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A NOTE ON NEAR-RING GROUP OF QUOTIENTS

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ABSTRACT

In this paper we present near-ring group structure of quotients of a near-ring group and is seen that near-ring of quotients may appear as a particular case in some cases. Bringing the notion of fractions of a near-ring group into our picture, we introduce the notion to meet our purpose in a broad aspect so that near-ring of quotients may appear as a particular case of what is available already.

***Keywords**: Near-ring; Near-ring group; Near-ring group of quotients.*

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1. INTRODUCTION

In case of a right near-ring N usually an N-group E is an algebraic structure such that (E,+) is a group (not necessarily abelian) together with an external composition $N \times E \to E$ operating E where N operates on E from left. In contrast what has been stated above we begin with the formal definition of an unusual module (N-group) structure because of the fact that to get the structure of right near-ring group of quotients or a near-ring group of quotients are more well behaved. Suppose (E, +) is a group and (N, +, .) is a right near-ring. A *complementary representation* of N on E is a semigroup homomorphism θ: (N,.) → (End (E), o). Suppose that (N, +, .) is a right near-ring. Then an *unusual near-ring N module* is a pair ((E,+) , *) where (E,+) is a group and $*: E \times N \to E$ is a function which satisfies (i) $x*(a.b)=(x*a)*b$ for all $x \in E$ and $a,b \in N$ and (ii) $(x+y)*a = x*a + y*a$ for all $x, y \in E$ and $a \in N$.

Suppose $((E,+),*)$ is an unusual near-ring N-module. Then a function

$$\theta : N \to End(E)$$
$$a \to a\theta$$

given by $\begin{matrix} a\theta : E \to E \\ x \to x(a\theta) = x*a \end{matrix}$ is a semigroup homomorphism.

Thus ‘ $*$ ’ induces the complimentary representation.

Conversely if $\theta : N \to End(E)$ is a complementary representation then define a right scalar multiplication $*: E \times N \to E$ by $x*a = x(a\theta)$ for all $a \in N$ and $x \in E$. Thus θ induces a right scalar multiplication that makes E an unusual near-ring N-module.

Here in this paper, our motivation to explore some unusual algebraic structure (other than usual ones) may lead us to some normally expected more well be have structures that appear as a very natural one in some specific extended generalize structures to give a better explanation of what is stated above may be read as follows .

In case of a ring we have the notion of ring of right quotients as well as that of a left quotients. Similarly we have the structures of module of quotients where that is a right module or a left module. In case of near-ring we have the structure of the right near-ring of right quotients and the right near-ring of left quotients. It is interesting to note that in case of right near-ring of right quotients incidentally some unusual module theoretic or near-ring group theoretic structures come up to the front line . It is very interesting to note that with a matching to Barua’s [[1]] work we find Grainger[[5]] has presented in his Doctoral thesis specifically in his discussion on unusual near-ring module structures.

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Incontrast what is available in ring theory it is here observed that to have the structure of near-ring group of quotients it is difficult to give up the notion of what Grainger [[5]] has termed as unusual near-ring module structures. One may hope fore very interesting structure theory in such type of near-ring group of quotients.

The definition of unusual near-ring group structure of an additive group (not necessarily Abelian) over a right **near-ring with identity 1 is** as follows.

Let (E, +) be a group and N be near- ring with a map

$$\mu : E \times N \to E, \ (x,q) \to xq$$

such that for all x , y $\in$ E and q, r $\in$ N , we have

$$(x + y) q = xq + yq$$
$$x(qr) = (xq)r$$
$$x0 = 0$$

and x1 = x , where zero in the left is the zero of N and the zero in the right is the zero in E.

Then E_N = (E , + , μ) is called the *near-ring group*.

As for example we consider the near-ring *N* (=Z_8) without unity w. r. t. addition modulo 8 and multiplication defined by the following table.

•	***0***	***1***	***2***	***3***	***4***	***5***	***6***	***7***
0	0	0	0	0	0	0	0	0
1	0	1	0	1	1	1	1	1
2	0	2	0	2	2	2	2	2
3	0	3	0	3	3	3	3	3
4	0	4	0	4	4	4	4	4
5	0	5	0	5	5	5	5	5
6	0	6	0	6	6	6	6	6
7	0	7	0	7	7	7	7	7

Here N is a near-ring group over itself.

A subset *A* of a near-ring group E_N is a sub-near-ring group of E_N , if $x - y, xn \in A$ for all $x, y \in A, n \in N$.

