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A COMMON FIXED POINT THEOREM FOR THREE SELF MAPPINGS IN A FUZZY METRIC SPACE WITH CONTINUOUS FUZZY METRIC

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ABSTRACT

In this paper we introduce the notion of a continuous fuzzy metric and prove a common fixed point theorem for three self maps on a complete fuzzy metric space with continuous fuzzy metric, under the influence of a contractive control function of type (AS).

Mathematical Subject Classification: *47H10, 54H25.*

Key words: *Common fixed point, Hadzic type t-norm, $\varphi -$ weakly commuting, Fuzzy metric spaces, Contractive control function of type (AS), Contractive control function of type (A). Continuous fuzzy metric.*

0. INTRODUCTION

Vasuki [13] proved a common fixed point theorem for two R-weakly commutative self maps on a complete fuzzy metric space with certain condition A.K.Sarma.*et.al* [9] extended this result to three self maps. In this paper, we make use of contractive control function of type (AS) to prove a common fixed point theorem for three self maps on a continuous complete fuzzy metric space.

1. PRELIMINARIES

Definition 1.1: [14] A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 1.2: [11] A binary operation $*:[0,1]\times[0,1]\longrightarrow[0,1]$ is called a continuous t-norm, if for each a,b,c,d in $[0,1]$, $*$ satisfies the following conditions

(i) $*$ is commutative and associative, i.e. $a*b=b*a$ and $a*(b*c)=(a*b)*c$,
(ii) $*$ is continuous ,
(iii) $a*1=a$ for all $a\in[0,1]$,
(iv) $a*b\le c*d$ whenever $a\le c$ and $b\le d$.

Examples of a continuous t-norm:

$$a*b=\min\{a,b\} \text{ and } a*b=ab$$

Definition 1.3: [5] The triplet $(X,M,*)$ is a fuzzy metric space, if X is a non empty set, $*$ is a continuous t-norm, M is a fuzzy set in $X^2\times[0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $s,t>0$,
(i) $M(x,y,0)=0$,
(ii) $M(x,y,t)=1\ \forall\ t>0 \Leftrightarrow x=y$,
(iii) $M(x,y,t)=M(y,x,t)\ for\ t>0$,
(iv) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$,
(v) $\lim_{t\to\infty}M(x,y,t)=1$ for all $x,y\in X$.
(vi) $M(x,y,\cdot):[0,\infty)\to[0,1]$ is left continuous,

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Note that $M(x,y,t)$ can be considered as the degree of nearness between x and y with respect to t. We identify $x=y$ with $M(x,y,t)=1 \ \forall \ t>0$.

The following Example shows that every metric space induces a fuzzy metric

Example 1.4: **[2]** Let (X,d) be a metric space. Let $a * b = \min\{a, b\}$ and $M(x,y,t)=\frac{t}{t+d(x,y)}$ $for \ t>0$ and for all $x,y,z \in X$. Then $(X, M, *)$ is called a fuzzy metric space. It is called the fuzzy metric space induced by d.

Lemma 1.5: **[3]** For all $x,y \in X, M(x,y,\cdot)$ is a non - decreasing function.

Definition 1.6:[3] A sequence $\{x_n\}_{n=1}^{\infty}$ in a fuzzy metric space $(X, M,*)$ is called Cauchy, if $\lim_{n\to\infty} M(x_{n+p}, x_n, t)=1 \quad for \ t>0$ and $p>0$.

Definition 1.7: **[3]** A sequence $\{x_n\}_{n=1}^{\infty}$ in a fuzzy metric space $(X, M,*)$ is called convergent to $x \in X$, if $\lim_{n\to\infty} M(x_n, x, t)=1 \quad for \ each \ t>0$. In this case x is called the limit of $\{x_n\}$.

Definition 1.8: **[3]** A fuzzy metric space $(X, M,*)$ is said to be complete if every Cauchy sequence in X converges in X.

Definition 1.9: Let $(X, M,*)$ be a fuzzy metric space. M is said to be a continuous fuzzy metric, if $x_n \to x$, $y_n \to y$ in X implies $M(x_n, y_n, t) \to M(x,y,t) \ \forall \ t>0$. In this case we say that $(X, M,*)$ is a continuous fuzzy metric space .

Definition 1.10: **[6]** Two mappings f and g of a fuzzy metric space $(X, M,*)$ into itself are said to be weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t)$ for each $x \in X$.

