International Journal of Mathematical Archive-2(4), Apr. - 2011, Page: 415-418
Available online through www.ijma.info ISSN 2229 – 5046

SOME THEOREMS DUE TO EPSTEIN AND SCHWARZENBERGER

Hamdi M. Genedi* and K. A. M. Kotb

Mathematics and Statistics Department, Faculty of Science, Taif University, Kingdom of Saudi Arabia
E-Mail: hmgenedy1954@hotmail.com, kamkotp2@yahoo.com

(Received on: 28-02-11; Accepted on: 21-03-11)

ABSTRACT

Epstein and Schwarzenberger said that if P_n is real projective space of dimension n, and f is homeomorphism of P_n into Euclidean m –space, then f is an embedding if it is differentiable and regular. They proved two theorems: (I) if $n = 2k$, $k > 1$ but it is not a power of 2, then P_n can be embedded in $(2n - 1)$ - space. (II) if $n = 4k + 1$, $k > 1$ but it is not a power of 2, then P_n can be embedded in $(2n - 2)$- space. In this paper we prove these two theorems in the complex case.

Keywords: *Embeddings, Complex projective spaces, Normal bundle, Complex vector bundle and Chern classes.*

---

1. INTRODUCTION:

Let $P_n(\mathbb{C})$ be complex projective space of dimension n, and let f be a homeomorphism of $P_n(\mathbb{C})$ into $\mathbb{C}^m$. We say that f is an embedding if it is analytic and regular. James has shown that there is an embedding of P_n in 2n-space [8]. The aim of this paper is to prove the two theorems of Epstein and Schwarzenberger [4] but in the complex case, i.e., in the first theorem we prove that if $n = 2k$, $k > 2$ and k is a power of 2, then $P_n(\mathbb{C})$ can be embedded in $2n$ space and as a consequence of this theorem. In the second theorem we prove that if $n = 4k + 1$, $k > 2$ and k is a power of 2, then $P_n(\mathbb{C})$ can be embedded in n- complex dimension. In each case the result would be false for k is not a power of 2. $P_n(\mathbb{C})$ cannot be embedded in $(2n - 1)$- space for $n = 2^r$, this is for the real case. There is a tentative conjecture, due to Atiyah [1], that $P_n(\mathbb{C})$ can be embedded in $(2n - \alpha(n) + 1)$- space but not in $(2n - \alpha(n))$- space. Here $\alpha(n)$ is the number of nonzero terms in the dyadic expansion of n. Our result agrees with the first part of this conjecture for the complex case $n = 2^r$, $n = 2^r + 1$, $2^r + 2^s$, $2^{r+1} + 2^{s+1} + 1$; $r > s > 0$.

The definition and fundamental concepts which will be required throughout the paper may be found in [2, 3, 5, 6, 7, 10]. Let H be the line bundle over projection $P_{n-1}(\mathbb{C})$ with bundle space $P_n(\mathbb{C}) - z$, $z \in P_n(\mathbb{C})$ and $T, T^`$ be the tangent bundles of $P_{n-1}(\mathbb{C})$, $P_n(\mathbb{C}) - z$. Then $T^`|P_{n-1}(\mathbb{C}) = T \oplus H$. Let I_r denote the trivial r- plane bundle, and write I for I_1. If L, M are two vector bundles, let $Hom(L, M)$ be the bundle whose fiber at z is $Hom(L_z, M_z)$

2. MAIN THEOREMS:

We need the following propositions:

Proposition: 1. The following two statements are equivalent:

(i) $P_n(\mathbb{C})$ can be embedded in complex m - space with normal bundle N so that $N \otimes H$ has a never zero section.

(ii) $P_n(\mathbb{C}) - z$ can be embedded in complex m – space.

