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TOTAL NUMBER OF SPECIAL KINDS OF NEIGHBOURHOOD SETS OF GRAPHS

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ABSTRACT

This paper is concerned with special kinds of neighbourhood sets of graphs P_n and C_n. The neighbourhood set, the split neighbourhood sets of a graph are considered using a recurrence relation, we give the number of all neighbourhood sets mentioned for the graphs P_n and C_n.

INTRODUCTION:

In this paper, we consider the path P_n and cycle C_n graphs of n vertices as graphs with $V(P_n)=\{x_1, x_2 ... x_n\}$, $E(P_n)=\{x_1x_2, x_2x_3, ... x_{n-1}x_n\}$ for $n \geq 3$ and $V(C_n)=\{x_1, x_2 ... x_n\}$, $E(C_n)=\{x_1x_2, x_2x_3, ... x_{n-1}x_n, x_nx_1\}$ for n ¤ 3. For $v \in V$, the closed neighbourhood of V is $N[v]=\{u \in v : uv \in E(G)\} \bigcup \{v\}$.

A subset S of $V(G)$ is a neighbourhood set of G if $G = \bigcup_{v \in S} \langle N(v) \rangle$. Where $\langle N(v) \rangle$ is the subgraph of G induced by $N[v]$. The neighbourhood number $\eta(G)$ of G is the minimum cardinality of a neighbourhood set of G.

Neighbourhood sets of P_n and C_n:

The results considered in this section will be used to determine the total number of split neighbourhood sets of P_n and C_n .To determine the number of all neighbourhood sets of a path of n vertices, for all $n \geq 1$, we introduce the following notations.

$\mathbf{N}(P_n) = \{D \subseteq V(G) : D$ is a neighbourhood set of $P_n\}$.

$\mathbf{N}_1(P_n) = \{D \in \mathbf{N}(P_n) : X_n \in D\}$.

$\mathbf{N}_2(P_n) = \mathbf{N}(P_n) - \mathbf{N}_1(P_1)$

Denoting the cardinalities of the families $\mathbf{N}(P_n), \mathbf{N}_1(P_n)$ and $\mathbf{N}_2(P_n)$ respectively by $\mathbf{N}'(P_n)$, $\mathbf{N}_1'(P_n)$, $\mathbf{N}_2'(P_n)$. We obtain the following equality,

$$\mathbf{N}'(P_n) = \mathbf{N}_1'(P_n) + \mathbf{N}_2'(P_n), n \geq 1 \qquad (1)$$

of course $\mathbf{N}'(P_n)$ is the total number of neighbourhood sets of a path P_n on n vertices. It is easy to see that $\mathbf{N}'(P_1)=1$, $\mathbf{N}'(P_2)=3$, $\mathbf{N}'(P_3)=5$. These inequalities are the initial conditions for the recurrence relation given in the following theorem.

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Theorem 1: *For* $n \geq 3$,

$$\mathbf{N}^{'}(P_{n+1}) = \mathbf{N}^{'}(P_n) + \mathbf{N}^{'}(P_{n-1})$$

Proof: Assume the $\mathbf{N}$ is a neighborhood set of P_{n+1} for $n \geq 3$. Then we have three possibilities:

(1) If $X_n, X_{n+1} \in D$, then $D - \{X_{n+1}\} \in \mathbf{N}_1(P_n)$.

(2) If $X_n \in D$ and $X_{n+1} \notin D$,then $D \in \mathbf{N}_1(P_n)$.

(3) If $X_n \notin D$ and $X_{n-1}, X_{n+1} \in D$, then $D - \{X_{n+1}\} \in \mathbf{N}_2(P_n)$.