The notion immediately leads us to the following,

If E and F are near-rings groups, then, a mapping $f : E \to F$ is *an N-homomorphism* if

i) f is a group homomorphism

ii) $f(en) = f(e)n$, for $e \in E$ and $n \in N$

in a usual way the notion of kernel of f follows.

The notion of *an N-map* follows when the condition (i) is absent.

In what follows it contains the notion of essential as well as rational extensions together with some relevant results.

An N-subgroup A($\neq$0) of E (i.e.(A,+) is a subgroup of (E,+), with AN$\subseteq$A) is an *essential N-subgroup* of E or E is an essential extension of A , if for every N-subgroup X ($\neq$ 0) of E , we have A $\cap$ X $\neq$0 and is denoted by A $\subseteq_e$ E.

An N-subgroup D of E is a dense (or rational) N-subgroup of E or E is a *rational extension* of D , written D $\subseteq_r$ E if given u , v $\in$ E with u $\neq$ 0 there exists t$\in$ N such that vt $\in$D and ut $\neq$ 0.

An N-subset D of L where N is a sub-near-ring of the near-ring L is $D \subseteq_r L$ if and only if given $k, l \in L$ with $l \neq 0$ there exists $x \in N$ such that $kx \in D$ and $lx \neq 0$.

An N-subgroup D of F where F is a near-ring group over L (where N is a sub-near-ring of near-ring L) is $D \subseteq_r F$ if and only if given $p, q \in L$ with $q \neq 0$ there exists $x \in D$ such that $px \in D$ and $qx \neq 0$.

If $D \subseteq F \subseteq E$, where E is a near-ring group, F is a sub-near-ring group and D is a normal sub-near-ring group of F such that $D \subseteq_r E$, then zero homomorphism is the only homomorphism from F/D to E [i.e., **Hom (F/D , E) = (0)].**

Clearly, a rational extension is an essential extension. Moreover if $D \subseteq G \subseteq E$, where E is a near-ring group, G, an N-subgroup of E and D an N-subset of G then, $D \subseteq_r E$ implies $D \subseteq_r G \subseteq_r E$.

In this paper we mainly present the formal structure of near-ring group of quotients of a near-ring group E_N as mentioned above.

A *fraction of* E_N is a N- map f: $A_N \rightarrow E_N$ where A_N is a dense N-subgroup of E_N.

Given two fractions f and g of E_N, f and g are '~' related (denoted 'f ~ g') if and only if they agree on the common part of their domains.

It is to be noted that if f and g are fractions of E_N, then f ~ g if and only if there exists a dense N-sub-near-ring group D of E_N such that f(x) = g(x) for all $x \in D$ and the relation '~' is an equivalence relation on the set of all fractions of E_N.

Let $Q(E)$ be the set of all equivalence classes of $\overline{f}, \overline{g}, \ldots$ in to which the fractions
f, g, ... of E_N are partitioned by the relation ~ .
Defined in $Q(E)$ by the rule $\overline{f} + \overline{g} = \overline{f+g}$.
We see that '+' in $Q(E)$ is justified. For, $f \sim h$ and $g \sim l$ let $x \in \text{Dom } f \cap \text{Dom } g \cap \text{Dom } h \cap \text{Dom } l$.

$$\begin{aligned}\text{Now } (f+g)(x) &= f(x) + g(x)\\ &= h(x) + l(x)\\ &= (h+l)(x)\end{aligned}$$

Hence $f+g \sim h+l$. Thus if $\overline{f} = \overline{h}$ and $\overline{g} = \overline{l}$, we have $\overline{f} + \overline{g} = \overline{h} + \overline{l}$.

Again, let $Q(N)$ be the set of all equivalence classes $\overline{\frac{n}{1}}, \overline{\frac{\alpha}{1}}, \ldots$ into which the fractions $\frac{n}{1}, \frac{\alpha}{1}, \ldots$ of N are partitioned by the relation ~ . Then define in $Q(E)$ the rule as follows

$$\overline{f}\,\overline{\frac{n}{1}} = \overline{f\frac{n}{1}}.$$

We see that it is well-defined in $Q(E)$. Suppose $f \sim g$ and $\frac{n}{1} \sim \frac{\alpha}{1}$

