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FIXED POINT THEOREMS FOR CERTAIN CONTRACTIVE MAPPINGS IN CONE METRIC SPACES

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ABSTRACT

In the present paper we prove some common fixed point theorems by using the Reich and Rhoades type contractive conditions in complete cone metric spaces which generalize and extend some well known previous results.

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1. INTRODUCTION:

Recently, Huang and Zhang [4] generalize the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtain some fixed point theorems for mappings satisfying different contraction conditions. The study of fixed point theorems in such spaces is followed by some other mathematicians, see [1], [5], [10], [11] and [12]. The purpose of this paper is to analyze the existence and uniqueness of fixed points of *T*-Reich contractive mappings S defined on a complete cone metric space (X, d) as well as *T*-Rhoades contractive mappings (see Definition 3.1). Our results generalize and extend the respective theorems of Morales and Rojas [10] and others.

2. PRELIMINARIES:

In this section we recall the definition of cone metric space and some of their properties (see c.f., [4]). The following notions will be used in order to prove the main results.

Definition: 2.1. Let E be a real Banach Space and P a subset of E. The set P is called a cone if and only if:

(i) P is closed, non-empty and $P \neq \{0\}$

(ii) $a, b \in R, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$

(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \rightarrow y$ but $x \neq y$, while $x << y$ will stand for $y - x \in \operatorname{int} P$, where int. P denotes the interior of the set P.

Definition: 2.2. Let E be a Banach Space and $P \subset E$ a cone. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ for all $x, y \in E$.The least positive number K satisfying the above inequality is called the normal constant of P.

In the following suppose that E is a Banach space, P is a cone in E with $\operatorname{int} P \neq \phi$ and $\leq$ is partial ordering with respect to P.

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Definition: 2.3. Let X be a non-empty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(a) 0 < d(x, y) for all $x, y \in X$ and d (x, y) = 0 if and only if x = y;

(b) d (x, y) = d (y, x) for all $x, y \in X$;

(c) $d(x, y) \le d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.
Notice that the notion of cone metric space is more general than the corresponding of metric space followed by an example:

Example: 2.1. Let $E = R^2, P = \{(x, y) \in E : x, y \ge 0\} \subset R^2, X = R$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha |x - y|)$, where $\alpha \ge 0$ is a constant. Then (X, d) is a cone metric space.

Definition: 2.4. Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(i) $\{x_n\}$ Converges to x if for every $c \in E$ with 0 << c, there is an n_0 such that for all $n \ge n_0$, d (x_n , x) << c. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x, (n \to \infty)$.

(ii) If for any $c \in E$ with 0 << c, there is an n_0 such that for all $n, m \ge n_0$, $d(x_n, x_m) << c$, then $\{x_n\}$ is called a Cauchy sequence in X. (X, d) is called a complete cone metric space, if every Cauchy sequence in X is convergent in X.

Lemma: 2.1. Let (X, d) be a cone metric space, $P \subset E$ a normal cone with normal constant K. Let $\{x_n\}$, $\{y_n\}$ be sequences in X and x, y $\in$ X.

(i) $\{x_n\}$ converges to x if and only if $\lim_{n \to \infty} d(x_n, x) = 0$;

(ii) If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y then x = y. That is the limit of $\{x_n\}$ is unique.

(iii) If $\{x_n\}$ converges to x, then $\{x_n\}$ is Cauchy sequence.

(iv) $\{x_n\}$ is a Cauchy sequence if and only if $\lim_{n,m \to \infty} d(x_n, x_m) = 0$

(v) If $x_n \to x$ and $y_n \to y$, $(n \to \infty)$ then $d(x_n, y_n) \to d(x, y)$.

Definition: 2.5. Let (X, d) be a cone metric space, P a normal cone with normal constant K and T: X → X. Then

(i) T is said to be continuous if $\lim_{n \to \infty} x_n = x$ implies that $\lim_{n \to \infty} T(x_n) = T(x)$, for all $\{x_n\}$ in X.

(ii) T is said to be sub-sequentially convergent, if we have for every sequence $\{y_n\}$ that $T(y_n)$ is convergent, implies $\{y_n\}$ has a convergent subsequence;

(iii) T is said to be sequentially convergent if for every sequence $\{y_n\}$, $T(y_n)$ is convergent, and then $\{y_n\}$ also is convergent.