Definition 1.11:Let $\varphi : [0,\infty) \to [0,\infty)$ be such that φ is increasing and $\varphi(t)=0 \Leftrightarrow t=0$. Two mappings f and g of a fuzzy metric space $(X, M,*)$ into itself are said to be $\varphi-$ weakly commuting if , $M(fgx, gfx, t) \geq M(fx, gx, \varphi(t)) \ \forall x \in X$.

Remark 1.12: $\varphi-$ weakly commutativity implies weak-commutativity only when $\varphi(t) \geq t$.

The following Example shows that a pair (f,g) may be $\varphi-$ weakly commutative but not weakly – commutative.

Example 1.13: Let $X=R$ be the set of all real numbers. Define $a * b = ab$ and

$$M(x,y,t)=\left[e^{\left(\frac{|x-y|}{t}\right)}\right]^{-1} \text{for all } x, y \in X \text{ and } t>0.$$

$$M(x,y,0)=0 . \text{ Then } (X, M,*) \text{ is a fuzzy metric space.}$$

Define $f(x) = 2x-1$ and $g(x)=x^2$. Then

$$M(fgx, gfx, t) = \left[e^{\left(\frac{2|x-y|^2}{t}\right)}\right]^{-1}$$

$$M(fx, gx, \tfrac{t}{2}) = \left[e^{\left(\frac{2|x-y|^2}{t}\right)}\right]^{-1}$$

Let $\varphi(t) = \frac{t}{2}$. Then f and g are $\varphi-$ weakly commuting. But f and g are not weakly commuting since exponential function is strictly increasing.

Definition 1.14: **[4]** Let $*$ be a continuous t-norm. For any $a \in [0,1]$, write
$*_0(a)=1$ and
$*_1(a) = *(*_0(a), a) = *(1, a) = a$. In general
$*_{n+1}(a) = *(*_n(a), a)$ for $n = 0,1,2,3,\ldots$

If the sequence $\{*_n\}$ is equicontunious at 1, that is given $\epsilon>0, \exists \ \delta>0 \ni x>1-\delta \Rightarrow *_n(x) > 1-\epsilon \ \forall \ n \in N$, then we say that $*$ is a Hadzic type t – norm.

We observe that 'min't-norm is of Hadzic type.

Definition 1.15: [10] If $\varphi : R^+ \to R^+$ is such that

(i) φ is increasing ,
(ii) $\varphi(t) > t \ \forall \ t > 0$,
(iii) $\varphi(\varphi(t) - t) \geq \varphi^2(t) - \varphi(t)$ *for every* $t > 0$, then φ is called a contractive control function of type (A) .

Definition 1.16:[10] If $\varphi : R^+ \to R^+$ is a contractive control function which is strictly increasing, φ is onto and $\varphi\left(t - \varphi^{-1}(t)\right) \geq \varphi(t) - t$ *for every* $t > 0$, then φ is called a contractive control function of type (AS).

Example 1.17: [10] If $\varphi : R^+ \to R^+$ is defined by

$$\varphi(t) = \begin{cases} n+1 & if \ t \in [n, n+1) \\ 1 & if \ t \in (0,1) \\ 0 & if \ t = 0 \end{cases}$$

then φ is a contractive control function of type (A) but not type (AS) .

Example 1.18: [10] If $\varphi : R^+ \to R^+$ is defined by $\varphi(t) = kt \ \forall \ t > 0$ *and for some* $k > 0$, then φ is a contractive control function of type (AS) .

Vasuki [13] proved the following theorem.

Theorem 1.19: Let $(X, M, *)$ be a complete fuzzy metric space, let $R > 0$ *and* $\varphi(t) = \frac{t}{R} \ \forall \ t > 0$. Let f *and* g be φ −weakly commuting mappings of X satisfying the condition $M(fx, fy, t) \geq r\{M(gx, gy, \varphi(t))\}$ for all $x, y \in X$.

where $r : [0,1] \to [0,1]$ is a continuous function such that

$r(t) > t$ *for* $0 < t < 1$. The sequences $\{x_n\}$ and $\{y_n\}$ *in* X are such that

$x_n \to x$, $y_n \to y$, $t > 0$ implies that $M(x_n, y_n, t) \to M(x, y, t)$ *as* $n \to \infty$.

If the range of g contains the range of f and if either f *or* g is continuous, then f and g have a unique common fixed point in X .

A. K. Sarma , V. H. Badshah , V. K. Gupta and A. Sarma [9] generalized the above result for three weakly commuting maps instead of two maps .

Theorem 1.20: [9] Let $(X, M, *)$ be a complete fuzzy metric space, let f, g *and* h be three self maps on X satisfying
(i) $f(X) \cap g(X) \subset h(X)$ and
(ii) $M(fx, gy, t) \geq r\{M(hx, hy, t)\}$ for all $x, y \in X$,

where $r : [0,1] \to [0,1]$ is a continuous function such that

$r(t) > t$ *for each* $0 < t < 1$.