Proof: For line bundle L and vector bundle M, $Hom(L, I) \approx L$ and $M \otimes Hom(L, I) \approx Hom(L, M)$. A global section of $Hom(L, M)$ is a bundle homomorphism $L \to M$; a never zero section is an embedding of L as a sub-complex bundle of M complex. Now suppose $P_{n-1}(\mathbb{C})$ can be embedded in m - space so that $N \otimes H$ has a never zero section. Then H is embedded in N as a sub-bundle. Therefore $P_n(\mathbb{C}) - z$ is embedded in a tubular neighbourhood of $P_{n-1}(\mathbb{C})$ in complex m - space. Suppose that $P_{n-1}(\mathbb{C}) \subset P_n(\mathbb{C}) - z$ is embedded in complex m – space. Let N and $N^`$ be the normal bundles of $P_{n-1}(\mathbb{C})$ and $P_n(\mathbb{C}) - z$ respectively. Then

$$N^`\big|P_{n-1}(\mathbb{C}) \oplus T \oplus H = N^`\big|P_{n-1}(\mathbb{C}) \oplus T^`|P_{n-1}(\mathbb{C}) = I_m = N \oplus T.$$

Therefore $N = N^`|P_{n-1}(\mathbb{C}) \oplus H$. The proposition follows.

**Corresponding author: Hamdi M. Genedi*, *E-mail: hmgenedy1954@hotmail.com*

Proposition: 2. Let $P_n(\mathbb{C})$ be embedded in complex m - space with normal bundle N. If $N \otimes H$ has never zero section, then there is a topological embedding of $P_{n-1}(\mathbb{C})$ in $(m+1)$ – space.

Proof: In view of proposition 1, the hypotheses imply that $P_{n+1}(\mathbb{C}) - z$ can be embedded in complex m - space. Let $P^{n-1} \subset P_{n+1}(\mathbb{C}) - z$ be a sphere which is the boundary of small round ball in $P_{n+1}(\mathbb{C})$ containing z. The proposition follows by placing a cone on this sphere by clutching.

Corollary: 3 If the hypotheses of proposition 2 are satisfied, and if $2m > 3n$, then $P_n(\mathbb{C})$ can be analytically embedded in $\mathbb{C}^m$.

The following example shows clearly how we apply proposition 2.

Example: 4. consider an embedding of $P_{n-1}(\mathbb{C})$ in $2n$ - space, with normal bundle N. Then $N \otimes H$ is an n - plane bundle. Therefore it has a never zero section and we obtain the well-known fact that $P_n(\mathbb{C})$ can be embedded in $\mathbb{C}^n$. Now suppose that $P_n(\mathbb{C})$ is embedded in complex m - space with normal bundle N. As a first step towards finding the primary obstruction to the existence of a never zero section of $N \otimes H$, we have to compute the chern classes $c_k(N \otimes H) \in H^{2k}(P_n(\mathbb{C}); Z)$. Let z be the generator of $H^*(P_n(\mathbb{C}); Z)$. We recall that $z = c_1(H)$.

Proposition: 5. Let $P_n(\mathbb{C})$ be embedded in complex m - space with normal bundle N. Then

$c(N \otimes H) = (1+z)^{m+1}$, where c denotes the total chern class.

Proof: Let T be the tangent bundle of $P_n(\mathbb{C})$ and I_r the trivial r – bundle. The well-known isomorphism $H \otimes I_{n+1} = T \oplus I$ (see e.g. [1]), implies that :

$I_{n+1} = (T \otimes H) \oplus H$. We also have $N \oplus T = I_m$. Therefore $(N \otimes H) \oplus I_{n+1} = (N \otimes H) \oplus (T \otimes H) \oplus H$ $= (N \oplus T \oplus I) \otimes H = I_{m+1} \otimes H$ and $c(N \otimes H) = c(H)^{m+1} = (1+z)^{m+1}$.