Thus, the number $\mathbf{N}^{'}(P_{n+1})$ of all neighbourhood sets of P_{n+1} is described by the following equality.

$$\mathbf{N}^{'}(P_{n+1}) = \mathbf{N}_1^{'}(P_n) + \mathbf{N}_1^{'}(P_n) + \mathbf{N}_2^{'}(P_{n-1}), \qquad (2)$$

it follows that for $n \geq 3$, we have

$$\mathbf{N}_1(P_n) = \mathbf{N}_1(P_{n-1}) + \mathbf{N}_2(P_{n-1})$$

and

$$\mathbf{N}_2(P_n) = \mathbf{N}_1(P_{n-1}) \qquad (3)$$

Applying these equalities in equation (3) to the first of the sum in equation (2), we obtain

$$\mathbf{N}^{'}(P_{n+1}) = \mathbf{N}_1(P_{n-1}) + \mathbf{N}_2(P_{n-1}) + \mathbf{N}_1(P_n) + \mathbf{N}_2(P_n)$$

Hence $\mathbf{N}^{'}(P_{n+1}) = \mathbf{N}^{'}(P_{n-1}) + \mathbf{N}^{'}(P_n)$.

The following results concern the total number of neighbourhood sets of the cycle C_n on n vertices for $n \geq 4$ .First we introduce the following notations:

$\mathbf{N}(C_n) = \{D\ subseteq V(G) :$ $\mathbf{N}$ is a neighbourhood set of C_n }.

$\mathbf{N}_{01}(c_n) = \{D \in \mathbf{N}(Cn) : (X_1 \in D$ and $X_n \notin D)$ or $(X_1 \notin D$ and $X_n \in D)\}$

$\mathbf{N}_{11}(c_n) = \{D \in \mathbf{N}(Cn) :\ X_1, X_n \in D\}$

By $\mathbf{N}^{'}(C_n)$, $\mathbf{N}_{01}^{'}(C_n)$ and $\mathbf{N}_{11}^{'}(C_n)$ we mean the cardinalities of families $\mathbf{N}(C_n)$, $\mathbf{N}_{01}(c_n)$ and $\mathbf{N}_{11}(c_n)$ respectively.Using these numbers, we obtain the following equality.

$$N'(C_n) = N'_{01}(C_n) + N'_{11}(C_n) \qquad n \geq 4, \qquad (4)$$

It is easy to check that $\mathbf{N}^{'}(C_4) = 7$ and $\mathbf{N}^{'}(C_5) = 11.$

Theorem 2: *For* $n \geq 5,$

$$\mathbf{N}^{'}(C_{n+1}) = \mathbf{N}^{'}(C_{n-1}) + \mathbf{N}^{'}(C_n)$$

Proof: Let n be a neighbourhood set of C_n for $n \geq 5$. Consider the following cases:

(i) If $X_1 \notin D$ and $X_2, X_n \in D$, then $D \in \mathbf{N}_{11}(H_1)$ where

$V(H_1) = V(C_n) - (X_1)$ and $E(H_1) = \{E(C_n) - (X_n X_1, X_1 X_2)\} \bigcup (X_n X_2)$

Hence $H_1 \approx C_{n-1}$

(ii) $If X_n \notin D$ and $X_1, X_{n-1} \in D$ then $D \in \mathbf{N}_{11}(H_2)$, where

$V(H_2) = V(C_n) - (X_n)$ and $E(H_2) = \{E(C_n) - (X_{n-1}X_n, X_n X_1)\} \bigcup (X_{n-1}X_1)$

