# TOTAL NUMBER OF SPECIAL KINDS OF NEIGHBOURHOOD SETS OF GRAPHS 

*M.P Sumathi and N. D. Soner<br>Department of Studies in Mathematics, University of Mysore, Mysore 570006, India<br>E-mail: sumathideepak@gmail.com, ndsoner@yahoo.co.in

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ABSTRACT
This paper is concerned with special kinds of neighbourhood sets of graphs $P_{n}$ and $C_{n}$. The neighbourhood set, the split neighbourhood sets of a graph are considered using a recurrence relation, we give the number of all neighbourhood sets mentioned for the graphs $P_{n}$ and $C_{n}$.

## INTRODUCTION:

In this paper, we consider the path $P_{n}$ and cycle $C_{n}$ graphs of n vertices as graphs with $V\left(P_{n}\right)=\left\{x_{1}, x_{2} \ldots x_{n}\right\}$, $E\left(P_{n}\right)=\left\{x_{1} x_{2}, x_{2} x_{3}, \ldots x_{n-1} x_{n}\right\}$ for $n \geq 3$ and $V\left(C_{n}\right)=\left\{x_{1}, x_{2} \ldots x_{n}\right\}, E\left(C_{n}\right)=\left\{x_{1} x_{2}, x_{2} x_{3}, \ldots x_{n-1} x_{n}, x_{n} x_{1}\right\}$ for $n \boxtimes 3$. For $v \in V$, the closed neighbourhood of $V$ is $N[v]=\{u \in v: u v \in E(G)\} \bigcup\{v\}$.

A subset $S$ of $V(G)$ is a neighbourhood set of G if $G=\bigcup_{v \in S}\langle N(v)\rangle$. Where $\langle N(v)\rangle$ is the subgraph of $G$ induced by $N[v]$. The neighbourhood number $\eta(G)$ of $G$ is the minimum cardinality of a neighbourhood set of $G$.

Neighbourhood sets of $P_{n}$ and $C_{n}$ :

The results considered in this section will be used to determine the total number of split neighbourhood sets of $P_{n}$ and $C_{n}$.To determine the number of all neighbourhood sets of a path of $n$ vertices, for all $n \geq 1$, we introduce the following notations.
$\mathbf{N}\left(P_{n}\right)=\left\{D \subseteq V(G): D\right.$ is a neighbourhood set of $\left.P_{n}\right\}$.
$\mathbf{N}_{1}\left(P_{n}\right)=\left\{D \in \mathbf{N}\left(P_{n}\right): X_{n} \in D\right\}$.
$\mathbf{N}_{2}\left(P_{n}\right)=\mathbf{N}\left(P_{n}\right)-\mathbf{N}_{1}\left(P_{1}\right)$

Denoting the cardinalities of the families $\mathbf{N}\left(P_{n}\right), \mathbf{N}_{1}\left(P_{n}\right)$ and $\mathbf{N}_{2}\left(P_{n}\right)$ respectively by $\mathbf{N}\left(P_{n}\right), \mathbf{N}_{1}\left(P_{n}\right)$, $\mathbf{N}_{2}\left(P_{n}\right)$. We obtain the following equality,
$\mathbf{N}\left(P_{n}\right)=\mathbf{N}_{1}\left(P_{n}\right)+\mathbf{N}_{2}\left(P_{n}\right), n \geq 1$
of course $\mathbf{N}\left(P_{n}\right)$ is the total number of neighbourhood sets of a path $P_{n}$ on $n$ vertices. It is easy to see that $\mathbf{N}\left(P_{1}\right)=1, \mathbf{N}\left(P_{2}\right)=3, \mathbf{N}\left(P_{3}\right)=5$. These inequalities are the initial conditions for the recurrence relation given in the following theorem.

