International Journal of Mathematical Archive-4(2), 2013, 231-236
Available online through www.ijma.info ISSN 2229 – 5046

ON G/A – SZEGED INDEX OF STANDARD GRAPHS

K. V. S. Sarma[1*] & I. V. N. Uma[2]

[1]Associate Professor, Regency Institute of Technology, Yanam, India
[2]Senior Faculty, Devi Academy, Chennai, India

(Received on: 23-01-13; Revised & Accepted on: 20-02-13)

ABSTRACT

A modern graph invariant is Szeged Index and it has considerable applications in molecular chemistry. A recently introduced graph invariant is G/A – Szeged Index and it has a numerable applications in chemistry. In this paper, the G/A – Szeged indices of standard graphs are calculated. A modified G/A – Szeged Index of a graph is also introduced in which all the vertices of the graph are taken into consideration, thereby the variations in these indices of standard graphs are identified.

Keywords: *Szeged Index, G/A – Szeged Index, modified G/A – Szeged Index.*

1. INTRODUCTION

The useful concepts related to a molecular graph associated with alkanes are mainly Wiener index (See[5]) and Szeged index (see[3]). A recently introduced concept (see [2]) is GA_2 Index and we coined it as G/A– Szeged Index (Geometric mean by Arthimetic mean – Szeged Index). As usual, no standard formula is available to find out this index as well, for any connected graph. In §2, we calculate the G/A – Szeged indices of standard graphs and in §3, we introduce modified G/A – Szeged index and observe the variation of these indices for the standard graphs.

Throughout this paper, we consider only non-empty, simple, finite and connected graph to avoid trivialities.

For the standard notation and results we refer Bondy & Murthy [1].

For ready reference, we give the following:

Definition 1.1 [3]: G is a (non-empty, simple, finite and connected) graph with vertex set V(G) and edge set E(G). Then the Szeged index of G, denoted by Sz(G), is defined to be $\sum_{e\in E(G)} n_1(e).n_2(e)$, where e = uv, $N_1(e|G)=\{w \in V(G):d(w, u)<d(w, v)\}$, $N_2(e|G) = \{w\in V(G): d(w, v) < d(w, u)\}$ and $n_1(e|G) = | N_1(e|G |$, $n_2(e|G) = | N_2(e|G) |$.
('d' denotes the distance function and '| |' , the cardinality function).

When there is only one graph G, under consideration, we write 'e' only instead of 'e|G'.

Observations 1.2 [4]:

a) For the graph K_n $(n \geq 2)$, $Sz(K_n) = n(n-1)/2$.
b) For the complete graph $K_{m,n}$ $(m, n \geq 1)$, $Sz(K_{m,n}) = (mn)^2$.
c) For the cycle C_k $(k \geq 3)$, $Sz(C_k) = k\left[\frac{k}{2}\right]^2$ ([ ] denotes integral part).
d) For the path P_n $(n \geq 2)$, $Sz(P_n) = n(n^2 – 1)/6$.
e) For the wheel $(K_1 \vee C_n)$ $(n \geq 3)$, $Sz(K_1 \vee C_n) = n\left\{(n-2)+\left[\frac{n}{2}\right]^2\right\}$.

2. G/A – SZEGED INDEX OF STANDARD GRAPHS

For convenience, we recollect the following:

Corresponding author: *K. V. S. Sarma[1*]*
[1]Associate Professor, Regency Institute of Technology, Yanam, India

Definition 2.1 [2]: G is a graph (i.e non- empty, simple, finite and connected graph). Let 'e' be any edge of G. Then the G/A – Szeged index of G is defined to be $\sum_{e \in E(G)} \frac{\sqrt{n_1(e)n_2(e)}}{[n_1(e)+n_2(e)]/2}$.

Theorem 2.2: For the complete graph K_n ($n \geq 2$), G/A - Sz(K_n) = n(n-1)/2.