Let $R > 0$ *and* $\varphi(t) = \frac{t}{R}$ for $t > 0$.

Suppose h is continuous and the pairs (f, h) *and* (g, h) are $\varphi -$ weakly commuting on X. Then f, g *and* h have a unique common fixed point in X .

2. MAIN RESULT

In this section we prove our main result and obtain the result of A. K. Sarma, V. H. Badshah, V. K. Gupta and A. Sarma [9] (Theorem 1.20) as a corollary. Sastry *et al.* [10] used the notion of contractive control function of type (AS) to prove a sufficient condition for a sequence $\{y_n\}$ in a Menger space $(X, F, *)$ with t-norm $*$ assumed to be of Hadzic type, to be Cauchy.

We use these notions in fuzzy metric spaces and prove the following Lemma, which we use in our main result.

Lemma 2.1: Let $(X, M, *)$ be a complete fuzzy metric space, where $*$ is a Hadzic type t – norm. Let φ be a contractive control function of type (AS) such that $\varphi^n(t) - \varphi^{n+1}(t) \to \infty$ as $n \to \infty$ *for all* $t > 0$. Suppose $\{x_n\}$ is a sequence *in* X such that $M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, \varphi(t)) \ \forall \ t > 0$ Then $\{x_n\}$is a Cauchy sequences *in* X .

Proof: By hypotheses

$$\begin{aligned} M(x_n, x_{n+1}, t) &\geq M(x_{n-1}, x_n, \varphi(t)) \\ &\geq \\ &\vdots \\ &\geq M(x_0, x_1, \varphi^n(t)) \\ &\geq M(x_0, x_1, \varphi^n(t) - \varphi^{n-1}(t)) \quad \rightarrow (2.1.1) \\ &= \lambda_n(t) \end{aligned}$$

Since $\varphi \in \Phi$, $\lambda_n(t) \rightarrow 1$ $as\ n \rightarrow \infty$. Now we show that

$$M(x_n, x_{n+k}, t) \geq *_k(\lambda_n(t)).$$

This is true for for k = 1 and any $n \in N$ by (2.1.1) assume the truth for k

$$\begin{aligned} M(x_n, x_{n+k+1}, t) &\geq * M(x_n, x_{n+1}, t - \varphi^{-1}(t)), M(x_{n+1}, x_{n+k+1}, \varphi^{-1}(t)) \\ &\geq * M(x_0, x_1, \varphi^n(t - \varphi^{-1}(t)), *_k M(x_0, x_1, \varphi^{n+1}(\varphi^{-1}(t)) - \varphi^n(\varphi^{-1}(t))) \\ &\geq * M(x_0, x_1, \varphi^n(t) - \varphi^{n-1}(t)), *_k M(x_0, x_1, \varphi^n(t) - \varphi^{n-1}(t)) \end{aligned}$$

by definition (1.15) and (1.16) we have

$$\begin{aligned} &= *_{k+1} M(x_0, x_1, \varphi^n(t) - \varphi^{n-1}(t)) \\ &= *_{k+1}(\lambda_n(t)) \quad \rightarrow (2.1.2) \end{aligned}$$

Let $\epsilon > 0$. Since $*$ is a Hadzic type t – norm, $*$ is equicontinuous at 1.

Hence there exists $\eta \in (0,1)$ such that $1 \geq s > 1 - \eta \Rightarrow *_{k+1}(s) > 1 - \epsilon$

Since $\lambda_n(t) \rightarrow 1$ $as\ n \rightarrow \infty$ there exists N such that $n \geq N$

$$\Rightarrow \lambda_n(t) > 1 - \eta.$$

Hence by (2.1.2), we have

$$\begin{aligned} M(x_n, x_{n+k+1}, t) &\geq *_{k+1}(\lambda_n(t)) \\ &> 1 - \epsilon \ \forall\ n \geq N \end{aligned}$$

Consequently

$M(x_n, x_m, t) > \epsilon$, whenever $m > n \geq N$

Hence $\{x_n\}$ is a Cauchy sequence .

Now we are sufficiently equipped with the tools to prove our main result.