Let $x \in \text{Dom}\frac{n}{1} \cap \text{Dom}\frac{\alpha}{1} \cap \left(\frac{n}{1}\right)^{-1}(\text{Dom } f) \cap \left(\frac{\alpha}{1}\right)^{-1}(\text{Dom } g)$

$$\begin{aligned}\text{Then } \left(f\frac{n}{1}\right)(x) &= f\left(\frac{n}{1}(x)\right)\\ &= f\left(\frac{\alpha}{1}(x)\right)\\ &= g\left(\frac{\alpha}{1}(x)\right) = \left(g\frac{\alpha}{1}\right)(x)\end{aligned}$$

Hence $f\frac{n}{1} \sim g\frac{\alpha}{1}$. Thus if $\overline{f} = \overline{g}$ and $\overline{\frac{n}{1}} = \overline{\frac{\alpha}{1}}$ we have $\overline{f}\overline{\frac{n}{1}} = \overline{g}\overline{\frac{\alpha}{1}}$.

A near-ring group F_L with E_N as its sub-near –ring group is a *near-ring group of quotients of* E_N if $E_N \subseteq_r F_L$, where N is a sub-near-ring of near-ring L. In a domain R, let S = R-{0}. Then S is a multiplicatively closed set. Then we can obtain the quotient ring RS^{-1} = Q(R) which is a rational extension of R.

The near-ring group E_N satisfies the *N-Ore condition w.r.t a subset S of N*, if given $(x, r) \in E \times S$, there exists a common right multiple

$$xr' = rx'$$

such that $(x', r') \in E \times S$.

An ordered family $\{A_1, A_2, ..., A_n\}$ of sub-near-ring groups of E_N is called an independent family if $(A_1 + ... + \hat{A_t} + ... + A_n) \cap A_t = 0$, for some $1 \le t \le n$. The symbol ^ denotes omission of A_t.

2. PRELIMINARIES

Lemma 2.1. Given two fractions f and g of E_N with domains Dom f and Dom g respectively, then the map

$$f + g: \text{Dom } f \cap \text{Dom } g \to E_N$$
$$x \to f(x) + g(x)$$

is also a fraction of E_N.

Proof: Since Dom f and Dom g are dense N-sub-near-ring group of E_N, Dom f $\cap$ Dom g is also dense N-sub-near-ring group of E_N. Now for any x $\in E_N$, n $\in$ N we have,

$$(f + g)(x\,n) = f(x\,n) + g(x\,n) = f(x)n + g(x)n = (f(x) + g(x))n = (f + g)(x)n.$$

Thus f + g is a N-map. Hence f + g is a fraction of E_N.

Lemma 2.2. Given two fractions f and $\frac{n}{1}$ with domains Dom f and Dom $\frac{n}{1}$, and also $\frac{n}{1}$: Dom $\frac{n}{1} \to E$, $x \to nx$. Then the map

$$f\frac{n}{1} : \left(\frac{n}{1}\right)^{-1}(\text{Dom } f) \to E_N$$
$$x \to f\left(\frac{n}{1}(x)\right)$$

is also a fraction of E_N.

Proof: Let u, v $\in E_N$, v $\neq 0$. Since Dom $\frac{n}{1}$ is dense N-sub-near-ring group of E_N, there exists t $\in$ N such that ut $\in$ Dom $\frac{n}{1}$ and vt $\neq$ 0. We note that

$$\frac{n}{1}(ut),\ vt \in E_N.$$

Since Dom f is dense in E_N, it follows that $\exists$ p $\in$ N such that

$$\frac{n}{1}(ut)p \in \text{Dom } f,\ (vt)p \neq 0.$$

Again since $\frac{n}{1}$ is N-map, $\frac{n}{1}(ut)p = \frac{n}{1}(utp)$. Thus given any u, v $\in E_N$, v $\neq 0$, $\exists$ p $\in$ N such that (ut)p $\in \left(\frac{n}{1}\right)^{-1}$ (Dom f) and (vt)p $\neq$ 0.

Thus, $\left(\frac{n}{1}\right)^{-1}$(Dom f) is dense N-sub-near-ring group of E_N. Further, x $\in \left(\frac{n}{1}\right)^{-1}$(Dom f), m $\in$ N

$$\left(f\frac{n}{1}\right)(xm) = f\left(\frac{n}{1}(x)m\right) = f\left(\frac{n}{1}(x)\right)m = \left(f\frac{n}{1}\right)(x)m.$$

Thus $f\frac{n}{1}$ is a fraction of E_N.

