

ON THE ORDER AND TYPE OF ENTIRE FUNCTIONS
 REPRESENTED BY DIRICHLET SERIES

K. V. RAMANA¹ & I. H. NAGARAJA RAO^{2*}

¹HEAD OF GENERAL SECTION, GOVT. MODEL RESIDENTIAL POLYTECHNIC, PADERU,
 VISAKHAPATNAM DIST, INDIA

²DIRECTOR & Sr. PROF., G. V. P. COLLEGE FOR DEGREE AND P. G. COURSES, RUSHIKONDA,
 VISAKHAPATNAM-530045, INDIA

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ABSTRACT

In this paper we consider entire functions represented by Dirichlet series. We obtained some relationships involving orders and types of two or more entire Dirichlet series [2].

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1. INTRODUCTION:

Let 'f' be an entire function represented by the Dirichlet series

$$f(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}, \tag{1.1}$$

Where $a_n \in \mathbb{C}$ and λ_n 's satisfy the following conditions:

(i) $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n \rightarrow \infty$ as $n \rightarrow \infty$.

(ii) $\limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} = 0$.

If ' ρ ' is the Ritt order [3] of 'f', then

$$\rho = \limsup_{n \rightarrow \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} \quad (0 \leq \rho \leq \infty). \tag{1.2}$$

Further λ_n satisfy the conditions:

(iii) $\lambda_{n+1} \sim \lambda_n$ as $n \rightarrow \infty$ and

(iv) $\left\{ \log \left| \frac{a_n}{a_{n+1}} \right| / (\lambda_{n+1} - \lambda_n) \right\}$, eventually, is non-decreasing sequence.

If ' λ ' is the lower order of 'f', then

$$\lambda = \liminf_{n \rightarrow \infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} \quad (0 \leq \lambda \leq \infty). \tag{1.3}$$

Let 'f' be an entire function of finite, positive order ' ρ '.

Corresponding author: I. H. NAGARAJA RAO^{2*}, ²DIRECTOR & Sr. PROF., G. V. P. COLLEGE FOR DEGREE AND P. G. COURSES, RUSHIKONDA, VISAKHAPATNAM-530045, INDIA

If 'T' is the Ritt type of 'f' then

$$T = \limsup_{n \rightarrow \infty} \frac{\lambda_n}{\rho e} |a_n|^{\rho/\lambda_n} \tag{1.4}$$

If 't' is lower type of 'f', then P. K. Kamthan[1] showed that

$$t = \liminf_{n \rightarrow \infty} \frac{\lambda_n}{\rho e} |a_n|^{\rho/\lambda_n} \tag{1.5}$$

In this paper we investigate certain relationships between two or more entire functions.

2. MAIN RESULTS

Theorem 1: Let $\{\mu_n\}$ and $\{\nu_n\}$ be real sequences such that conditions (i) and (ii) are satisfied with ' μ_n ' in the place of ' λ_n ' or ' ν_n ' in the place of ' λ_n ', for every n. f_1 and f_2 are entire functions represented by the Dirichlet series

$$f_1(z) = \sum_{n=1}^{\infty} b_n e^{\mu_n z}, \quad f_2(z) = \sum_{n=1}^{\infty} c_n e^{\nu_n z} \quad \forall z \in C,$$

of orders ρ_1, ρ_2 ; lower orders λ_1, λ_2 , types T_1, T_2 and lower types t_1, t_2 respectively, each being positive and finite.

Further condition (iv) is satisfied with ' b_n ' and ' c_n ' in the place of ' a_n '. Then the function 'f' defined by

$$f(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}, \quad \forall z \in C,$$

where $\{\lambda_n\}$ satisfies (i) and (ii),

(v) $\mu_n \sim \lambda_n \sim \nu_n$ as $n \rightarrow \infty$ and

(vi) For some positive reals α_1, α_2 with $\alpha_1 + \alpha_2 = 1$,

$$\log|a_n|^{-1} \sim (\log|b_n|^{-1})^{\alpha_1} \sim (\log|c_n|^{-1})^{\alpha_2} \text{ as } n \rightarrow \infty,$$

is an entire function. Further if ' ρ, λ ' are the order and lower order of 'f' and 'T', 't' are the type and lower type of 'f' respectively, then

(a) $\lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \leq \lambda \leq \rho \leq \rho_1^{\alpha_1} \rho_2^{\alpha_2}$,

(b) $t_1^{\alpha_1} t_2^{\alpha_2} \leq t$ and

(c) $T \leq T_1^{\alpha_1} T_2^{\alpha_2}$, provided $\rho = \rho_1^{\alpha_1} \rho_2^{\alpha_2}$.