Theorem 1: For $n \geq 3$,

$$
\mathbf{N}\left(P_{n+1}\right)=\mathbf{N}\left(P_{n}\right)+\mathbf{N}\left(P_{n-1}\right)
$$

Proof: Assume the $\mathbf{N}$ is a neighborhood set of $P_{n+1}$ for $n \geq 3$. Then we have three possibilities:
(1) If $X_{n}, X_{n+1} \in D$, then $D-\left\{X_{n+1}\right\} \in \mathbf{N}_{1}\left(P_{n}\right)$.
(2) If $X_{n} \in D$ and $X_{n+1} \notin D$,then $D \in \mathbf{N}_{1}\left(P_{n}\right)$.
(3) If $X_{n} \notin D$ and $X_{n-1}, X_{n+1} \in D$, then $D-\left\{X_{n+1}\right\} \in \mathbf{N}_{2}\left(P_{n}\right)$.

Thus, the number $\mathbf{N}\left(P_{n+1}\right)$ of all neighbourhood sets of $P_{n+1}$ is described by the following equality.
$\mathbf{N}^{\prime}\left(P_{n+1}\right)=\mathbf{N}_{1}^{\prime}\left(P_{n}\right)+\mathbf{N}_{1}^{\prime}\left(P_{n}\right)+\mathbf{N}_{2}^{\prime}\left(P_{n-1}\right)$,
it follows that for $n \geq 3$, we have

$$
\mathbf{N}_{1}\left(P_{n}\right)=\mathbf{N}_{1}\left(P_{n-1}\right)+\mathbf{N}_{2}\left(P_{n-1}\right)
$$

and

$$
\begin{equation*}
\mathbf{N}_{2}\left(P_{n}\right)=\mathbf{N}_{1}\left(P_{n-1}\right) \tag{3}
\end{equation*}
$$

Applying these equalities in equation (3) to the first of the sum in equation (2), we obtain
$\mathbf{N}^{\prime}\left(P_{n+1}\right)=\mathbf{N}_{1}\left(P_{n-1}\right)+\mathbf{N}_{2}\left(P_{n-1}\right)+\mathbf{N}_{1}\left(P_{n}\right)+\mathbf{N}_{2}\left(P_{n}\right)$
Hence $\mathbf{N}\left(P_{n+1}\right)=\mathbf{N}^{\prime}\left(P_{n-1}\right)+\mathbf{N}^{\prime}\left(P_{n}\right)$.

The following results concern the total number of neighbourhood sets of the cycle $C_{n}$ on n vertices for $n \geq 4$.First we introduce the following notations:
$\mathbf{N}\left(C_{n}\right)=\left\{D\right.$ subseteq $V(G): \mathbf{N}$ is a neighbourhood set of $\left.C_{n}\right\}$.
$\mathbf{N}_{01}\left(c_{n}\right)=\left\{D \in \mathbf{N}(C n):\left(X_{1} \in D\right.\right.$ and $\left.X_{n} \notin D\right)$ or $\left(X_{1} \notin D\right.$ and $\left.\left.X_{n} \in D\right)\right\}$
$\mathbf{N}_{11}\left(c_{n}\right)=\left\{D \in \mathbf{N}(C n): X_{1}, X_{n} \in D\right\}$

By $\mathbf{N}^{\prime}\left(C_{n}\right), \mathbf{N}_{01}^{\prime}\left(C_{n}\right)$ and $\mathbf{N}_{11}^{\prime}\left(C_{n}\right)$ we mean the cardinalities of families $\mathbf{N}\left(C_{n}\right), \mathbf{N}_{01}\left(c_{n}\right)$ and $\mathbf{N}_{11}\left(c_{n}\right)$ respectively.Using these numbers, we obtain the following equality.
$\mathrm{N}^{\prime}\left(\mathrm{C}_{\mathrm{n}}\right)=\mathrm{N}_{01}^{\prime}\left(\mathrm{C}_{\mathrm{n}}\right)+\mathrm{N}_{11}^{\prime}\left(\mathrm{C}_{\mathrm{n}}\right) \quad n \geq 4$,
It is easy to check that $\mathbf{N}\left(C_{4}\right)=7$ and $\mathbf{N}\left(C_{5}\right)=11$.
Theorem 2: For $n \geq 5$,

$$
\mathbf{N}\left(C_{n+1}\right)=\mathbf{N}\left(C_{n-1}\right)+\mathbf{N}\left(C_{n}\right)
$$

