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VECTOR METRIC SPACES AND SOME FIXED POINT THEOREMS

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ABSTRACT

In this paper it is shown that a vector metric space bears a metric-like Topology. Cantor intersection like Theorem is given, by an application of which a useful fixed point Theorem is proved. The paper closes with study of Ćirić operators in respect of their fixed points.

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1. INTRODUCTION:

Let S denote the collection of all real sequences $\{\alpha = (\alpha_1, \alpha_2, ..., \alpha_n, ...)\}$, then S is a real vector space in which zero vector θ equals to $(0, 0, 0, ...)$. Let us partially order S by $\alpha \leq \beta$ (equivalently, $\beta \geq \alpha$); $\alpha, \beta \in S$ if and only if $\alpha_n \leq \beta_n$ for all n where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n, ...)$ and $\beta = (\beta_1, \beta_2, ..., \beta_n, ...)$.
Define $\max(\alpha, \beta) = (\max(\alpha_1, \beta_1), \max(\alpha_2, \beta_2), ... \max(\alpha_n, \beta_n) ...)$ and similarly one defines $\min(\alpha, \beta)$. Clearly $\max(\alpha, \beta), \min(\alpha, \beta) \in S$.

Let X be a non-empty set. Then $V : X \times X \to S$ is said to be a vector metric if following conditions are met :

(i) $V(x, y) \geq \theta$ for all $x, y \in X$ and $V(x, y) = \theta$ if and only if $x = y$.
(ii) $V(x, y) = V(y, x)$ for all $x, y \in X$
(iii) $V(x, z) \leq V(x, y) + V(y, z)$ for all x, y and $z \in X$.

Thus a metric space is a vector metric space.

Example 1.1: Let X be the collection of all real polynomials $p(t) = a_0 + a_1 t + ... + a_r t^r$ of degree $r \leq n$, and let $V : X \times X \to S$ be taken as

$$V(p, q) = (|a_0 - b_0|, |a_1 - b_1|, ..., |a_r - b_r|, ...),$$

where $q(t) = b_0 + b_1 t + b_2 t^2 + ... + b_r t^r$, then (X, V) is a vector metric space.

The study of vector metric spaces had been initiated long back by T. K. Sreenivasan in 1947 [7]; In our knowledge added in the literature is a paper of Branciari as late as in 2000 [1], and recently in 2003 one finds that Lahiri and Das [4] have proved Banach Contraction Principle like Theorem in a vector-metric space. In all these works no metric- like Topological structure had ever been incorporated into the space, where so-called convergence had been taken care of

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through author's definition. In this paper we have invited a metric like Topology in a vector metric space, and with its aid some useful fixed point Theorems have been established wherefrom all front-line known fixed point Theorem could be derived.

2. VECTOR-METRIC TOPOLOGY:

Let (*X, V*) be a vector metric space. A member $(\alpha_1, \alpha_2, ..., \alpha_n, ...) \in S$ with $\alpha_n > 0$ for all *n* is said to be a positive member of *S*. A positive real member ε is taken as a positive member $(\varepsilon, \varepsilon, \varepsilon, ...)$ of *S*. Let $x_0 \in X$ and *r* be a positive member of *S*. Then the set denoted by $B_r(x_0) = \{x \in X : V(x, x_0) < r\}$ is called an open ball in *X*.

Theorem 2.1: The family *B* of all open balls in (*X, V*) together with empty set forms a base for a Topology τ_V on *X*.

Proof: Take two members $B_{r1}(x_1)$ and $B_{r2}(x_2)$ in *B* and $x_0 \in B_{r1}(x_1) \cap B_{r2}(x_2)$. Suppose $V(x_0, x_1) = (\alpha_1(x_0, x_1), \alpha_2(x_0, x_1), ...)$ and we have $\alpha_n(x_0, x_1) < r_{1n}$ for all *n*, where $r_i = (r_{i1}, r_{i2}, ..., r_{in}, ...)(i = 1, 2)$ are two positive members of *S*.