Hence $H_2 \approx C_{n-1}$

(iii) $If X_1, X_n \in D$ and $X_2 \notin D$ then $D \in \mathbf{N}_{01}(H_1)$

(iv) $If X_1, X_2, X_n \in D$ then $D \in \mathbf{N}_{11}(H_1)$

Hence from case (i),(ii),(iii) and (iv) we have that

$$\mathbf{N}'_{01}(C_n) = \mathbf{N}'_{11}(H_1) + \mathbf{N}'_{11}(H_2)$$
$$\mathbf{N}'_{01}(C_n) = 2\mathbf{N}'_{11}(C_{n-1})$$
$$\mathbf{N}'_{11}(C_n) = \mathbf{N}'_{01}(C_{n-1})/2 + \mathbf{N}'_{11}(C_{n-1})$$
$$\begin{aligned}\mathbf{N}'(C_n) &= \mathbf{N}'_{01}(C_n) + \mathbf{N}'_{11}(C_n)\\ &= 2\mathbf{N}'_{11}(C_{n-1}) + \mathbf{N}'_{01}(C_{n-1})/2 + \mathbf{N}'_{11}(C_{n-1})\\ &= 2(\mathbf{N}'_{01}(C_{n-2})/2 + \mathbf{N}'_{11}(C_{n-2})) + \mathbf{N}'_{01}(C_{n-1})/2 + \mathbf{N}'_{11}(C_{n-1})\\ &= \mathbf{N}'_{01}(C_{n-2}) + 2\mathbf{N}'_{11}(C_{n-2}) + 1/2(2\mathbf{N}'_{11}(C_{n-2})) + \mathbf{N}'_{11}(C_{n-1})\\ &= \mathbf{N}'_{01}(C_{n-2}) + 2\mathbf{N}'_{11}(C_{n-2}) + \mathbf{N}'_{11}(C_{n-2}) + \mathbf{N}'_{11}(C_{n-1})\\ &= \mathbf{N}'_{01}(C_{n-2}) + \mathbf{N}'_{01}(C_{n-1}) + \mathbf{N}'_{11}(C_{n-2}) + \mathbf{N}'_{11}(C_{n-1})\end{aligned}$$

From equation (4) the above equation reduces to

$$\mathbf{N}'(C_n) = \mathbf{N}'_{01}(C_{n-2}) + \mathbf{N}'_{11}(C_{n-2}) + \mathbf{N}'_{01}(C_{n-1}) + \mathbf{N}'_{11}(C_{n-1})$$
$$\mathbf{N}'(C_n) = \mathbf{N}'(C_{n-2}) + \mathbf{N}'(C_{n-1})$$
$$\mathbf{N}'(C_{n+1}) = \mathbf{N}'(C_{n-1}) + \mathbf{N}'(C_n)$$

Split neighbourhood set of P_n and C_n:

Using the numbers $\mathbf{N}(P_n)$ and $\mathbf{N}(C_n)$ we determine the number of split neighbourhood sets of path and the cycle of graphs on n vertices. First we give supportive theorem that characterize the split neighbourhood set of P_n and C_n.

Lemma 3: Any neighbourhood set D of P_n $n \geq 3$, is a split neighbourhood set of P_n if and only if $V(P_n) - D \neq \emptyset$, $\langle V(P_n - D)\rangle_{(P_n)} \neq K_1$.

Proof: Let S be a split neighbourhood set D of P_n .According to the definition of S , it follows that $\langle V(P_n) - D\rangle$ is disconnected. Thus $V(P_n) - D \neq \emptyset$ $\langle V(P_n) - D\rangle_{(P_n)} not \approx K_1$, proving necessity.

For sufficiency, let D be a neighbourhood set of P_n , $n \geq 3$, and suppose $V(P_n) - D \neq \emptyset$, $\langle V(P_n - D)\rangle_{(P_n)} \neq K_1$. Since $H = \langle V(P_n - D)\rangle_{(P_n)}$ is an induced subgraph of P_n , then any connected component of H is isomorphic to the path P_k , $1 \leq k \leq n-1$ (where $P_1 = K_1$). Let H_1 be a connected component of the subgraph H. We show that H_1 is not a unique connected component of H. First, observe that H_1 contains atmost two vertices. Otherwise there would exist a vertex of $H_1 \subset H = \langle V(P_n - D)\rangle_{(P_n)}$ not neighbour of D in P_n. Consequently, $H_1 \approx P_1$ or $H_1 \approx P_2$. Hence H has at least two connected components, because $H_1 \neq P_1$ or $H_1 \neq P_2$ by premise. This shows that $H = \langle V(P_n) - D\rangle_{(P_n)}$ is disconnected. Moreover, since D is also a neighbourhood set of P_n, it is a split neighbourhood set of P_n, completes the proof of the theorem.

