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FORMATION OF TWO SUMMATION FORMULAE ALLIED WITH CONTIGIOUS RELATION

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ABSTRACT

The main aim of this paper is to evaluate two summation formulae involving Contiguous Relation associated with Recurrence relation and Hypergeometric function.

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Key words and phrases: *Pochhammer's symbol, Contiguous relation, Recurrence relation*

A. INTRODUCTION:

The Pochhammer's symbol:

$$(\alpha , k) =(\alpha)_k= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha+1)(\alpha+2)\,.....\,(\alpha+k-1); & \text{if } k = 1,2,3\,......\\ 1 & ; \quad \text{if } k = 0 \\ k! & ; \quad \text{if } \alpha = 1 \end{cases} \tag{1}$$

Generalized Gaussian Hypergeometric function of one variable:

$${}_AF_B(a_1,a_2,\ldots,a_A;b_1,b_2,\ldots b_B;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\ldots(a_A)_k\, z^k}{(b_1)_k(b_2)_k\ldots(b_B)_k\, k!} \tag{2}$$

or

$${}_AF_B((a_A);(b_B);z) \equiv {}_AF_B((a_j)^A_{j=1};(bj)^B_{j=1};z) = \sum_{k=0}^{\infty} \frac{((a_A))_k\, z^k}{((b_B))_k\, k!} \tag{3}$$

where the parameters $b_1 , b_2 , \ldots, b_B$ are neither zero nor negative integers and A , B are non negative integers.

Contiguous Relations:

[Andrews p.363 (9.16), E.D. p.51 (10), H.T.F.I. p.103 (32)]

$$(a-b)\, {}_2F_1(a, b ; c; z) = a\, {}_2F_1(a+1,b; c; z) - b\, {}_2F_1(a,b+1; c; z) \tag{4}$$

[Abramowitz p.558 (15.2.19)]

$$(a-b)(1-z)\, {}_2F_1(a, b ; c; z) = (c-b)\, {}_2F_1(a,b-1; c; z) + (a-c)\, {}_2F_1(a-1,b; c; z) \tag{5}$$

A New Summation Formula:

$${}_2F_1\left(a, b ; \frac{a+b-1}{2}; \frac{1}{2}\right) = 2^{b-1} \frac{\Gamma(\frac{a+b-1}{2})}{\Gamma(b)} \left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} \left\{ \frac{b+a-1}{a-1} \right\} + 2\, \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a}{2})} \right] \tag{6}$$

Recurrence relation:

$$\Gamma(z+1) = z\,\Gamma(z) \tag{7}$$

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B. MAIN SUMMATION FORMULAE:

For both the formulae $a \neq b$

For $a<1$ and $a>13$

$$ {}_2F_1\left(a, b ; \frac{a+b-13}{2}; \frac{1}{2}\right) = 2^{b-1} \frac{\Gamma\left(\frac{a+b-13}{2}\right)}{(a-b)\Gamma(b)}\left[\frac{\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a-13}{2}\right)}\left\{ \frac{-135135a+264207a^2-177331a^3+57379a^4-10045a^5}{(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} + \right.\right. $$

$$ + \frac{973a^6-49a^7+a^8+135135b-318983a^2b+298522a^3b-86415a^4b+18956a^5b-1225a^6b+90a^7b-264207b^2}{(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} $$

$$ + \frac{318983ab^2-93730a^3b^2+54145a^4b^2-5733a^5b^2+910a^6b^2+177331b^3-298522ab^3+93730a^2b^3-5005a^4b^3}{(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} + $$

$$ + \frac{2002a^5b^3-57379b^4+86415ab^4-54145a^2b^4+5005a^3b^4+10045b^5-18956ab^5+5733a^2b^5-2002a^3b^5}{(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} $$

$$ + \frac{-973b^6+1225ab^6-910a^2b^6+49b^7-90ab^7-b^8}{(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} \right\} + \frac{\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a-12}{2}\right)}\left\{ \frac{221874a-249056a^2+141866a^3-27200a^4}{(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \right. $$