Proof: For any e ϵ E(K_n), by Th.2.3 in [4], $n_1(e) = n_2(e) = 1$.

$$\text{So G/A – Sz}(K_n) = \sum_{e \in E(K_n)} 2\frac{\sqrt{(1)(1)}}{(1+1)}$$

$$= |E(K_n)|$$

$$= \frac{n(n-1)}{2}.$$

Theorem 2.3: For the complete bipartite graph $K_{m,\,n}$ (m, n ≥ 1),

$$\text{G/A – Sz}(K_{m,\,n}) = \frac{2(mn)^{3/2}}{m+n}.$$

Proof: For e ϵ E($K_{m,n}$), by Th.(2.5) in [4], $n_1(e) = n$ and $n_2(e) = m$, further $|E(K_{m,n})| = mn$.

$$\text{So G/A – Sz}(K_{m,n}) = \sum_{e \in E(K_{m,n})} 2\frac{\sqrt{(n)(m)}}{(n+m)}$$

$$= 2\frac{\sqrt{(m)(n)}}{(m+n)}|E(K_{m,n})|$$

$$= \frac{2(mn)^{3/2}}{m+n}.$$

Theorem 2.4: For the cycle C_k ($k \geq 3$), G/A – Sz(C_k) = k.

Proof: Clearly $|E(C_k)| = k$.

We divide this into two cases.

Case (i): k is even and say k = 2n (n ≥ 2).

For any e ϵ E(C_{2n}), by Th. (2.6) in [4], $n_1(e) = n_2(e) = n$.

$$\text{So G/A – Sz}(C_{2n}) = \sum_{e \in E(C_{2n})} \frac{2\sqrt{(n)(n)}}{n+n}$$

$$= |E(C_{2n})| = 2n.$$

Case (ii): k is odd and any say k = 2n-1 (n ≥ 2). For any e ϵ E(C_{2n-1}), by Th. 2.6 in [4], $n_1(e) = n_2(e) = n-1$.

$$\text{So G/A – Sz}(C_{2n-1}) = \sum_{e \in E(C_{2n})} \frac{2\sqrt{(n-1)(n-1)}}{(n-1)+(n-1)}$$

$$= |E(C_{2n-1})|$$

= 2n – 1. Hence follows the result.

Theorem 2.5: For the path P_n ($n \geq 2$), G/A – Sz(P_n) = $\frac{2}{n}\sum_{i=1}^{n-1}\sqrt{i(n-i)}$.

Proof: Let V(P_n) = {$v_1, v_2, \ldots, v_n$}. Any edge of P_n is of the form $e_i = v_iv_{i+1}$ for i = 1… (n-1). For any e_i ϵ E(P_n), by Th.2.2 in [4], $n_1(e_i) = i$ and $n_2(e_i) = (n-i)$.

So, G/A – Sz(P_n) = $\sum_{i=1}^{n-1} 2\frac{\sqrt{i(n-i)}}{(i+n-i)} = \frac{2}{n}\sum_{i=1}^{n-1}\sqrt{i(n-i)}$.

Corollary 2.6:

a) $$\mathrm{G/A}-Sz(\mathrm{P}_{2\mathrm{m}}) = \begin{cases} 1 \text{ if } \mathrm{m}=1 \\ \frac{4}{2m}\left(\sum_{i=1}^{m-1}\sqrt{i(2m-i)}+\frac{m}{2}\right) \text{ for } \mathrm{m}\geq 2. \end{cases}$$

Observe that, with the convention $\sum_{i=r}^{s}\ldots$ = 0 if s < r, we get the index of P_2 from the later formula by taking m = 1).

b) $$\mathrm{G/A}-Sz(\mathrm{P}_{2\mathrm{m}+1}) = \frac{4}{2m+1}\left\{\left(\sum_{i=1}^{m-1}\sqrt{i(2m-i)}\right) \text{ for } \mathrm{m}\geq 1.\right.$$

Proof: By taking n = 2 in Theorem (2.5), we get that G/A – Sz(P_2) = $\frac{2}{2}\sqrt{(1)(1)}$ = 1.

