\left\|\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}\right\|_2 = \sqrt{2t} \Rightarrow \varepsilon = \max_{t\in[0,1]}\sqrt{2t} + \delta = \sqrt{2} + 0.1 \approx 1.5142,$$

$$K = 2(2n^{\frac{1}{2}}\alpha + 8\gamma\varepsilon + \eta\varepsilon \approx 11.574,\ M = q + 2n^{\frac{1}{2}}\alpha\varepsilon + (4\gamma + \frac{1}{2}\eta)\varepsilon^2 \approx 11.19172,$$

$$h = \min\left\{1-0, \frac{0.1}{11.19172}\right\} \approx 0.008935.$$

The first two approximate solutions of $X_1(t)$ are:

$$X_1^{(1)}(t) = X_{11} + \int_1^t \left[Q_1(u) + B_1(u) * X_1^{(0)}(u)\right]du$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \int_1^t \left\{ \begin{bmatrix} u & u \\ u & u \end{bmatrix} + \begin{bmatrix} u & 2u \\ 2u & 4u \end{bmatrix} * \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} du = \frac{1}{2}\begin{bmatrix} t^2-1 & t^2-1 \\ t^2-1 & t^2-1 \end{bmatrix}.$$

$$X_1^{(2)}(t) = X_{11} + \int_1^t \left[Q_1(u) + B_1(u) * X_1^{(1)}(u)\right]du$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \int_1^t \left\{ \begin{bmatrix} u & u \\ u & u \end{bmatrix} + \frac{1}{2}\begin{bmatrix} u & 2u \\ 2u & 4u \end{bmatrix} * \begin{bmatrix} u^2-1 & u^2-1 \\ u^2-1 & u^2-1 \end{bmatrix} \right\} du$$
$$= \int_1^t \left\{ \begin{bmatrix} u & u \\ u & u \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{1}{4}u^4 - \frac{3}{2}u^2 & \frac{1}{4}u^4 - \frac{3}{2}u^2 \\ \frac{1}{2}u^4 - 3u^2 & \frac{1}{2}u^4 - 3u^2 \end{bmatrix} \right\} du$$
$$= \frac{1}{2}\begin{bmatrix} t^2-1 & t^2-1 \\ t^2-1 & t^2-1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{1}{20}t^5 - \frac{1}{2}t^3 + \frac{9}{20} & \frac{1}{20}t^5 - \frac{1}{2}t^3 + \frac{9}{20} \\ \frac{1}{10}t^5 - t^3 + \frac{9}{10} & \frac{1}{10}t^5 - t^3 + \frac{9}{10} \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} \frac{1}{20}t^5 - \frac{1}{2}t^3 + t^2 - \frac{11}{20} & \frac{1}{20}t^5 - \frac{1}{2}t^3 + t^2 - \frac{11}{20} \\ \frac{1}{10}t^5 - t^3 + t^2 - \frac{1}{10} & \frac{1}{10}t^5 - t^3 + t^2 - \frac{1}{10} \end{bmatrix}.$$

The first two approximate solutions of $X_2(t)$ are:

$$X_2^{(1)}(t) = X_{21} + \int_1^t \left[Q_2(u) + B_2(u) * X_2^{(0)}(u)\right] .du$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \int_1^t \left\{ \begin{bmatrix} 5u & 3u \\ u & u \end{bmatrix} + \begin{bmatrix} 2u & u \\ 0 & 2u \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} du$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \int_1^t \begin{bmatrix} \frac{3}{2}u^2 + 5u & \frac{3}{2}u^2 + 3u \\ u^2 + u & u^2 + u \end{bmatrix} du$$
$$= \begin{bmatrix} \frac{1}{2}t^3 + \frac{5}{2}t^2 - 2 & \frac{1}{2}t^3 + \frac{3}{2}t^2 - 1 \\ \frac{1}{3}t^3 + \frac{1}{2}t^2 + \frac{1}{6} & \frac{1}{3}t^3 + \frac{1}{2}t^2 + \frac{1}{6} \end{bmatrix}.$$

$$X_2^{(2)}(t) = X_{21} + \int_1^t \left[Q_2(u) + B_2(u) * X_2^{(1)}(u)\right]du$$

$$=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}+\int_1^t\left\{\begin{bmatrix}5u & 3u\\ u & u\end{bmatrix}+\begin{bmatrix}2u & u\\ 0 & 2u\end{bmatrix}*\begin{bmatrix}\frac{1}{2}u^3+\frac{5}{2}u^2-2 & \frac{1}{2}u^3+\frac{3}{2}u^2-1\\ \frac{1}{3}u^3+\frac{1}{2}u^2+\frac{1}{6} & \frac{1}{3}u^3+\frac{1}{2}u^2+\frac{1}{6}\end{bmatrix}\right\}du$$

$$=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}+\int_1^t\left\{\begin{bmatrix}5u & 3u\\ u & u\end{bmatrix}+\begin{bmatrix}\frac{1}{15}u^5+\frac{11}{24}u^4-\frac{23}{12}u^2 & \frac{1}{15}u^5+\frac{7}{24}u^4-\frac{11}{12}u^2\\ \frac{1}{30}u^5+\frac{1}{12}u^4+\frac{1}{6}u^2 & \frac{1}{30}u^5+\frac{1}{12}u^4+\frac{1}{6}u^2\end{bmatrix}\right\}du$$

$$=\begin{bmatrix}\frac{1}{90}t^6+\frac{11}{120}t^5-\frac{23}{36}t^3+\frac{5}{2}t^2+\frac{1067}{360} & \frac{1}{90}t^6+\frac{7}{120}t^5-\frac{11}{36}t^3+\frac{3}{2}t^2+\frac{815}{360}\\ \frac{1}{180}t^6+\frac{1}{60}t^5+\frac{1}{18}t^3+\frac{1}{2}t^2+\frac{71}{45} & \frac{1}{180}t^6+\frac{1}{60}t^5+\frac{1}{18}t^3+\frac{1}{2}t^2+\frac{71}{45}\end{bmatrix}.$$

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