$$ + \frac{4886a^5-224a^6+14a^7-221874b+224578a^2b-101760a^3b+37650a^4b-2688a^5b+350a^6b+249056b^2}{(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} $$

$$ + \frac{224578ab^2+39260a^3b^2-6240a^4b^2+1638a^5b^2-141866b^3+101760ab^3-39260a^2b^3+1430a^4b^3+27200b^4}{(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} $$

$$ \left.\left. + \frac{-37650ab^4+6240a^2b^4-1430a^3b^4-4886b^5+2688ab^5-1638a^2b^5+224b^6-350ab^6-14b^7}{(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} \right\}\right] \tag{8} $$

For $a<1$ and $a>14$

$$ {}_2F_1\left(a, b ; \frac{a+b-14}{2}; \frac{1}{2}\right) = 2^{b-1} \frac{\Gamma\left(\frac{a+b-14}{2}\right)}{(a-b)\Gamma(b)}\left[\frac{\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a-13}{2}\right)}\left\{ \frac{-645120a+1326336a^2-833920a^3}{(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} \right.\right. $$

$$ + \frac{330480a^4-49600a^5+7224a^6-280a^7+15a^8+645120b-489984ab-721792a^2b+887360a^3b-258240a^4b}{(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} $$

$$ + \frac{70256a^5b-4088a^6b+440a^7b-836352b^2+1135488ab^2-351328a^2b^2-141440a^3b^2+117000a^4b^2-12792a^5b^2}{(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} $$

$$ + \frac{2548a^6b^2+420224b^3-758208ab^3+268160a^2b^3-47840a^3b^3-5720a^4b^3+3432a^5b^3-108304b^4+165440ab^4}{(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} $$

$$ + \frac{-115000a^2b^4+11960a^3b^4-1430a^4b^4+15680b^5-30352ab^5+9240a^2b^5-3640a^3b^5-1288b^6+1624ab^6}{(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} $$

$$ + \frac{-1260a^2b^6+56b^7-104ab^7-b^8}{(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)} \right\} + \frac{\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a-14}{2}\right)}\left\{ \frac{-645120a+836352a^2-420224a^3+108304a^4}{(a-14)(a-12)(a-10)(a-8)(a-6)(a-2)} \right. $$

$$ + \frac{-15680a^5+1288a^6-56a^7+a^8+645120b+489984ab-1135488a^2b+758208a^3b-165440a^4b+30352a^5b}{(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} $$

$$ + \frac{-1624a^6b+104a^7b-1326336b^2+721792ab^2+351328a^2b^2-268160a^3b^2+115000a^4b^2-9240a^5b^2}{(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} $$

$$ + \frac{1260a^6b^2+833920b^3-887360ab^3+141440a^2b^3+47840a^3b^3-11960a^4b^3+3640a^5b^3-330480b^4}{(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} $$

$$ + \frac{258240ab^4-117000a^2b^4+5720a^3b^4+1430a^4b^4+49600b^5-70256ab^5+12792a^2b^5-3432a^3b^5}{(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)} $$

$$+\frac{-7224b^6+4088ab^6-2548a^2b^6+280b^7-440ab^7-15b^8}{(a-14)(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)}\}] \qquad (9)$$

C. DERIVATIONS OF SUMMATION FORMULAE (8) TO (9):

Derivation of (7): Substituting $c = \frac{a+b-13}{2}$ and $z = \frac{1}{2}$ in equation (4) , we get

$$(a-b)\ {}_2F_1(a, b; \tfrac{a+b-13}{2}; \tfrac{1}{2}) = (a-b-13)\ {}_2F_1(a, b-1; \tfrac{a+b-13}{2}; \tfrac{1}{2}) + (a-b+13)\ {}_2F_1(a-1, b; \tfrac{a+b-13}{2}; \tfrac{1}{2})$$