Proof: In view of the hypothesis it follows that

$$\limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} = \limsup_{n \rightarrow \infty} \frac{\log n}{\mu_n} = \limsup_{n \rightarrow \infty} \frac{\log n}{\nu_n} = 0.$$

Let $k > 0$. Since f_1 and f_2 are entire functions it follows that

$$\frac{\log|b_n|^{-1}}{\mu_n} > k \text{ and } \frac{\log|c_n|^{-1}}{\nu_n} > k \text{ eventually.}$$

Hence, for sufficiently large 'n',

$$\left(\frac{\log|b_n|^{-1}}{\mu_n} \right)^{\alpha_1} \left(\frac{\log|c_n|^{-1}}{\nu_n} \right)^{\alpha_2} > k^{\alpha_1 + \alpha_2} = k.$$

By (v) and (vi) we get that

$$\frac{\log|a_n|^{-1}}{\lambda_n} > k \text{ eventually. Thus 'f' is an entire function.}$$

$$\text{Let } 0 < \epsilon < \min\left\{\frac{1}{\rho_1}, \frac{1}{\rho_2}\right\};$$

By the definition of $\rho_j (j = 1, 2)$, we have

$$\left(\log|b_n|^{-1}\right)^{\alpha_1} > (\mu_n \log \mu_n)^{\alpha_1} (\rho_1^{-1} - \epsilon)^{\alpha_1}$$

and

$$\left(\log|c_n|^{-1}\right)^{\alpha_2} > (\nu_n \log \nu_n)^{\alpha_2} (\rho_2^{-1} - \epsilon)^{\alpha_2} \text{ eventually.}$$

$$\text{Hence } \left(\log|b_n|^{-1}\right)^{\alpha_1} \left(\log|c_n|^{-1}\right)^{\alpha_2} > (\mu_n \log \mu_n)^{\alpha_1} (\nu_n \log \nu_n)^{\alpha_2} (\rho_1^{-1} - \epsilon)^{\alpha_1} (\rho_2^{-1} - \epsilon)^{\alpha_2}$$

$$\Rightarrow \log|a_n|^{-1} > (\lambda_n \log \lambda_n) (\rho_1^{-1} - \epsilon)^{\alpha_1} (\rho_2^{-1} - \epsilon)^{\alpha_2}$$

$$\Rightarrow \limsup_{n \rightarrow \infty} \frac{\lambda_n \log \lambda_n}{\log|a_n|^{-1}} = \rho \leq \frac{1}{(\rho_1^{-1} - \epsilon)^{\alpha_1} (\rho_2^{-1} - \epsilon)^{\alpha_2}};$$

Since $\epsilon > 0$ is arbitrary, follows that

$$\rho \leq \frac{1}{(\rho_1^{-1})^{\alpha_1} (\rho_2^{-1})^{\alpha_2}} = \rho_1^{\alpha_1} \rho_2^{\alpha_2}.$$

It is trivial that $\lambda \leq \rho$.

Let $\epsilon > 0$; by the definition of $\lambda_j (j = 1, 2)$, we have

$$\left(\log|b_n|^{-1}\right)^{\alpha_1} > (\mu_n \log \mu_n)^{\alpha_1} (\lambda_1^{-1} + \epsilon)^{\alpha_1}$$

and

$$\left(\log|c_n|^{-1}\right)^{\alpha_2} > (\nu_n \log \nu_n)^{\alpha_2} (\lambda_2^{-1} + \epsilon)^{\alpha_2} \text{ eventually.}$$

$$\Rightarrow \log|a_n|^{-1} > (\lambda_n \log \lambda_n) (\lambda_1^{-1} + \epsilon)^{\alpha_1} (\lambda_2^{-1} + \epsilon)^{\alpha_2} \text{ eventually.}$$

$$\Rightarrow \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} = \frac{1}{(\lambda_1^{-1})^{\alpha_1} (\lambda_2^{-1})^{\alpha_2}} \leq \liminf_{n \rightarrow \infty} \frac{\lambda_n \log \lambda_n}{\log|a_n|^{-1}} = \lambda.$$

Thus $\lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \leq \lambda \leq \rho \leq \rho_1^{\alpha_1} \rho_2^{\alpha_2}$. This proves (a).

$$\text{Let } 0 < \epsilon < \min\{t_1, t_2\}.$$

By the definition of $t_j (j = 1, 2)$, we have

$$\frac{\mu_n}{e\rho_1} |b_n|^{\rho_1/\mu_n} > t_1 - \epsilon \text{ and } \frac{\nu_n}{e\rho_2} |c_n|^{\rho_2/\nu_n} > t_2 - \epsilon \text{ eventually; these imply}$$

$$\left(\log|b_n|^{-1}\right)^{\alpha_1} < \left[\frac{\mu_n}{\rho_1} \log\left\{\frac{\mu_n}{e\rho_1(t_1 - \epsilon)}\right\}\right]^{\alpha_1}$$

and

$$\left(\log|c_n|^{-1}\right)^{\alpha_2} < \left[\frac{\nu_n}{\rho_2} \log\left\{\frac{\nu_n}{e\rho_2(t_2 - \epsilon)}\right\}\right]^{\alpha_2} \text{ eventually;}$$