Lemma 2.3.If f and g are fractions of E_N , then f ~ g if and only if there exists a dense N-sub-near-ring group D of E_N such that f(x) = g(x) for all x $\in$ D.

Proof: If D = Dom f $\cap$ Dom g , then 'only if ' follows.

To see 'if' part, let x $\in$ Dom f $\cap$ Dom g. Suppose f(x) $\neq$ g(x) . Then we have x, f(x) – g(x) $\in$ E and f(x) – g(x) $\neq$ 0.

Hence, there exists y $\in$ N such that xy $\in$ D and (f(x)-g(x)) y $\neq$ 0

But we have, f(x)y = f(xy) = g(xy), since xy $\in$ D
= g(x) y,

or (f(x) – g(x))y = 0. Which is a contradiction. From this contradiction it follows that f(x) = g(x) for all x $\in$ Dom f $\cap$ Dom g .

Lemma 2.4. The relation '~' is an equivalence relation on the set of all fractions of E_N .

Proof: The relation ~ is reflexive and symmetric trivially. To see that it is transitive, let f, g, h be fractions of E_N such that f ~ g and g ~ h. Then
f(x) = g(x) for all x $\in$ Dom f $\cap$ Dom g and g(x) = h(x) for all x $\in$ Dom g $\cap$ Dom h.

Consequently we get f(x) = h(x) for all x $\in$ Dom f $\cap$ Dom g $\cap$ Dom h.

Where Dom f $\cap$ Dom g $\cap$ Dom h is a dense N –sub-near-ring group of E_N . Thus f ~ h.

Lemma 2.5. Let F_L be a near-ring group with E_N as a sub-near-ring group (where N is a sub-near-ring of near-ring L). Then F_L is a near-ring group of quotients of E_N if and only if for every $q \in L, q \neq 0$ we have $q^{-1}E_N \subseteq_r E_N, q(q^{-1}E_N) \supset (0)$ where $q^{-1}E_N = \{x \in E_N \mid qx \in E_N\}$.

Proof: Suppose $E_N \subseteq_r F_L$. Given $z \in q^{-1}E_N$ and $n \in N$. Then we get,
$q(zn) \in E_N$
$\Rightarrow zn \in q^{-1}E_N$

Hence $q^{-1}E_N$ is subset of F_L .
$u, v \in F_L, v \neq 0$. Then $qu \in F_L$. Since $E_N \subseteq_r F_L$, there exists t $\in$ N such that
$(qu)t \in E_N$ and vt $\neq$ 0

The first condition implies that $ut \in q^{-1}E_N$. Thus, given $u, v \in F_L, v \neq 0$, there exists $t \in N$ such that $ut \in q^{-1}E_N$ and $vt \neq 0$ leading there by $q^{-1}E_N \subseteq_r F_L$.

Since we have $q^{-1}E_N \subseteq E_N \subseteq F_L$ and $q^{-1}E_N \subseteq_r F_L$, it follows that $q^{-1}E_N \subseteq_r E_N$.

Now $a \in E_N \cap qE_N$ gives $a \in E_N, qE_N$ and hence there exists $b \in E_N$ such that *a=qb*. Since $qb = a \in E_N$,

we have $b = q^{-1}E_N$. Hence, $a(= qb) \in q(q^{-1}E_N)$. Thus $q(q^{-1}E_N) \supseteq E_N \cap qE_N$.

Recalling $E_N \subseteq_r F_L$, and $E_N \subseteq_e F_L$, we see that $E_N \cap qE_N \supset (0)$, which gives $q(q^{-1}E_N) \supset (0)$.

Suppose $q^{-1}E_N \subseteq_r E_N$ and $q(q^{-1}E_N) \supset (0)$ hold and $p, q \in L$ with $q \neq 0$.

Now we show that there exists $x \in E_N$ for which $px \in E_N$ and $qx \neq 0$. Keeping in note $q(q^{-1}E_N) \subseteq qE_N$ and

$$
\begin{aligned}
q(q^{-1}E_N) &= \{qx \mid x \in q^{-1}E_N\} \\
&= \{qx \mid qx \in E_N\} \\
&\subseteq E_N
\end{aligned}
$$

We get $q(q^{-1}E_N) \subseteq E_N \cap qE_N$ which in turn gives $E_N \cap qE_N \supset (0)$.

Thus, there exists b $\in E_N$ such that $a = qb$. We note that $qb \neq 0$.