Let n ≥ 4 and be even. We can write n = 2m (m ≥ 2).

So, G/A – Sz(P_{2m}) = $\frac{2}{2m}\left(\sum_{i=1}^{2m-1}\sqrt{i(2m-i)}\right)$

$$= \frac{2}{2m}\left(\sum_{i=1}^{m-1}\sqrt{i(2m-i)}+\sqrt{m(2m-m)}+\sum_{i=m+1}^{2m-1} i(2m-i)\right) \quad (2.6.1)$$

Replacing i by (2m – i') in the third sum of (2.6.1), we observe that

$$\sum_{i=m+1}^{2m-1} i(2m-i) = \sum_{i^1=1}^{m-1}(2m-i^1)(i^1) = \sum_{i=1}^{m-1}\sqrt{i(2m-i)} .$$

Hence, from (2.6.1) we have

$$\text{G/A – Sz}(P_{2m}) = \frac{2}{2m}\left[2\sum_{i=1}^{m-1}\sqrt{i(2m-i)}+m\right]$$
$$= \frac{4}{2m}\left[\sum_{i=1}^{m-1}\sqrt{i(2m-i)}+\frac{m}{2}\right].$$

This proves (a).

Let n = 2m + 1 (m ≥ 1)

$$\text{Now G/A – Sz}(P_{2m+1}) = \frac{2}{2m+1}\left[\sum_{i=1}^{2m}\sqrt{i(2m+1-i)}\right]$$
$$= \frac{2}{2m+1}\left[\sum_{i=1}^{m}\sqrt{i(2m+1-i)}+\sum_{i=m+1}^{2m}\sqrt{i(2m+1-i)}\right]$$

Replacing 'i' by (2m+1-i') in the second sum, as in the previous case, we observe that this sum is same as the first one.

So, G/A – Sz(P_{2m+1}) = $\frac{4}{2m+1}\left[\sum_{i=1}^{m}\sqrt{i(2m+1-i)}\right]$.

This proves (b) and thus the proof of the theorem is complete.

Theorem 2.7: For the wheel

$$K_1 \vee C_n (n \geq 3),\ G/A - Sz(K_1 \vee C_n) = n\left(1+\frac{2\sqrt{n-2}}{n-1}\right).$$

Proof: Let $V(K_1 \text{ v } C_n) = \{ u_0, v_1, v_2, \ldots, v_n\}$ where u_0 is the centre (hub) of the wheel.

Now, $E(K_1 \text{ v } C_n) = \{u_0v_i : i = 1, 2, \ldots, n\}$ U $\{v_iv_{i+1} : i = 1, 2,, \ldots, n\}$ (with the convention $v_{n+1} = v_1$).
Denote $e_i = u_0v_i$ and $f_i = v_iv_{i+1}$ (i = 1, 2, , n}. Now, from Th. 2.7 in [4], for any $e \in E(K_1 \text{ v } C_n)$, $n_1(e) = n - 2$, $n_2(e) = 1$ and observe that the f_i's constitute C_n.

$$\text{So, } G/A - Sz(K_1 \text{ v } C_n) = \sum_{i=1}^{n} \frac{2\sqrt{n_1(e_i)n_2(e_i)}}{n_1(e_i)+n_2(e_i)} + G/A - Sz(Cn)$$

$$= 2\sum_{i=1}^{n} \frac{\sqrt{(n-2)(1)}}{(n-2)+1} + n \quad \text{(by virtue of Th.(2.4))}$$

$$= \frac{2n}{n-1}\sqrt{(n-2)} + (n)$$

$$= n\left(1+\frac{2\sqrt{n-2}}{n-1}\right).$$

3. MODIFIED G/A – SZEGED INDEX OF STANDARD GRAPHS

In the calculations of G/A – Szeged indices of K_n $(n \geq 2)$, C_{2n-1} $(n \geq 2)$ and K_1 v C_n $(n \geq 3)$ the contribution of all the vertices of the corresponding graphs are not there. To avoid this, we propose the following modified index that involves all the vertices.