Now with the help of the derived result from equation (6), we get

$$\text{L.H.S} = (a-b-13)\ 2^{b-2}\frac{\Gamma(\frac{a+b-13}{2})}{(a-b+1)\Gamma(b-1)}\Big[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-11}{2})}\Big\{\frac{135135-118677a-96005a^2+112567a^3-3875a^4}{(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}$$

$$\frac{6097a^5-455a^6+13a^7-264207b+363186ab-58929a^2b-56004a^3b+27183a^4b-3822a^5b+273a^6b}{(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}$$

$$+\frac{177331b^2-274531ab^2+104702a^2b^2+858a^3b^2-4433a^4b^2+1001a^5b^2-57379b^3+91292ab^3-40274a^2b^3}{(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}$$

$$+\frac{5148a^3b^3+429a^4b^3+10045b^4-16443ab^4+6279a^2b^4-1001a^3b^4-973b^5+1330ab^5-637a^2b^5+49b^6}{(a\ \ 11)(a\ \ 9)(a\ \ 7)(a\ \ 5)(a\ \ 3)(a\ \ 1)}$$

$$+\frac{-77ab^6-b^7}{(a-11)(a-9)(a-7)(a-5)(a-3)(a-1)}\Big\}+\frac{\Gamma(\frac{b-1}{2})}{\Gamma(\frac{a-12}{2})}\Big\{\frac{211479-41633a-144357a^2+85507a^3-20923a^4}{(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$+\frac{2205a^5-119a^6+a^7-480059b+292450ab+69059a^2b-82052a^3b+20139a^4b-2142a^5b+77a^6b+398307b^2}{(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$+\frac{-320047ab^2+44590a^2b^2+22386a^3b^2-6097a^4b^2+637a^5b^2-160407b^3+144924ab^3-32890a^2b^3+572a^3b^3}{(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$+\frac{1001a^4b^3+35061b^4-32799ab^4+8151a^2b^4-429a^3b^4-4641b^5+3458ab^5-1001a^2b^5+273b^6-273ab^6-13b^7}{(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)}\Big\}\Big]$$

$$+\ (a-b+13)\ 2^{b-1}\frac{\Gamma(\frac{a-b-13}{2})}{(a-b-1)\Gamma(b)}\Big[\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-12}{2})}\Big\{\frac{-211479+480059a-398307a^2+160407a^3-35061a^4+4641a^5}{(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$+\frac{-273a^6+13a^7+41633b-292450ab+320047a^2b-144924a^3b+32799a^4b-3458a^5b+273a^6b+144357b^2}{(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$+\frac{-69059ab^2-44590a^2b^2+32890a^3b^2-8151a^4b^2+1001a^5b^2-85507b^3+82052ab^3-22386a^2b^3-572a^3b^3}{(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)}$$

$$+\frac{429a^4b^3+20923b^4-20139ab^4+6097a^2b^4-1001a^3b^4-2205b^5+2142ab^5-637a^2b^5+119b^6-77ab^6-b^7}{(a-12)(a-10)(a-8)(a-6)(a-4)(a-2)}\Big\}$$

$$+\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-13}{2})}\Big\{\frac{-135135+264207a-177331a^2+57379a^3-10045a^4+973a^5-49a^6+a^7+118677b-363186ab}{(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)}$$

$$+\frac{274531a^2b-91292a^3b+16443a^4b-1330a^5b+77a^6b+96005b^2+58929ab^2-104702a^2b^2+40274a^3b^2}{(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)}\Big]$$

$$+\frac{-6279a^4b^2+637a^5b^2-112567b^3+56004ab^3-858a^3b^3-5148a^2b^3+1001a^4b^3+38675b^4-27183ab^4}{(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)}$$

$$+\frac{4433a^2b^4-429a^3b^4-6097b^5+3822ab^5-1001a^2b^5+455b^6-273ab^6-13b^7}{(a-13)(a-11)(a-9)(a-7)(a-5)(a-3)}\}]$$

On simplification, we get the result (8)

Similarly, we can prove the result (9).

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