If $0 < \varepsilon_n < \min\{(r_{1n} - \alpha_n(x_0, x_1)), (r_{2n} - \alpha_n(x_0, x_2))\}$ for *n* = 1, 2 . . . then $\varepsilon = (\varepsilon_1, \varepsilon_2 ... \varepsilon_n ...)$ is a positive member of *S* such that $B_{\in}(x_0) \subset B_{r1}(x_1) \cap B_{r2}(x_2)$. The proof is now complete.

Note: This Topology τ_V is termed as a vector metric Topology on (*X, τ*).

Theorem 2.2: The vector metric space (*X, V*) is a T_2-space.

Proof is a routine exercise and is left out.

Definition 2.1: A subset *B* of a vector metric space (*X, V*) is called bounded if there is a positive member *K* in *S* such that $V(b_1, b_2) \le K$ for all $b_1, b_2 \in B$.

Definition 2.2: Diameter of a bounded set *B*, denoted by Diam (B) is defined as,

$$\text{Diam}B = \left(\sup_{b_1, b_2 \in B} \alpha_1(b_1, b_2), \sup_{b_1, b_2 \in B} \alpha_2(b_1, b_2) ... \sup_{b_1, b_2 \in B} \alpha_n(b_1, b_2) ... \right)$$

where for each $i, \sup_{b_1, b_2 \in B} \alpha_i(b_1, b_2) < +\infty$ as *B* is bounded.

Following Lahiri and Das [4] we have

Definition 2.3 (a): A sequence $\{x_k\}$ in (*X, V*) is said to be cauchy if $\lim_{k \to \infty} V(x_{k+p}, x_k) = \theta, p = 1, 2, ...$

(b) (*X, V*) is said to be complete if every cauchy sequence in (*X, V*) converges to a member of *X* i.e. there is a member $u \in X$ such that $\lim_{k \to \infty} V(x_k, u) = \theta$.

Theorem 2.3: A necessary and sufficient condition that a vector metric space (*X, V*) to be complete is that every nested sequence of nonempty closed subsets $\{G_n\}$ with diameters tending to zero has $\bigcap_{n=1}^{\infty} G_n$ as a singleton.

To prove this theorem we need a lemma that we prove first.

Lemma 2.1: If *G* is a nonempty subset of (*X, V*) then $\text{Diam}G = Diam(\bar{G}), \bar{G}$ denoting the closure of *G* in vector metric topology τ_V on *X*.

Proof: First of all, we note that if ε is an arbitrary positive member of *S* and $l \in S$ with $\theta \le l$ satisfying $l \le \varepsilon$; Then $l = \theta$.

we always have $\text{Diam}(G) \le \text{Diam}(\bar{G})$ (1)

Let ε be an arbitrary positive member of *S*, If $a, b \in \bar{G}$, we find $u, v \in G$ such that

$$V(u,a) < \frac{\varepsilon}{2} \text{ and } V(v,b) < \frac{\varepsilon}{2}.$$

Now
$$\begin{aligned} V(a,b) &\le V(a,u) + V(u,v) + V(v,b) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + V(u,v) \\ &= \varepsilon + V(u,v) \end{aligned}$$

This gives $V(a,b) \le \varepsilon + \text{Diam}(G)$ and hence,

$$\sup_{(a,b)\in \bar{G}} V(a,b) \le \varepsilon + \text{Diam}(G)$$

$$\text{or } \text{Diam}(\bar{G}) \le \varepsilon + \text{Diam}(G)$$

As ε is arbitrary, it follows that

$$\text{Diam}(\bar{G}) \le \text{Diam}(G) \tag{2}$$

From (1) and (2) we have,

$$Diam(G) = Diam(\bar{G}).$$

Proof of Theorem 2.3: Take $a_n \in G_n$; Then for $p \ge 1, a_{n+p} \in G_{n+p} \subset G_n$ and $V(a_n, a_{n+p}) \le \text{Diam}(G_n) \to \theta$ as $n \to \infty$; Then $\{a_n\}$ becomes cauchy in (*X, V*), and by completeness of (*X, V*) let $\lim_{n\to\infty} a_n = u \in X$. Now $a_{n+p} \in G_n$ and by closure property of G_n we have $\lim_{p\to\infty} a_{n+p} = u \in G_n$. Therefore, $u \in \bigcap_{n=1}^{\infty} G_n$. If *v* is a member of $\bigcap_{n=1}^{\infty} G_n$ we have $u, v \in G_n$ and $V(u,v) \le \text{Diam}(G_n)$ that tends to θ as $n \to \infty$. Therefore, $u = v$. Hence $\bigcap_{n=1}^{\infty} G_n$ is a singleton.