Kannan [6, 7] has introduced the following contractive mapping condition:

If there exists a number a, $0 < a < \frac{1}{2}$, such that for each $x, y \in X$,

$$d(f(x), f(y)) \le a[d(x, f(x)) + d(y, f(y))].$$

Further Reich [13] extended it and gave the following contractive mapping condition:

If there exist non-negative numbers a, b, c satisfying a + b + c < 1 such that for each $x, y \in X$,

$$d(f(x), f(y)) \le ad(x, f(x)) + bd(y, f(y)) + cd(x, y).$$

Chatterjea [3] has introduced the following contractive mapping condition:
If there exists a number a, $0 < a < \frac{1}{2}$, such that for each $x, y \in X$,

$$d(f(x), f(y)) \le a[d(x, f(y)) + d(y, f(x))].$$

Further Rhoades [15] extended it and gave the following contractive mapping condition:

If there exist non-negative numbers a, b, c satisfying a + b + c < 1 such that for each $x, y \in X$, $d(f(x), f(y)) \le ad(x, f(y)) + bd(y, f(x)) + cd(x, y)$.

3. MAIN RESULTS:

First, we give definitions of *T*-Reich contractive and *T*-Rhoades contractive mappings on cone metric spaces which are based on the ideas of Moradi [9] and Morales and Rojas [10].

Definition: 3.1. Let (X, d) be a cone metric space and T, S: X → X two functions.

***T*R_1-** A mapping S is said to be *T*-Reich contraction, (*T*R_1-Contraction) if there is a + b + c < 1 such that

$$d(TSx, TSy) \le ad(Tx, TSx) + bd(Ty, TSy) + cd(Tx, Ty)$$

for all $x, y \in X$ and $a, b, c \ge 0$.

***T*R_2-** A mapping S is said to be *T*-Rhoades contraction, (*T*R_2-Contraction) if there is a + b + c < 1 such that

$$d(TSx, TSy) \le a\, d(Tx, TSy) + b\, d(Ty, TSx) + c\, d(Tx, Ty)$$

for all $x, y \in X$ and $a, b, c \ge 0$.

Theorem: 3.1 Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K, in addition let T: X → X be a one to one continuous function and S: X → X a *T*R_1- Contraction. Then

(1) For every $x_0 \in X$

$$\lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = 0;$$

(2) There is a $v \in X$ such that

$$\lim_{n\to\infty} TS^n x_0 = v;$$

(3) If T is subsequentially convergent, then $\{S^n x_0\}$ has a convergent subsequence;

(4) There is a unique $u \in X$ such that Su = u;

(5) If T is sequentially convergent, then for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to u.

Proof: Let x_0 be any arbitrary point in X. We define the sequence $\{x_n\}$ by $x_{n+1} = Sx_n = S^n x_0$. Therefore by TR_1,

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(TSx_{n-1}, TSx_n) \\ &\le a.d(Tx_{n-1}, TSx_{n-1}) + bd(Tx_n, TSx_n) + cd(Tx_{n-1}, Tx_n) \\ &\le ad(Tx_{n-1}, Tx_n) + bd(Tx_n, Tx_{n+1}) + cd(Tx_{n-1}, Tx_n) \end{aligned}$$

(1-b) $d(Tx_n, Tx_{n+1}) \le (a+c)d(Tx_{n-1}, Tx_n)$

$$d(Tx_n, Tx_{n+1}) \le \left(\frac{a+c}{1-b}\right) d(Tx_{n-1}, Tx_n)$$

Hence $d(Tx_n, Tx_{n+1}) \le \left(\frac{a+c}{1-b}\right)^n d(Tx_o, TSx_o)$

On repeating the same argument, we have

$$d(TS^n x_0, TS^{n+1} x_0) \le \left(\frac{a+c}{1-b}\right)^n d(Tx_0, TSx_o) \tag{3.1}$$

From (3.1) we have $\left\| d\left(TS^n x_0, TS^{n+1} x_0\right)\right\| \le \left(\frac{a+c}{1-b}\right)^n K \left\| d(Tx_0, TSx_0)\right\|$

where K is the normal constant of E. By above inequality we get

$$\lim_{n\to\infty} \left\| d(TS^n x_0, TS^{n+1} x_0)\right\| = 0$$

Hence $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = 0$. (3.2)

By (3.1), for every $m, n \in N$ with m > n we have

$$\begin{aligned} d(Tx_n, Tx_m) &\le d(Tx_n, Tx_{n+1}) + \dots\dots + d(Tx_{m-1}, Tx_m) \\ &\le \left(\frac{a+c}{1-b}\right)^n d(Tx_0, TSx_0) + \dots\dots\dots + \left(\frac{a+c}{1-b}\right)^{m-1} d(Tx_0, TSx_0) \\ &= \left(\frac{a+c}{1-b}\right)^n \times \frac{1}{1-\frac{a+c}{1-b}} d(Tx_0, TSx_0) \end{aligned}$$