Similar to the case of P_n, we have a result concerning the split neighbourhood sets of C_n.

Lemma 4: *Any neighbourhood set* D *of* C_n, $n \geq 4$ *is a split neighbourhood set of* C_n *if and only if*

$V(P_n) - D \neq \emptyset$, $\langle V(C_n) - D\rangle_{(C_n)} \neq K_1$.

Additionally, observe that there is only one neighbourhood set D *of* P_n *such that* $V(P_n) - D \neq \emptyset$*, and there are exactly* n *neighbourhood sets* D *of* P_n *such that* $\langle V(P_n) - D\rangle_{(P_n)} \approx K_1$.

In special cases of C_n, we have one Neighbouhood set D such that $V(C_n) - D = \emptyset$ and n for $\langle V(C_n) - D\rangle_{(C_n)} \approx K_1$.

For $n \geq 3$, we introduce the notation

$S(G) = \{D \subseteq V(G) :\ D$ is a split neighbourhood set of $G\}$

$S(G) = | S(G) |$

From the above, we have the following corollary, which will be used in proving theorem.

Corollary 5: $s(P_n) = d(P_n) - (1+n)$ for $n \geq 3$

and $s(C_n) = d(C_n) - (1+n)$ It is easy to see that $s(P_3) = 1$, $s(P_4) = 3$ and $s(P_5) = 7$.

Theorem 6: *For* $n \geq 5$,

$$s(P_{n+1}) = s(P_{n-1}) + s(P_n) + (n-1)$$

Proof: Let $n \geq 5$, according to corollary (3), for P_{n+1} we have that

$s(P_{n+1}) = d(P_{n+1}) - (2+n)$ since $d(P_{n+1}) = d(P_{n-1}) + d(P_n)$

by theorem (1), we obtain

$$\begin{aligned} s(P_{n+1}) &= d(P_{n+1}) - (2+n) \\ &= d(P_{n-1}) + d(P_n) - (2+n) \\ &= d((P_{n-1}) - n + n) + d((P_n) - (n+1) + (n+1)) - (2+n) \\ &= d((P_{n-1}) - n) + d((P_n) - (n+1)) - (2+n+n+(n+1)) \\ &= d((P_{n-1}) - n) + d((P_n) - (n+1)) + (n+1+n-2-n) \end{aligned}$$

since $(n+1+n-2-n=(n-1)$, it follows that

$$=d((P_{n-1})-n)+d((P_n)-(n+1))+(n-1)$$

Finally applying corollary (3)to the expressions in brackets,

$$s(P_{n+1})=s(P_{n-1})+s(P_n)+(n-1) \ .$$

Theorem 7: *For* $n\geq 5$,
$$s(C_{n+1})=s(C_{n-1})+s(C_n)+(n-1)$$

Proof: Let $n\geq 5$, putting $n+1$ in place of n in corollary 3, it follows

$s(C_{n+1})=d(C_{n+1})-(2+n)$ according to theorem (2),
$d(C_{n+1})=d(C_{n-1})-d(C_n)$ hence,

$$\begin{aligned} s(C_{n+1}) &= d(C_{n+1})-(2+n) \\ &= d(C_{n-1})+d(C_n)-(2+n) \\ &= d((C_{n-1})-n+n)+d((C_n)-(n+1)+(n+1))-(2+n) \\ &= d((C_{n-1})-n)+d((C_n)-(n+1))+(n+1+n-2-n) \end{aligned}$$

since $(n+1+n-2-n=(n-1))$, it follows that

$$=d((C_{n-1})-n)+d((C_n)-(n+1))+(n-1)$$

Finally applying corollary (3) to the expressions in brackets,

$$s(C_{n+1})=s(C_{n-1})+s(C_n)+(n-1) \ .$$

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