Conversely, let $\{x_n\}$ be a Cauchy sequence in (*X, V*); Put $H_n = (x_n, x_{n+1}, x_{n+2} \ldots)$; Then $\{\bar{H}_n\}$ is a decreasing sequence of nonempty closed sets in (*X, V*) such that

$\text{Diam}(\bar{H}_n) = \text{Diam}(H_n)$ (by lemma 2.1) which tends to θ as $n \to \infty$. Then $\bigcap_{n=1}^{\infty} \bar{H}_n$ is a singleton, say $\{u\}$. Now x_n and $u \in \bar{H}_n$ for all *n* and $V(x_n, u) \le \text{Diam}(H_n) \to \theta$ as $n \to \infty$. Therefore $\lim_{n\to\infty} x_n = u \in X$. Proof is now complete.

3. Theorem 3.1: Let (*X, V*) be a complete vector metric space and $T : X \to X$ be an operator such that

$$V(T(x), T(y)) \le \alpha V(x, T(x)) + \beta V(y, T(y)) + \gamma V(x, y)$$

with $0 \le \alpha, \beta, \gamma$ and $\alpha + \beta + \gamma < 1$ and for all $x, y \in X$. Then *T* has a unique fixed point in *X*.

Proof: If x_0 is an arbitrary point in *X* and $x_n = T^n(x_0)\, n = 1, 2, \ldots, (T^0(x_0) = x_0)$, we have

$$V(x_2,x_1)=V(T(x_1),T(x_0))$$
$$\leq \alpha V(x_1,x_2)+\beta V(x_0,x_1)+\gamma V(x_0,x_1)$$

or, $V(x_2,x_1)\leq \dfrac{\beta+\gamma}{1-\alpha}V(x_0,x_1)$

and $V(x_3,x_2)=V(T(x_2),T(x_1))$

$$\leq \alpha V(x_2,x_3)+\beta V(x_1,x_2)+\gamma V(x_1,x_2)$$

or, $V(x_3,x_2)\leq \dfrac{\beta+\gamma}{1-\alpha}V(x_1,x_2)$

$$\leq \left(\frac{\beta+\gamma}{1-\alpha}\right)^2 V(x_0,x_1)$$

By induction,

$$V(x_n,x_{n+1})\leq \left(\frac{\beta+\gamma}{1-\alpha}\right)^n V(x_0,x_1)$$

or, $V(x_n,T(x_n))=\delta^n V(x_0,T(x_0))$ (1)

where $\delta=\dfrac{\beta+\gamma}{1-\alpha}<1$ and therefore $\lim\limits_{n\to\infty}\delta_n=0$.

If h_k is a positive member of S such that $\lim\limits_{k\to\infty}h_k=\theta$, and $h_{k+1}\leq h_k$ for all k.

Put $G_k=\{x\in X : V(x,T(x))\leq h_k\}$ where $h_k=(h_k,h_k...)\in S$

From (1) it follows that for large $k, G_k\neq\phi$. Suppose $G_k\neq\phi$ for all k. It is an easy exercise to see that each G_k is closed and further, each G_k is bounded and if $x, y\in G_k$ we have

$$V(x,y)\leq V(x,T(x))+V(T(x),T(y))+V(T(y),y)$$

$$\leq 2h_k+\alpha V(x,T(x))+\beta V(y,T(y))+\gamma V(x,y)$$

$$\leq \frac{\alpha+\beta+2}{1-\gamma}h_k$$

This gives $\text{Diam}(G_k)\leq \dfrac{\alpha+\beta+2}{1-\gamma}$. h_k and right hand side tends to θ as $k\to\infty$.

By routine exercise we show that $T(G_k)\subset G_k, k=1,2,\ldots$. Thus $\{G_k\}$ is a decreasing chain of non-empty closed sets in (X, V) with $\text{Diam}(G_k)\to\theta$ as $k\to\infty$.