So, $d(TS^n x_0, TS^m x_0) \le \left(\frac{a+c}{1-b}\right)^n \times \frac{1}{1-\frac{a+c}{1-b}} d(Tx_0, TSx_0)$ (3.3)

From (3.3) we have

$$\left\| d\left(TS^n x_0, TS^m x_0\right) \right\| \le \left(\frac{a+c}{1-b}\right)^n \times \frac{K}{1-\frac{a+c}{1-b}} \left\| d\left(Tx_0, TSx_0\right) \right\|$$

Where K is the normal constant of E. Taking limit and keeping in mind that $\frac{a+c}{1-b}<1$, we obtain

$$\lim_{n,m\to\infty} \left\| d\left(TS^n x_0, TS^m x_0\right) \right\| = 0$$

Hence we have, $\lim_{n\to\infty} d\left(TS^n x_0, TS^m x_0\right) = 0,$

Implying thereby $\{TS^n x_0\}$ is a Cauchy sequence in X. By the completeness of X, there is $v \in X$ such that

$$\lim_{n\to\infty} TS^n x_0 = v \tag{3.4}$$

Now, if T is subsequentially $\{S^n x_0\}$ has a convergent subsequence. So, there are $u \in X$ and $\{x_{n_i}\}$ such that

$$\lim_{i\to\infty} S^{n_i} x_0 = u \tag{3.5}$$

Since T is continuous, then by (3.5) we have

$$\lim_{i\to\infty} TS^{n_i} x_0 = Tu \tag{3.6}$$

By (3.4) and (3.6) we conclude that

$$Tu = v. \tag{3.7}$$

On the other hand,

$$\begin{aligned} d(TSu, Tu) &\le d\left(TSu, TS^{n_i} x_0\right) + d\left(TS^{n_i} x_0, TS^{n_i+1} x_0\right) + d\left(TS^{n_i+1} x_0, Tu\right) \\ &\le ad(Tu, TSu) + bd\left(TS^{n_i-1} x_0, TS^{n_i} x_0\right) + cd\left(Tu, TS^{n_i-1} x_0\right) \\ &\quad + \left(\frac{a+c}{1-b}\right)^{n_i} d\left(Tx_0, TSx_0\right) + d\left(TS^{n_i+1} x_0, Tu\right) \end{aligned}$$

Therefore,

$$\begin{aligned} (1-a)d(TSu, Tu) &\le bd\left(TS^{n_i-1} x_0, TS^{n_i} x_0\right) + cd\left(Tu, TS^{n_i-1} x_0\right) \\ &\quad + \left(\frac{a+c}{1-b}\right)^{n_i} d\left(Tx_0, TSx_0\right) + d\left(TS^{n_i+1} x_0, Tu\right) \end{aligned}$$

$$\begin{aligned} d(TSu, Tu) &\le \left(\frac{b}{1-a}\right) d\left(TS^{n_i-1} x_0, TS^{n_i} x_0\right) + \left(\frac{c}{1-a}\right) d\left(Tu, TS^{n_i-1} x_0\right) \\ &\quad + \frac{1}{1-a}\left(\frac{a+c}{1-b}\right)^{n_i} d\left(Tx_0, TSx_0\right) + \left(\frac{1}{1-a}\right) d\left(TS^{n_i+1} x_0, Tu\right) \end{aligned}$$

$$\|d(TSu,Tu)\| \le \frac{bK}{1-a}\left\|d\left(TS^{n_i-1}x_0, TS^{n_i}x_0\right)\right\| + \frac{c}{1-a}K\left\|d\left(Tu, TS^{n_i-1}x_0\right)\right\|$$
$$+\frac{1}{1-a}\left(\frac{a+c}{1-b}\right)^{n_i}K\|d(TSx_0,Tx_0)\| + \frac{1}{1-a}K\left\|d\left(TS^{n_i+1}x_0, Tu\right)\right\| \to 0 \text{ as } i\to\infty$$

where K is the normal constant of X. Hence d (TSu, Tu) = 0, which implies that TSu = Tu. Since T is one to one, we have

Su = u.

Hence S has a fixed point. Because S is a TR$_1$-Contraction, we have

$$d(TSu,TSv) \le a[d(Tu,TSu)] + b[d(Tv,TSv)] + c[d(Tu,Tv)].$$

If v is another fixed point of S then from injectivity of T we get Su = Sv.