Proof: Let $n$ be a neighbourhood set of $C_{n}$ for $n \geq 5$. Consider the following cases:
(i) If $X_{1} \notin D$ and $X_{2}, X_{n} \in D$, then $D \in \mathbf{N}_{11}\left(H_{1}\right)$ where
$V\left(H_{1}\right)=V\left(C_{n}\right)-\left(X_{1}\right)$ and $E\left(H_{1}\right)=\left\{E\left(C_{n}\right)-\left(X_{n} X_{1}, X_{1} X_{2}\right)\right\} \bigcup\left(X_{n} X_{2}\right)$

Hence $H_{1} \approx C_{n-1}$
(ii) If $X_{n} \notin D$ and $X_{1}, X_{n-1} \in D$ then $D \in \mathbf{N}_{11}\left(H_{2}\right)$, where

$$
V\left(H_{2}\right)=V\left(C_{n}\right)-\left(X_{n}\right) \text { and } E\left(H_{2}\right)=\left\{E\left(C_{n}\right)-\left(X_{n-1} X_{n}, X_{n} X_{1}\right)\right\} \bigcup\left(X_{n-1} X_{1}\right)
$$

Hence $H_{2} \approx C_{n-1}$
(iii) If $X_{1}, X_{n} \in D$ and $X_{2} \notin D$ then $D \in \mathbf{N}_{01}\left(H_{1}\right)$
(iv) $I f X_{1}, X_{2}, X_{n} \in D$ then $D \in \mathbf{N}_{11}\left(H_{1}\right)$

Hence from case (i),(ii),(iii) and (iv) we have that

$$
\begin{aligned}
\mathbf{N}_{01}^{\prime}\left(C_{n}\right) & =\mathbf{N}_{11}^{\prime}\left(H_{1}\right)+\mathbf{N}_{11}^{\prime}\left(H_{2}\right) \\
\mathbf{N}_{01}^{\prime}\left(C_{n}\right) & =2 \mathbf{N}_{11}^{\prime}\left(C_{n-1}\right) \\
\mathbf{N}_{11}^{\prime}\left(C_{n}\right) & =\mathbf{N}_{01}^{\prime}\left(C_{n-1}\right) / 2+\mathbf{N}_{11}^{\prime}\left(C_{n-1}\right) \\
\mathbf{N}^{\prime}\left(C_{n}\right) & =\mathbf{N}_{01}^{\prime}\left(C_{n}\right)+\mathbf{N}_{11}^{\prime}\left(C_{n}\right) \\
& =2 \mathbf{N}_{11}^{\prime}\left(C_{n-1}\right)+\mathbf{N}_{01}^{\prime}\left(C_{n-1}\right) / 2+\mathbf{N}_{11}^{\prime}\left(C_{n-1}\right) \\
& =2\left(\mathbf{N}_{01}^{\prime}\left(C_{n-2}\right) / 2+\mathbf{N}_{11}^{\prime}\left(C_{n-2}\right)\right)+\mathbf{N}_{01}^{\prime}\left(C_{n-1}\right) / 2+\mathbf{N}_{11}^{\prime}\left(C_{n-1}\right) \\
& =\mathbf{N}_{01}^{\prime}\left(C_{n-2}\right)+2 \mathbf{N}_{11}^{\prime}\left(C_{n-2}\right)+1 / 2\left(2 \mathbf{N}_{11}^{\prime}\left(C_{n-2}\right)\right)+\mathbf{N}_{11}^{\prime}\left(C_{n-1}\right) \\
& =\mathbf{N}_{01}^{\prime}\left(C_{n-2}\right)+2 \mathbf{N}_{11}^{\prime}\left(C_{n-2}\right)+\mathbf{N}_{11}^{\prime}\left(C_{n-2}\right)+\mathbf{N}_{11}^{\prime}\left(C_{n-1}\right) \\
& =\mathbf{N}_{01}^{\prime}\left(C_{n-2}\right)+\mathbf{N}_{01}^{\prime}\left(C_{n-1}\right)+\mathbf{N}_{11}^{\prime}\left(C_{n-2}\right)+\mathbf{N}_{11}^{\prime}\left(C_{n-1}\right)
\end{aligned}
$$