(1) Suppose p = 0 and x = b. Then we get px (= 0b = 0) $\in E_N$ and $qx (= qb) \neq 0$

(2) Suppose $p \neq 0$. Then $q^{-1}E_N \subseteq_r E_N$ gives $p^{-1} E_N \subseteq_r E_N$

And we have b, qb $\in E_N$ with $qb \neq 0$. Hence there exists y $\in$ N such that $by \in p^{-1}E_N$ and $(qb)y \neq 0$.

Again x = by gives $x \in E_N$ such that $px \in E_N$.

Thus in both the cases p = 0 and $p \neq 0$, there exists x $\in E_N$ with $px \in E_N$ and $qx \neq 0$. Hence $E_N \subseteq_r F_L$.

As in Lemma 2.1.1 [2] we have

Lemma 2.6. Let $C(Q(N))$ be the complete near-ring of quotients of N and $C(Q(N))$ exists.If $s_1, s_2, ... s_n \in S$, then there exists $x_1, x_2, ..., x_n \in N$ and $s \in S$ such that $s_i^{-1} = x_i s^{-1}, i = 1,2,...,n$

3. MAIN RESULTS

Now we present the important notion of what we are intending.

We see in this section that the fractions of E_N yield a near-ring group of quotients of E_N.

Theorem 3.1. The set $Q(E)$ of all equivalence classes of $\overline{f}, \overline{g}, \ldots$ in to which the fractions *f*, g, ... of E_N are partitioned by the relation ~ defined by the rule $\overline{f} + \overline{g} = \overline{f + g}$ and $\overline{f}\ \overline{\alpha} = \overline{f\alpha}$ is a near-ring group. Where $\overline{\alpha} \in Q(N)$, the near-ring of right quotients.

Proof: Let us define μ: $Q(E) \times Q(N) \rightarrow Q(E), (\overline{f}, \overline{\alpha}) = \overline{f}\ \overline{\alpha} = \overline{f\alpha}$, for all $\overline{\alpha} \in Q(N)$, $\overline{f} \in Q(E)$.

Now for all $\overline{\alpha_1}, \overline{\alpha_2} \in$ N, $\overline{f}, \overline{g} \in Q(E)$ we have

$$
\begin{aligned}
(\overline{f} + \overline{g})(\overline{\alpha_1}) &= \overline{(f+g)\alpha_1} \\
&= \overline{(f\alpha_1 + g\alpha_1)} \\
&= \overline{f\alpha_1} + \overline{g\alpha_1} \\
&= \overline{f}\ \overline{\alpha_1} + \overline{g}\ \overline{\alpha_1}
\end{aligned}
$$

$$
\begin{aligned}
\overline{f}(\overline{\alpha_1}\ \overline{\alpha_2}) &= \overline{f}\,\overline{(\alpha_1\alpha_2)} \\
&= \overline{f(\alpha_1\alpha_2)} \\
&= (\overline{(f\alpha_1)})\overline{\alpha_2} \\
&= (\overline{f}\ \overline{\alpha_1})\overline{\alpha_2}
\end{aligned}
$$

$Q(E)$ has an additive identity $\overline{0}$ given by the fraction 0: $E_N \rightarrow E_N$, x$\rightarrow$ 0.

For every $\overline{f} \in Q(E)$ we get a fraction

$$-f : \text{Dom } f \to E_N \text{ , x} \to -f\text{(x)}$$

Given x $\in$ Dom f, we have

$(f + (-f))(x) = f(x) - f(x) = 0$

$0(x) = 0,$

$((-f) + (f))(x) = -f(x) + f(x) = 0$

Hence, $\overline{f + (-f)} = \overline{0} = \overline{(-f) + f}$ Or, $\overline{f} + \overline{(-f)} = \overline{0} = \overline{(-f)} + \overline{f}$

Thus, every $\overline{f} \in Q(E)$ has an inverse $\overline{(-f)} \in Q(E)$.

And so we have

$$\overline{f}\, 0 = 0.$$

Hence $Q(E)$= ($Q(E)$, +, μ) is a near-ring group over $Q(N)$.

In particular $Q(N)$ =($Q(N)$,+,.) is a near ring and $Q(E)$ is a near-ring $Q(N)$ group.