By Theorem 2.3, $\bigcap\limits_{k=1}^{\infty}G_k$ is a singleton, $\text{say}=\{u\}$ for some $u\in X$. So $T(u)=u$; Uniqueness of u is now clear.

Corollary: Theorem 3.1 gives Theorem 1 of Lahiri and Das [4] and well-known Kannan fixed point Theorem [3].
We now invite a Ćirić operator T over a vector metric space (X, V).

$T : X \to X$ is said to be a Ćirić operator if

$$V\left(T^n\left(x\right),T^n\left(y\right)\right) \le q^n\left(x,y\right)\delta\left(x,y\right), n = 1,2...$$

and $x, y \in X$ where $q : X \times X \to R^+$ and $\delta : X \times X \to R^+$ (R^+ = set of non-negative reals) satisfy $q\left(x,y\right) < 1$ with $\sup_{x,y\in X} q\left(x,y\right) = 1$ and $\delta\left(x,y\right)$ is a member $\left\{\delta\left(x,y\right),\delta\left(x,y\right),...\delta\left(x,y\right)....\right\}$ in *S* for $\left(x,y\right) \in X \times X$.

Theorem 3.2: Let *T* be a Ćirić operator over a complete vector metric space satisfying

$$V\left(T\left(x\right),T\left(y\right)\right) \le \alpha\left[V\left(x,T\left(x\right)\right)+V\left(y,T\left(y\right)\right)\right]+\beta V\left(x,y\right)$$
$$+\gamma \max\left\{V\left(x,T\left(y\right)\right),V\left(y,T\left(x\right)\right)\right\}$$

for all $x, y \in X$ where α, β and $\gamma \ge 0$ are such that $\max\left\{\alpha,\beta\right\}+\gamma < 1$, then *T* has a unique fixed point in *X*.

Proof: Take any $x_0 \in X$ and any natural numbers *m, n*; Then by a routine calculation,

$$V\left(T^m\left(x_0\right),T^n\left(x_0\right)\right) \le \frac{2\max\left\{\alpha,\beta\right\}+\gamma}{1-\beta-\gamma}\left[V\left(T^{m-1}\left(x_0\right),T^m\left(x_0\right)\right)+V\left(T^{n-1}\left(x_0\right),T^n\left(x_0\right)\right)\right]$$
$$\le \frac{2\max\left\{\alpha,\beta\right\}+\gamma}{1-\beta-\gamma}\left[q^{m-1}\left(x_0,T\left(x_0\right)\right)+q^{n-1}\left(x_0,T\left(x_0\right)\right)\right]\times\delta\left(x_0,T\left(x_0\right)\right)$$

and right hand side tends to θ as $m,n \to \infty$. That makes $\left\{T^n\left(x_0\right)\right\}$ cauchy in (*X, V*) and if $\lim_{n\to\infty} T^n\left(x_0\right) = u$ for some $u \in X$, we write

$$V\left(T^n\left(x_0\right),T\left(u\right)\right) \le \alpha\left[V\left(T^{n-1}\left(x_0\right),T^n\left(x_0\right)\right)+V\left(u,T\left(u\right)\right)\right]$$
$$+\beta V\left(T^{n-1}\left(x_0\right),u\right)+\gamma\max\left\{V\left(T^{n-1}\left(x_0\right),T\left(u\right)\right),V\left(u,T^n\left(x_0\right)\right)\right\}.$$

As $n \to \infty$, we derive $V\left(u,T\left(u\right)\right) \le \left(\alpha+\gamma\right)V\left(u,T\left(u\right)\right)$ and hence $u = T\left(u\right)$; Further if, $v = T\left(v\right)$ for some $v \in X$, we have,

$$V\left(u,v\right) = V\left(T^n\left(u\right),T^n\left(v\right)\right) \le q^n\left(u,v\right)\delta\left(u,v\right) \to \theta \text{ as } n \to \infty .$$

So $u = v$. The proof is complete.

Corollary: Theorem 3.2 gives Theorem 1 of Saha and Baisnab (See [6]) as a special case.

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