Hence fixed point is unique .Finally, if T is sequentially convergent, by replacing for n_i by n, we conclude that

$\lim_{n\to\infty} S^n x_0 = u$ This shows that $\{S^n x_0\}$ converges to the fixed point of S.

Corollary: 3.1. Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose that the mapping S: X→ X satisfies the contractive condition

$d(Sx,Sy) \le ad(x,Sx) + bd(y,Sy) + cd(x,y)$ for all $x, y \in X$ and $0 \le a + b + c < 1$

Then S has a unique fixed point in X and for any $x_0 \in X$ .Iterative sequence $\{S^n x_0\}$ converges to the fixed point.

Corollary: 3.2. Let (X, d) be a complete metric space and T, S: X → X be mappings such that T is continuous one to one and subsequentially convergent. If $0 \le a + b + c < 1$ and $d(TSx,TSy) \le ad(Tx,TSx) + bd(Ty,TSy) + cd(Tx,Ty)$ for all $x, y \in X$ .Then S has a unique point and if T is sequentially convergent then for every $x_0 \in X$ the sequence iterates $\{S^n x_0\}$ converges to the fixed point of S.

Remark: 3.1. If we put a = b and c = 0 in Theorem 3.1, then we get Theorem 3.1 of Morales and Rojas [10].

Remark: 3.2. Again if we take a = b and c = 0 in Corollary 3.1 and Corollary 3.2, then we get the Corollary 3.2 and Corollary 3.3 of [10].

Theorem: 3.2. Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K, let in addition T: X → X be a one to one continuous function and S: X → X a TR$_2$ Contraction then (1) , (2) , (3) , (4) and (5) of Theorem 3.1 hold.

Proof: Let x_0 be an arbitrary point in X. we define the iterative sequence $\{x_n\}$ by $x_{n+1} = Sx_n = S^n x_0$. Since S is a TR$_2$-Contraction, we have

$$d(TSx_n, TSx_{n+1}) \le a\, d(Tx_n, TSx_{n+1}) + b\, d(Tx_{n+1}, TSx_n) + c\, d(Tx_n, Tx_{n+1})$$
$$\le a\, d(TSx_{n-1}, TSx_{n+1}) + b\, d(TSx_n, TSx_n) + c\, d(TSx_{n-1}, TSx_n) \text{ or}$$

$$\le a\,\{d(TSx_{n-1}, TSx_n) + d(TSx_n, TSx_{n+1})\} + c\, d(TSx_{n-1}, TSx_n)$$

$$d(TSx_n, TSx_{n+1}) \le \left(\frac{a+c}{1-a}\right) d(TSx_{n-1}, TSx_n) = h\, d(TSx_{n-1}, TSx_n)$$

where $h = \left(\frac{a+c}{1-a}\right)$ .Recursively, we obtain

$$d(TSx_n, TSx_{n+1}) \le h^n\, d(TSx_0, TSx_1)\,. \tag{3.8}$$

Therefore

$$\|d(TSx_n, TSx_{n+1})\| \le h^n K\, \|d(TSx_0, TSx_1)\|$$

where K is the normal constant of X. Hence

$$\lim_{n\to\infty} \|d(TSx_n, TSx_{n+1})\| = 0,$$

this implies that

$$\lim_{n\to\infty} \left\|d(TS^n x_0, TS^{n+1} x_0)\right\| = 0.$$

By (3.8), for every $m, n \in N$ with $n > m$ we have

$$\begin{aligned} d(TSx_m, TSx_n) &\le d(TSx_n, TSx_{n+1}) + \ldots + d(TSx_{m-1}, TSx_m) \\ &\le [h^{n-1} + h^{n-2} + \ldots + h^m] d(TSx_0, TSx_1) \\ &\le \frac{h^m}{1-h} d(TSx_0, TSx_1)\,. \end{aligned}$$

Taking norm we get

$$\|d(TSx_m, TSx_n\| \le \frac{h^m}{1-h} K \|d(TSx_0, TSx_1)\|,$$

Consequently, we have

$$\lim_{n,m\to\infty} d(TSx_n, TSx_m) = 0\,.$$

Hence $\{TS^n x_0\}$ is a Cauchy sequence in X and since X is a complete cone metric space, there is $v \in X$ such that

$$\lim_{n\to\infty} TS^n x_0 = v\,.$$

The rest of the proof is similar to the proof of Theorem 3.1.

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