Theorem 3.2. E_N is embedded in the near-ring group $Q(E)_{Q(N)}$.(and so N is embedded in the near-ring $Q(N)$

Proof: For every n $\in$ N, we get a left multiplication in the near-ring group of transformations of E_N ,

$$\frac{n}{1} : E_N \to E_N \text{ , x} \to \text{n x .}$$

Given x $\in E_N$, p $\in$ N we see that

$$\left(\frac{n}{1}\right) \text{(xp) = n (xp)}$$

$$= \text{(nx) p}$$

$$= \left(\frac{n}{1}\right)\text{(x)p}$$

Thus the left multiplication $\left(\frac{n}{1}\right)$ is a N-map and hence a fraction of E_N .

Consider the map α: $E_N \to Q(E)_{Q(N)}$, x→ $\frac{\overline{x}}{1}$, for x $\in E_N$.

Then for x, y $\in E_N$ we have,

$$\alpha\,(x + y) = \frac{\overline{x+y}}{1}$$

$$= \frac{\overline{x}}{1} + \frac{\overline{y}}{1}$$

$$= \alpha\,(x) + \alpha\,(y)$$

Again, for n $\in$ N, x $\in E_N$ we have,

α (xn) = $\overline{\frac{xn}{1}}$ = $\overline{\frac{x}{1}\frac{n}{1}}$

= $\overline{\frac{x}{1}n}$

= $\overline{\frac{x}{1}}\,n$

= α (x) n

Thus α is a N-homomorphism.

Now, Kernel α = { x $\in$ E_N | $\overline{\frac{x}{1}}$ = 0 }

= {x $\in$ E_N | x = 0 }

= (0)

Hence α is a N-monomorphism.

Note 3.3. Since the map α: $E_N \to Q(E)_{Q(N)}$ is a monomorphism , we shall identify $\alpha(E_N)$ with E_N and $\overline{\frac{n}{1}}$ with n, for simplicity of notation.

Theorem 3.4. If $D \subseteq_r E_N$ and $q \in Q(N),\ q \neq 0$, then $qD = (0)$ implies $q = 0$.

Proof: Let $q = \bar{t}$, where t is a fraction of N, and $n \in N$. Then $qD = (0)$ implies that

$\overline{t(n/1)}$ = (0)

Or, $t((n/1)(x)) = 0$ for every $x \in (n/1)^{-1}$ (Domt)

$\Rightarrow t$ (Domt) = 0

Thus $q = \bar{t} = 0$.

Theorem 3.5. If $q \in Q(N)$, t is a fraction of N, and f is a fraction of E_N such that $q = \bar{t}$, then $Domf \subseteq q^{-1}E_N$.

Proof: Let $r \in Domf$, then

$qr = \bar{t}\overline{\frac{r}{1}}$ = $\overline{t\left(r/1\right)}$ = $\overline{t(r)/1} \in E_N$

$\Rightarrow r \in q^{-1}E_N$

Hence $Domf \subseteq q^{-1}E_N$.

As a corollary we get

Corollary 3.6. If $q \in Q(N)$, then $q^{-1}E_N \subseteq_r E_N$,where $q^{-1}E_N = \{ x \in E_N \mid qx \in E_N \}$. Also

Theorem 3.7. If $q \in Q(N),\ q \neq 0$, then

$$q(q^{-1}E_N) \supset (0).$$

The proof is immediately follows from Theorem 3.4.

Using the last two results and the Lemma 2.5. , we have the

Theorem3.8. $Q(E)_{Q(N)}$ is a near-ring group of right quotients of E_N .

Theorem 3.9. Let E_N satisfy the N-Ore condition with respect to a multiplicatively closed subset S of N and have N-homomorphisms

$$\alpha : N \to Q(N)$$
$$\beta : E_N \to Q(E)_{Q(N)}$$

such that r∈S implies $\alpha(r)^{-1} \in Q(N)$. Then the subset
$ES^{-1} = \{\beta(a)\alpha(r)^{-1} \mid (a,r) \in E \times S\}$ is a sub-near-ring group of $Q(E)_{Q(N)}$.