Definition 3.1: Let G be a graph and e = uv $\in$ E(G).

Denote $N_1^*(e/G) = \{w \in V(G) : d(w, u) \leq d(w, v)\}$,
$N_2^*(e/G) = \{w \in V(G) : d(w, v) < d(w, u)\}$
and
$n_1^*(e/G) = |N_1^*(e/G)|$ & $n_2^*(e/G) = |N_2^*(e/G)|$.

The refined G/A – Szeged Index of G, denoted by $G/A - Sz^*(G)$ is defined as,

$$\sum_{e \in E(G}\left\{2\frac{\sqrt{n_1^*(e/G).n_2^*(e/G)}}{n_1^*(e/G)+n_2^*(e/G)}\right\}$$

(Another way of defining this modified index is to keep < as it is in $N_1^*(e/G)$ and changing < into $\leq$ in $N_2^*(e/G)$.

Observation 3.2: For the graphs, $P_n(n \geq 2)$, $K_{m,n}$ $(m, n \geq 1)$, C_{2n} $(n \geq 2)$, we observe that this modified index is same as the previous one, since there are no leftout vertices.

Theorem 3.3: For $n \geq 2$, $G/A - Sz^*(K_n) = (n-1)^{3/2}$.

Proof: For any $e \in E(K_n)$, by Th. 3.3 in [4], $n_1^*(e) = n-1$ and $n_2^*(e) = 1$.

$$\text{So, } G/A - Sz(K_1 \vee C_n) = \sum_{e \in E(K_n)} \frac{2\sqrt{(n-1)(1)}}{n-1+1}$$

$$= \frac{2}{n}\sqrt{n-1}\left(\frac{n(n-1)}{2}\right)$$

$$= (n-1)^{3/2}.$$

Theorem 3.4: For n ≥ 2, $\text{G/A - Sz}^*(C_{2n-1}) = 2\sqrt{n(n-1)}\ \ (\le 2n-1)$

Proof: For any e ϵ E(C_{2n-1}), by Th. (3.4) in [4], $n_1^*(e_i)$ = n and $n_2^*(e_i)$ =n - 1.

$$\text{So, G/A - Sz}^*(C_{2n-1}) = \sum_{e\in E(C_{2n-1})} \frac{2\sqrt{n(n-1)}}{(n+n-1)}$$

$$= \frac{2}{2n-1}\sqrt{n(n-1)}\,|E(C_{2n-1})|$$

$$= \frac{2}{2n-1}\sqrt{n(n-1)}(2n-1)$$

$$= 2\sqrt{n(n-1)}(\le 2n-1) \text{ (since G.M.} \le \text{A.M.)}$$

Observation 3.5: In the other way of defining the modified index, we get the same indices for K_n and C_{2n-1} since $n_1^*(e)$ and $n_2^*(e)$ are interchanged in the corresponding calculations.

Theorem 3.6: For the wheel K_1 v C_n (n ≥ 3),

$$\text{G/A} - Sz^*(K_1 \vee C_n) = \begin{cases} \dfrac{2n}{n+1}\left[\sqrt{n} + \sqrt{\dfrac{n}{2}(\dfrac{n}{2}+1)}\right] & \text{if n is even,} \\ \dfrac{n}{n+1}\left[2\sqrt{n} + \sqrt{n^2+2n-3}\right] & \text{if n is odd.} \end{cases}$$

Proof: With the same notation of Th 2.7, $n_1^*(e_i)$ = n – 2 + 2 = n and $n_2^*(e_i)$ = 1.