From equation (4) the above equation reduces to
$\mathbf{N}\left(C_{n}\right)=\mathbf{N}_{01}\left(C_{n-2}\right)+\mathbf{N}_{11}\left(C_{n-2}\right)+\mathbf{N}_{01}\left(C_{n-1}\right)+\mathbf{N}_{11}\left(C_{n-1}\right)$
$\mathbf{N}\left(C_{n}\right)=\mathbf{N}\left(C_{n-2}\right)+\mathbf{N}\left(C_{n-1}\right)$
$\mathbf{N}\left(C_{n+1}\right)=\mathbf{N}\left(C_{n-1}\right)+\mathbf{N}\left(C_{n}\right)$

Split neighbourhood set of $P_{n}$ and $C_{n}$ :

Using the numbers $\mathbf{N}\left(P_{n}\right)$ and $\mathbf{N}\left(C_{n}\right)$ we determine the number of split neighbourhood sets of path and the cycle of graphs on n vertices. First we give supportive theorem that characterize the split neighbourhood set of $P_{n}$ and $C_{n}$.

Lemma 3: Any neighbourhood set $D$ of $P_{n} \quad n \geq 3$, is a split neighbourhood set of $P_{n}$ if and only if $V\left(P_{n}\right)-D \neq \emptyset,\left\langle V\left(P_{n}-D\right)\right\rangle_{\left(P_{n}\right)} \neq K_{1}$.

Proof: Let S be a split neighbourhood set D of $P_{n}$.According to the definition of $S$, it follows that $\left\langle V\left(P_{n}\right)-D\right\rangle$ is disconnected. Thus $V\left(P_{n}\right)-D \neq \emptyset\left\langle V\left(P_{n}\right)-D\right\rangle_{\left(P_{n}\right)}$ not $\approx K_{1}$, proving necessity.
For sufficiency, let $D$ be a neighbourhood set of $P_{n}, \quad n \geq 3$, and suppose $V\left(P_{n}\right)-D \neq \emptyset$, $\left\langle V\left(P_{n}-D\right)\right\rangle_{\left(P_{n}\right)} \neq K_{1}$. Since $H=\left\langle V\left(P_{n}-D\right)\right\rangle_{\left(P_{n}\right)} \quad$ is an induced subgraph of $\quad P_{n}$, then any connected component of $H$ is isomorphic to the path $P_{k}, 1 \leq k \leq n-1$ (where $P_{1}=K_{1}$ ). Let $H_{1}$ be a connected
component of the subgraph $H$. We show that $H_{1}$ is not a unique connected component of $H$. First, observe that $H_{1} \quad$ contains atmost two vertices. Otherwise there would exist a vertex of $H_{1} \subset H=\left\langle V\left(P_{n}-D\right)\right\rangle_{\left(P_{n}\right)}$ not neighbour of $D$ in $P_{n}$. Consequently, $H_{1} \approx P_{1}$ or $H_{1} \approx P_{2}$. Hence $H$ has at least two connected components, because $H_{1} \neq P_{1}$ or $H_{1} \neq P_{2}$ by premise. This shows that $H=\left\langle V\left(P_{n}\right)-D\right\rangle_{\left(P_{n}\right)}$ is disconnected. Moreover, since $D$ is also a neighbourhood set of $P_{n}$, it is a split neighbourhood set of $P_{n}$, completes the proof of the theorem.

Similar to the case of $P_{n}$, we have a result concerning the split neighbourhood sets of $C_{n}$.
Lemma 4: Any neighbourhood set $D$ of $C_{n}, n \geq 4$ is a split neighbourhood set of $C_{n}$ if and only if
$V\left(P_{n}\right)-D \neq \emptyset,\left\langle V\left(C_{n}\right)-D\right\rangle_{\left(C_{n}\right)} \neq K_{1}$.
Additionally, observe that there is only one neighbourhood set $D$ of $P_{n}$ such that $V\left(P_{n}\right)-D \neq \emptyset$, and there are exactly $n$ neighbourhood sets $D$ of $P_{n}$ such that $\left\langle V\left(P_{n}\right)-D\right\rangle_{\left(P_{n}\right)} \approx K_{1}$.