Proof: Let $\beta(a)\alpha(r)^{-1}, \beta(b)\alpha(s)^{-1} \in ES^{-1}$. Then we have $(a,r),(b,s) \in E \times S$. Since E_N satisfies the N-Ore condition w.r.t. S and $(a,r),(b,s) \in E \times S$, we get that there exists $(r',s'),(b',r'') \in E \times S$ such that $rs' = sr'$ and $br'' = rb'$ and hence (i) $\beta(rs') = \beta(sr')$ and (ii) $\beta(br'') = \beta(rb')$

Again, as N satisfies the Ore condition w.r.t S and $(r,s),(p,r) \in N \times S$, we therefore get that there exists $(r',s'),(p',r') \in N \times S$ such that $rs' = sr'$ and $pr'' = rp'$
And hence (iii) $\alpha(rs') = \alpha(sr')$ and (iv) $\alpha(pr'') = \alpha(rp')$

Now $\beta(a)\alpha(r)^{-1} - \beta(b)\alpha(s)^{-1} = \beta(a)(\alpha(s')\alpha(sr')^{-1} - \beta(b)(\alpha(r')\alpha(rs')^{-1}))$ [using (iii)]
$= \beta(a)\alpha(s')\alpha(rs')^{-1} - \beta(b)\alpha(r')\alpha(rs')^{-1}$
$= (\beta(a)\alpha(s') - \beta(b)\alpha(r'))\alpha(rs')^{-1} \in ES^{-1}$

Again, let $\alpha(n)\alpha(r)^{-1} \in NS^{-1}$.

Then,
$\beta(b)\alpha(s)^{-1}\alpha(n)\alpha(r)^{-1} = \beta(b)\alpha(rs')\alpha(r')^{-1}\alpha(n)\alpha(r)^{-1}$ [using (iii)]
$= \beta(b)\alpha(rs')\alpha(n')\alpha\left(r_1'\right)^{-1}$, where $\alpha(r')^{-1}\alpha(n) = \alpha(n')\alpha\left(r_1'\right)^{-1}$
$= \beta(b)\alpha(rs'n')\alpha(rr_1')^{-1}$
$\in ES^{-1}$

Hence, ES^{-1} is a sub-near ring group of $Q(E)_{Q(N)}$.

Remark 3.10. We note that if in Theorem 3.9., 1∈S , then the near-ring group ES^{-1} has $\beta(1)$ as its identity.

Thus we get

Theorem 3.11. Let E_N be near-ring group and S be a multiplicatively closed subset of N containing 1, $\alpha : N \to Q(N), \beta : E_N \to Q(E)_{Q(N)}$ are homomorphisms satisfying the condition $\alpha(r)^{-1} \in Q(N)$, for r ∈S and the condition $\beta(a) = 0$, for $a \in E_N$ implies that there exists t ∈S with $at = 0$.

Also if the subset ES^{-1} is a sub-near-ring group of $Q(E)_{Q(N)}$, then E_N satisfies the *N-Ore* condition with respect to S.

Proof: Let $(a,r)\in E\times S$. Then $a\in E_N$ and r $\in$S . Since r $\in$S implies $\alpha(r)^{-1}\in Q(N)$, $\alpha(r)^{-1}$ and $\alpha(1)^{-1}$ exist and as ES^{-1} is a sub-near-ring group, we get,

$$\beta(1)\alpha(r)^{-1}\beta(a)\alpha(1)^{-1}\in ES^{-1} \Rightarrow \alpha(r)^{-1}\beta(a)\in ES^{-1}$$

It follows from the definition of ES^{-1} there exists $(b,s)\in E\times S$ such that

$$\alpha(r)^{-1}\beta(a)=\beta(b)\alpha(s)^{-1}\Rightarrow \beta(a\alpha(s))=\beta(\alpha(r)b)\Rightarrow \beta(a)\alpha(s)=\alpha(r)\beta(b)$$
$$\Rightarrow \beta((a\alpha(s))-\alpha(r)b)=0$$

Thus there exists t $\in$S such that $(a(\alpha(s))-\alpha(r)b)t=0$

$$\Rightarrow a\alpha(st)-\alpha(r)bt=0$$

Putting bt = a' and st = r', we see that given $(a,r)\in E\times S$, there exists $(a',r')\in E\times S$ such that $\alpha(r)a'=a\alpha(r')$, i.e., E_N satisfies N-Ore condition with respect to S.

Theorem 3.12. Let S be the set of non-zero divisors of N. If E_N satisfies the N-Ore condition with respect to S and s $\in$S, then $sE_N\subseteq_r E_N$.

Proof: Clearly s E_N is an N-subgroup of E_N . Let $a,b\in E_N, b\neq 0$. Then $(a,s)\in E\times S$ and hence there exists a common right multiple $as'=sa'$ such that $(a',s')\in E\times S$. Since $sa'\in sE_N$ and since s' is a non-zero divisor and $b\neq 0$, we have $bs'\neq 0$. Thus $sE_N\subseteq_r E_N$.