Notation 6 [6]. Let $P = \sum_{o \le k \le n} P_k z^k$ be a polynomial clutching for a complex vector bundle ξ cover X . Let $L^n(P)$ denote the linear polynomial clutching function for the complex vector bundle $L^n(\xi) = \underbrace{\xi \oplus \cdots \oplus \xi}_{n+1}$ given by the following matrix

$$L^n(P) = \begin{bmatrix} P_0 & P_1 & P_2 & \cdots & P_{n-1} & P_n \\ -z & 1 & 0 & \cdots & 0 & 0 \\ 0 & -z & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -z & 1 \end{bmatrix}.$$

Observe that $L^n(P)$ is the product of three matrices

$$\begin{bmatrix} 1 & P_1^*(z) & \cdots & P_n^*(z) \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} P(z) & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -z & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -z & 1 \end{bmatrix},$$

where

$$P_r^*(z) = \sum_{r \le k \le n} P_k z^{k-r} \quad \text{and} \quad P_r^*(z) - z\,P_{R+1}^*(z) = P_r(z).$$

Consequently, $L^n(P) = (1+N_1)(P \oplus I_n)(1+N_2)$, where N_1 and N_2 are nilpotent. Then

$$L_t^n(P) = (1+tN_1)(P \oplus I_n)(1+tN_2)$$

is a homotopy of clutching function of $L^n(\xi)$. This yields the following results.

Proposition: 7 For a polynomial clutching map $P(z) = \sum_{0 \le k \le n} P_k z^k$, for ξ over X, $[L^n(\xi), L^n(P)]$ and $[L^n(\xi), P \oplus I_n]$ are isomorphic vector bundles over $X \times S^2$.

Corollary: 8 If $m = 2n$, then $c_n(N \otimes H) \neq 0$ if and only if $n = 2^r$ for some r.

Let $P_n(\mathbb{C})$ be embedded in complex m-space with normal bundle N and $m = n + k$. There exists a never zero section of $N \otimes H$ over the $(k-1)$-skeleton of $P_n(\mathbb{C})$. The obstruction to extending this section over the k-skeleton of $P_n(\mathbb{C})$ is an element $c_k \in H^k(P_n\,;\,F_k)$ where F_k is a bundle of coefficients with fibre Z. If $m - n = k$ is odd, there is a co-boundary homomorphism δ such that $c_k = \sum_{0\le i\le k} a_i P^i\,(z)$ where $c_k \in H^k(P_n(\mathbb{C}); Z\,)$ [9].

Theorem: 8 If $n = 2k,\ k > 2$ and k is a power of 2, then $P_n(\mathbb{C})$ can be embedded in $2n$ space and as a consequence of this theorem.

Proof: There is an embedding of $P_n(\mathbb{C})$ in complex n-space [8] with normal n-plane bundle N. There is one obstruction, we put the chern class c_n as a polynomial clutching, i.e. $c_n = \sum_{0\le i\le n} a_i P^i\,(z)$, to existence of a never

zero section of $N \otimes H$. According to proposition 5, clutching polynomial is the mod 2 coefficient $\binom{2n-2}{n-2}$, which is non- zero if and only if $n = 2^r$. By proposition 2, and corollary 8, the theorem follows.

Theorem 8, shows that $P_n(\mathbb{C})$ can be embedded in $2n$ –space if n is odd, but $n \neq 2^r + 1,\ r \geq 0$. In addition James has given such an embedding if n is even [8].

We now feed this information into proposition 1, 2 and use secondary obstructions to embed $P_n(\mathbb{C})$ in complex n-space. Henceforward we assume n is of the form $n = 2^r$.
Let $P_n(\mathbb{C})$ be embedded in $2n$-space with normal bundle N. The obstruction to a never zero section of $N \otimes H$ on the complex n-cells of $P_n(\mathbb{C})$ is an element of $c_n \in H^n(P_n(\mathbb{C}); Z)$.

Proposition: 9 $c_n = 0$.