So,

$$\sum_{i=1}^{n} \frac{2\sqrt{n_1^*(e_i)n_2^*(e_i)}}{n_1^*(e_i)+n_2^*(e_i)} = \frac{2n}{n+1}\sqrt{n} \qquad (3.6.1)$$

Case (i): Suppose n is even.

By Th. 3.6.1 in [4], we have

$n_1^*(f_i)$ = n/2 + 1 and $n_2^*(f_i)$ = n/2

So

$$\sum_{i=1}^{n} \frac{2\sqrt{n_1^*(f_i)n_2^*(f_i)}}{n_1^*(f_i)+n_2^*(f_i)} = \frac{2}{n+1} n\sqrt{\frac{n}{2}(\frac{n}{2}+1)} \qquad (3.6.2)$$

By (3.6.1) and (3.6.2)

$$\text{G/A - Sz}^*(K_1 \text{ v } C_n) = \frac{2n}{n+1}\left[\sqrt{n} + \sqrt{\frac{n}{2}(\frac{n}{2}+1)}\right].$$

Case (ii): Suppose n is odd.

By Th.3.6.2 in [4],

$n_1^*(f_i) = \dfrac{n-1}{2} + 2 = \dfrac{n+3}{2}$ and $n_1^*(f_i) = \dfrac{n-1}{2}$.

$$\text{So, } \sum_{i=1}^{n} 2\frac{\sqrt{n_1^*(f_i)n_2^*(f_i)}}{n_1^*(f_i)+n_2^*(f_i)} = \frac{2n}{n+1}\sqrt{\left(\frac{n+3}{2}\right)\left(\frac{n-1}{2}\right)}$$

$$= \frac{n}{n+1}\sqrt{n^2+2n-3} \qquad (3.6.3)$$

By (3.6.1) and (3.6.3)

$$G/A - Sz^*(K_1 \vee C_n) = \frac{n}{n+1}\left[2\sqrt{n} + \sqrt{n^2+2n-3}\right]$$

This completes the proof of the theorem.

Observation 3.7: In the otherway of defining the modified Index, we get that $(n_i^*(e_i) = (n-2)$ and $(n_2^*(e_i) = 3$ and the other relations remain the same. So,

$$\sum_{i=1}^{n} \frac{2\sqrt{n_1^*(e_i)n_2^*(e_i)}}{n_1^*(e_i)+n_2^*(e_i)} = \frac{2n}{n+1}\left[\sqrt{3(n-2)}\right]$$

$$\text{Hence, G/A} - Sz^*(K_1 \vee C_n) = \begin{cases} \frac{2n}{n+1}\left[\sqrt{3(n-2)} + \sqrt{\frac{n}{2}(\frac{n}{2}+1)}\right] & \text{if } n \text{ is even,} \\ \frac{n}{n+1}\left[2\sqrt{3(n-2)} + \sqrt{n^2+2n-3}\right] & \text{if } n \text{ is odd .} \end{cases}$$

ACKNOWLEDGEMENTS

We are very thankful to Prof. I. H. Nagaraja Rao for his continuous support and encouragement in preparing this paper.

REFERENCES

[1] Bondy J.A. and Murthy U.S.R., Graph Theory with Applications, North Holand, New York, 1976.

[2] G. H. Fath-Tabar, B. Fortula and I.Gutman, A new geometricarithmetic index, J. Math. Chem., (2009) DOI: 10.1007/s10910-009-9584-7.

[3] KlavZar S., Rajapakse A. and Gutman I., The Szeged and the Wiener Index of Graphs, Appl. Math. Lett., **9**, (1996), 45-49.

[4] K.V.S. Sarma and I.V.N. Uma, On Szeged Index of Standard Graphs, International Journal of Mathematical Archive – 3(8), 2012, 2874-2880.

[5] Wiener, H., Structural determination of Paraffin Boiling points, Jour. Amer. Chemi. Soc., 69, 17-20 (1947).

Source of support: Nil, Conflict of interest: None Declared