Because of the note 3.3., we regard E_N as a sub near-ring group of $Q(E)_{Q(N)}$

Following result gives how the N-ore condition is connected with classical near-ring group of quotients

Theorem 3.13. If E_N satisfies the N-Ore condition w.r.t. S, then the subset $C(Q(E))=\{xr^{-1}\in Q(E)\,|\,(x,r)\in E\times S\}$ is a sub- near-ring group of $Q(E)_{Q(N)}$. $C(Q(E))$ is the *classical near-ring group of quotients of* E_N .

Proof: Let α and β be the embedding

$\alpha: N\to Q(N)$ and $\beta: E_N\to Q(E)_{Q(N)}$

r $\to$ r, $x\to\frac{\bar{x}}{1}$

Also for any $r\in S,\alpha(r)^{-1}\in Q(N)$

Thus $C(Q(E))=\{xr^{-1}\in Q(E)\,|\,(x,r)\in E\times S\}$

$$=\{\beta(x)\alpha(r)^{-1}\in Q(E)\,|\,(x,r)\in E\times S\}$$
$$=ES^{-1}$$

As ES^{-1} sub near-ring group of $Q(E)_{Q(N)}$, $C(Q(E))_{c(Q(N))}$ is a sub-near-ring group of $Q(E)_{Q(N)}$ (where $C(Q(N))$ is the classical near-ring of quotients of near-ring N).

Theorem 3.14. If J be a right N-sub-near-ring group of E_N, then the subset $JS^{-1}=\{xs^{-1}\in Q(E)\,|\,(x,s)\in J\times S\}$ is a right $C(Q(N))$ sub-near-ring group of $C(Q(E))_{C(Q(N))}$

Proof: Let p $\in C(Q(E))_{C(Q(N))}$, q $\in C(Q(N))$.

Then p = as^{-1}, q = xt^{-1} where $a \in J$, x$\in$N , s, t$\in$S

And $pq = (as^{-1})(xt^{-1})$

$= a(s^{-1}xt^{-1})$

Since $xt^{-1} \in C(Q(N))$, $s^{-1} = 1\, s^{-1} \in NS^{-1} = C(Q(N))$, we get

$s^{-1}(xt^{-1}) \in C(Q(N))$

Let $s^{-1}(xt^{-1}) = bu^{-1}$, $b \in$ N , $u \in$ S . Then, $ab \in$ J gives $pq \in JS^{-1}$.

Hence the result.

Theorem 3.15. If { $J_1, J_2, \ldots, J_t$ } be an independent family of right N-sub-near-ring groups of E_N , then { $J_1S^{-1}, J_2S^{-1}, \ldots, J_tS^{-1}$ } is an independent family of right $C(Q(N))$ –sub-near-ring group of $C(Q(E))$

Proof: If possible, let {$J_1S^{-1}, J_2S^{-1}, \ldots, J_tS^{-1}$ } be not an independent family. Then there is an m , $1 \le m \le t$ such that , $J_mS^{-1} \cap \sum_{n \neq m} J_nS^{-1} \neq 0$.

Then there is an non-zero element,

$j_ms_m^{-1} = j_1s_1^{-1} + \ldots + \hat{j_m s_m} + \ldots + j_ts_t^{-1}$ (^stands for deletion of the term underneath) in the intersection.

By Lemma 2.6., for $s_1, \ldots, s_t \in S$, we get $x_1, \ldots, x_t \in N$ and s$\in$S such that

$$s_i^{-1} = x_is^{-1}, \quad 1 \le i \le t$$

Now, $j_mx_ms^{-1} = j_1x_1s^{-1} + \ldots + \hat{j_m x_m s^{-1}} + \ldots + j_tx_ts^{-1}$

$= (j_1x_1 + \ldots + \hat{j_m x_m} + \ldots + j_tx_t)s^{-1}$

And this gives,

$j_mx_m = j_1x_1 + \ldots + \hat{j_m x_m} + \ldots + j_tx_t$

$\neq 0.$

So, $J_m \cap (\sum_{n \neq m} J_n) \neq (0)$ and is a contradiction, for { $J_1, \ldots, J_t$ } is an independent family of N-sub-near-ring group of E_N .

Therefore {$J_1S^{-1} \ldots J_tS^{-1}$} is an independent family of right $C(Q(N))$–sub-near-ring group of $C(Q(E))$.

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