Proof: For n odd, $H^n(P_n(\mathbb{C}); Z) = 0$ and there is nothing to prove. For n even, $c_n = \sum_{0\le i\le n} a_i P^i\,(z)$ which is zero unless $n = 2^r$ for some $r \geq 0$. The obstruction to extending the never zero section of $N \otimes H$ to $P_n(\mathbb{C})$ is a secondary obstruction, which can be computed as follows, let E be the total space of complex sphere bundle associated to $N \otimes H$. Since the Euler class χ_n of $N \otimes H$ is zero, the Gysin sequence of E breaks up into short exact sequences

$$0 \to H^i(P_n(\mathbb{C}); G) \xrightarrow{p^*} H^i(E\,; G) \xrightarrow{\psi} H^{i-n+2}(P_n(\mathbb{C}); G) \to 0,$$

Where G is Z and $p : E \to P_n(\mathbb{C})$ is the projection. We have the diagram

$$\begin{array}{ccccccccc} 0 \to & H^n(P_n(\mathbb{C}); Z) & \xrightarrow{p^*} & H^n(E\,; Z) & \xrightarrow{\psi} & H^0(P_n(\mathbb{C}); Z) & \to 0 \\ & \wr\vert & & \wr\vert & & \downarrow \boldsymbol{c_2} & \\ 0 \to & H^n(P_n(\mathbb{C}); Z) & \xrightarrow{p^*} & H^n(E\,; Z) & \xrightarrow{\psi} & H^2(P_n(\mathbb{C}); Z) & \to 0 \end{array}$$

The vertical maps include an initial reduction mod 2. The diagram is commutative by [9]. Let $a \in H^n(E\,; Z)$ be such that $\psi a = 1$. We have the splitting

$$H^n(E\,; Z) = p^* H^n(P_n(\mathbb{C})\,; Z) + a \cdot p^* H^2(E\,; Z)$$

And the map is "division by a" [8]. Then

$$c_2\, a + a \sum_{0\le i\le 2} a_i p^i(z) = c_2\, a + a \cdot \psi c_2\, a$$

Maps to $\psi c_2\, a + \psi c_2\, a = 0$ under ψ. Therefore, there is a class z $\in H^n(P_n(\mathbb{C}); Z)$ such that

$$p^* z = c_2\, a + a \sum_{0\le i\le 2} a_i p^i(z).$$

Since a can be varied, z is not uniquely determined. It may vary by any element in the image of

$$c_2 :\ H^{n-2}(P_n(\mathbb{C}); Z) \to H^n(P_n(\mathbb{C}); Z).$$

According to Liao [7], z is the secondary (and last) obstruction to constructing a never zero section $P_n \to N \otimes H$ $(n > 4)$. Therefore, if

$$c_2 : H^{n-2}(P_n(\mathbb{C}); Z) \rightarrow H^n(P_n(\mathbb{C}); Z)$$

Is bijective, it is clear that we can construct the section we want.

Theorem: 10 If $n = 4k + 1$, $k > 2$ and k is a power of 2, then $P_n(\mathbb{C})$ can be embedded in n- complex dimension.
Proof: We embed P_n in $2n$-space by using theorem 1. Then $H^{n-2}(P_n(\mathbb{C}); Z) = Z_2$ and $c_2 : H^{n-2}(P_n(\mathbb{C}) ; Z) \rightarrow H^n(P_n(\mathbb{C}) ; Z)$ is an isomorphism for $n = 4k$. The theorem then follows from proposition 2, and corollary 3.

REFERENCES:

[1] M. F. Atiyah, Thom complexes, Proc. London Math. Soc. 11 (1961), 291-310.

[2] S. S. Chern, Complex manifold, mimeographic note, 1955-1956.

[3] S. Elenbt and N. Steenrod, Algebraic topology, Princeton University Press, 1956.

[4] D. B. A. Epstein and R. L. E. Schwarzenberger, Imbeddings of real projective spaces, Annals of Math., 76 (1962) 180-184.

[5] M. W. Hirsch, Differential topology, Springer-verlag, 1976.

[6] D. Hvsemoller, Fibre bundles, Mc graw-Hill Book Company, New York, 1966.

[7] S. D. Liao, on the theory of obstruction of fibre bundles, Ann. of Math., 60 (1954), 146-191.

[8] I. M. James, SOME embeddings of projective spaces, proc. Cambridge Phil. Soc., 55 (1959), 294-298.

[9] N. E. Steenrod, The topology of Fibre Bundles, Princeton, 1951.

[10] D. M. Davis, Some new immersions and nonimmersion of real projective spaces, AMS Contemporary Math. 19 (1983) 51-64.

****