In special cases of $C_{n}$, we have one Neighbouhood set D such that $V\left(C_{n}\right)-D=\emptyset \quad$ and $n$ for $\left\langle V\left(C_{n}\right)-D\right\rangle_{\left(C_{n}\right)} \approx K_{1}$.

For $n \geq 3$, we introduce the notation
$S(G)=\{D \subseteq V(G): D$ is a split neighbourhood set of $G\}$
$S(G)=|S(G)|$
From the above, we have the following corollary, which will be used in proving theorem.
Corollary 5: $s\left(P_{n}\right)=d\left(P_{n}\right)-(1+n)$ for $n \geq 3$
and $s\left(C_{n}\right)=d\left(C_{n}\right)-(1+n)$ It is easy to see that $s\left(P_{3}\right)=1, s\left(P_{4}\right)=3$ and $s\left(P_{5}\right)=7$.
Theorem 6: For $n \geq 5$,

$$
s\left(P_{n+1}\right)=s\left(P_{n-1}\right)+s\left(P_{n}\right)+(n-1)
$$

Proof: Let $n \geq 5$, according to corollary (3), for $P_{n+1}$ we have that

$$
s\left(P_{n+1}\right)=d\left(P_{n+1}\right)-(2+n) \text { since } d\left(P_{n+1}\right)=d\left(P_{n-1}\right)+d\left(P_{n}\right)
$$

by theorem (1), we obtain

$$
\begin{aligned}
s\left(P_{n+1}\right) & =d\left(P_{n+1}\right)-(2+n) \\
& =d\left(P_{n-1}\right)+d\left(P_{n}\right)-(2+n) \\
& =d\left(\left(P_{n-1}\right)-n+n\right)+d\left(\left(P_{n}\right)-(n+1)+(n+1)\right)-(2+n) \\
& =d\left(\left(P_{n-1}\right)-n\right)+d\left(\left(P_{n}\right)-(n+1)\right)-(2+n+n+(n+1)) \\
& =d\left(\left(P_{n-1}\right)-n\right)+d\left(\left(P_{n}\right)-(n+1)\right)+(n+1+n-2-n)
\end{aligned}
$$

since $(n+1+n-2-n=(n-1)$, it follows that

$$
=d\left(\left(P_{n-1}\right)-n\right)+d\left(\left(P_{n}\right)-(n+1)\right)+(n-1)
$$

Finally applying corollary (3)to the expressions in brackets,

$$
s\left(P_{n+1}\right)=s\left(P_{n-1}\right)+s\left(P_{n}\right)+(n-1) .
$$

Theorem 7: For $n \geq 5$,
$s\left(C_{n+1}\right)=s\left(C_{n-1}\right)+s\left(C_{n}\right)+(n-1)$

Proof: Let $n \geq 5$, putting $n+1$ in place of $n$ in corollary 3, it follows

$$
\begin{aligned}
& s\left(C_{n+1}\right)=d\left(C_{n+1}\right)-(2+n) \text { according to theorem (2), } \\
& \begin{aligned}
d\left(C_{n+1}\right) & =d\left(C_{n-1}\right)-d\left(C_{n}\right) \text { hence, } \\
s\left(C_{n+1}\right) & =d\left(C_{n+1}\right)-(2+n) \\
& =d\left(C_{n-1}\right)+d\left(C_{n}\right)-(2+n) \\
& =d\left(\left(C_{n-1}\right)-n+n\right)+d\left(\left(C_{n}\right)-(n+1)+(n+1)\right)-(2+n) \\
& =d\left(\left(C_{n-1}\right)-n\right)+d\left(\left(C_{n}\right)-(n+1)\right)+(n+1+n-2-n)
\end{aligned}
\end{aligned}
$$

since $(n+1+n-2-n=(n-1))$, it follows that

$$
=d\left(\left(C_{n-1}\right)-n\right)+d\left(\left(C_{n}\right)-(n+1)\right)+(n-1)
$$

Finally applying corollary (3) to the expressions in brackets,

$$
s\left(C_{n+1}\right)=s\left(C_{n-1}\right)+s\left(C_{n}\right)+(n-1) .
$$

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