$$\begin{aligned} \Rightarrow \log|a_n|^{-1} &< \frac{\lambda_n}{\rho} \left[\log \left\{ \frac{\lambda_n}{A} \right\}^{\alpha_1} \log \left\{ \frac{\lambda_n}{B} \right\}^{\alpha_2} \right] \\ &= \frac{\lambda_n}{\rho} \left[\left(1 - \frac{\log A}{\log \lambda_n} \right)^{\alpha_1} \left(1 - \frac{\log B}{\log \lambda_n} \right)^{\alpha_2} \right] \log \lambda_n \\ &< \frac{\lambda_n}{\rho} \left[1 - \frac{\log(A^{\alpha_1} B^{\alpha_2})}{\log \lambda_n} + O((\log \lambda_n)^{-2}) \right] \log \lambda_n, \end{aligned}$$

where $A = e\rho_1(t_1 - \epsilon)$, $B = e\rho_2(t_2 - \epsilon)$.

$$\Rightarrow |a_n|^{\rho/\lambda_n} > \lambda_n \left[1 - \frac{\log(A^{\alpha_1} B^{\alpha_2}) + O((\log \lambda_n)^{-2})}{\log \lambda_n} \right].$$

$$\Rightarrow \frac{\lambda_n}{\rho e} |a_n|^{\rho/\lambda_n} \geq \frac{1}{\rho e} A^{\alpha_1} B^{\alpha_2} \left(e^M \right)^{\frac{1}{\log \lambda_n}}, \text{ where } M \text{ is a constant.}$$

$$\begin{aligned} \Rightarrow t = \liminf_{n \rightarrow \infty} \frac{\lambda_n}{\rho e} |a_n|^{\rho/\lambda_n} &\geq \frac{1}{\rho e} \left[(e\rho_1(t_1 - \epsilon))^{\alpha_1} (e\rho_2(t_2 - \epsilon))^{\alpha_2} \right] \quad (1) \\ &= \frac{\rho_1^{\alpha_1} \rho_2^{\alpha_2}}{\rho} \left[(t_1 - \epsilon)^{\alpha_1} (t_2 - \epsilon)^{\alpha_2} \right] \\ &\geq (t_1 - \epsilon)^{\alpha_1} (t_2 - \epsilon)^{\alpha_2} \end{aligned}$$

$\Rightarrow t \geq t_1^{\alpha_1} t_2^{\alpha_2}$ or $t \leq t_1^{\alpha_1} t_2^{\alpha_2}$. This proves (b).

Let $\epsilon > 0$ and $A_1 = e\rho_1(T_1 + \epsilon)$, $B_1 = e\rho_2(T_2 + \epsilon)$; as in (b) we get that

$$\frac{\lambda_n}{\rho e} |a_n|^{\rho/\lambda_n} < \frac{1}{\rho e} A_1^{\alpha_1} B_1^{\alpha_2} \left[\left(e^{M_1} \right)^{\frac{1}{\log \lambda_n}} \right] \text{ eventually, where } M_1 \text{ is a constant;}$$

$$\begin{aligned} \Rightarrow T = \limsup_{n \rightarrow \infty} \frac{\lambda_n}{\rho e} |a_n|^{\rho/\lambda_n} &\leq \frac{1}{\rho e} A_1^{\alpha_1} B_1^{\alpha_2} \quad (1) \\ &= \frac{\rho_1^{\alpha_1} \rho_2^{\alpha_2}}{\rho} (T_1 + \epsilon)^{\alpha_1} (T_2 + \epsilon)^{\alpha_2} \\ &= (T_1 + \epsilon)^{\alpha_1} (T_2 + \epsilon)^{\alpha_2}, \text{ provided } \rho = \rho_1^{\alpha_1} \rho_2^{\alpha_2} \end{aligned}$$

$\Rightarrow T \leq T_1^{\alpha_1} T_2^{\alpha_2}$. This proves (c).

Thus if $\rho = \rho_1^{\alpha_1} \rho_2^{\alpha_2}$, we have $t_1^{\alpha_1} t_2^{\alpha_2} \leq t \leq T \leq T_1^{\alpha_1} T_2^{\alpha_2}$.

Remark: Theorems (1), (2) and (3) of P. K. Kamthan [1] can be deduced from our theorem by taking $\alpha_1 = \alpha_2 = \frac{1}{2}$.

REFERENCES

[1]. KAMTHAN, P.K: On the order and type of entire Dirichlet series, Mathematics Student, XXXIII (1965), 89-94.
 [2]. RAMANA, K.V : A Study of Entire Dirichlet Series in One and Several Complex Variables, Ph.D. Thesis, Andhra University (2002).
 [3]. RITT, J.F: On certain points in the theory of Dirichlet series, Amer. Jour. Math., 50(1928), 73-86.

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