Theorem 2.2: Let $f, g\ and\ h$ be three self mappings on a continuous complete fuzzy metric space $(X, M, *)$, where $*$ is a Hadzic type t – norm. Suppose

(i) $f(X) \cap g(X) \subset h(X)$ (2.2.1)

and

(ii) $M(fx, gy, t) \geq M(hx, hy, \varphi(t))$ (2.2.2)

where φ is a contractive control function of type (AS) such that

$\varphi^n(t) - \varphi^{n-1}(t) \rightarrow \infty$ as $n \rightarrow \infty$ $\forall\ t > 0$.

Let $\psi : [0, \infty) \rightarrow [0, \infty)$ be as in the definition (1.11). Suppose that h is continuous and one of the pairs (f, h) and (g, h) is $\psi - weakly$ commuting on X . Then $f, g\ and\ h$ have a unique common fixed point in X.

Proof: Let $x_0 \in X$. By (2.2.1) we can choose $x_1 \in X$ such that $fx_0 = hx_1$ and for this $x_1 \in X$, $\exists\ x_2 \in X$ such that $gx_1 = hx_2$ and so on. Continuing in this manner

We can choose a sequence $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} = hx_{2n+1}$$

$$y_{2n+1} = gx_{2n+1} = hx_{2n+2}\ ,\ for\ n = 0,1,2,.... \tag{2.2.3}$$

Now $M(y_{2n}, y_{2n+1}, t) \geq M(fx_{2n}, gx_{2n+1}, \varphi(t))$

$= M(hx_{2n}, hx_{2n+1}, \varphi(t))$ by (2.2.2)

$$\therefore\ M(y_{2n}, y_{2n+1}, t) \geq M(y_{2n-1}, y_{2n}, \varphi(t)) \tag{2.2.4}$$

and

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, t) &= M(gx_{2n+1}, fx_{2n+2}, t) \\ &= M(fx_{2n+2}, gx_{2n+1}, t) \\ &\geq M(hx_{2n+2}, hx_{2n+1}, \varphi(t)) \\ &= M(y_{2n+1}, y_{2n}, \varphi(t)) \end{aligned}$$

$$\therefore\ M(y_{2n+2}, y_{2n+1}, t) \geq M(y_{2n+1}, y_{2n}, \varphi(t)) \tag{2.2.5}$$

From (2.2.4) and (2.2.5) we get

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, \varphi(t))\ \forall\ t > 0 \text{ and } n = 1,2,...$$

Now by Lemma (2.1) , the sequence $\{y_n\}$ is Cauchy sequence in *X*. But *X* is complete and so by completeness of X, $\{y_n\}$ converges to some point u in *X*.

Consequently the sequences $\{fx_{2n}\}$, $\{hx_{2n+1}\}$, $\{gx_{2n+1}\}$, $\{hx_{2n+2}\}$ of $\{y_n\}$ also converge to the same point u in *X*.

Suppose the pair (f, h) is $\psi - weakly$ commuting. Since h is continuous it follows that

$$M(fhx_n, hfx_n, t) \geq M(fx_n, hx_n, \psi(t))\ \ \forall x \in X,$$

On letting $n \to \infty$, we get

$$\lim_{n\to\infty} M(fhx_n, hu, t) \geq M(u, u, \psi(t))$$

Hence $fhx_n \to hu$ from (2.2.2.) We have

$$M(fhx_{2n}, gx_{2n+1}, t) \geq M(hhx_{2n}, hx_{2n+1}, \varphi(t))$$

On letting $n \to \infty$, we get

$$\begin{aligned} M(hu, u, t) &\geq M(hu, u, \varphi(t)) \\ &\geq M(hu, u, t), \text{ since } \varphi(t) > t \end{aligned}$$

Hence $M(hu, u, s) = M(hu, u, t)\ \forall s \in [t, \varphi(t)]$,

Since $\varphi(t)$ is strictly increasing and onto R^+, it follows that $\varphi(t) \to \infty$ as $t \to \infty$.

Hence $M(hu, u, t)$ is a constant in $(0, \infty)$.

Now by definition 1.3 (v) follows that
$M(hu, u, t) = 1\ \forall\ t > 0$. Consequently, $hu = u$ (by definition 1.3 (ii)).

Also by (2.2.2) we have
$M(fu, gx_{2n+1}, t) \geq M(hu, hx_{2n+1}, \varphi(t))$

On letting $n \to \infty$, we get

$$\begin{aligned} M(fu,u,t) &\geq M(hu,u,\varphi(t)) \\ &= M(u,u,\varphi(t)) \\ &= 1 \ \forall t > 0 . \end{aligned}$$

Hence $fu = u$.

Now consider

$$\begin{aligned} M(u,gu,t) &\geq M(fu,gu,t) \\ &\geq M(hu,hu,\varphi(t)) \\ &= 1 \ \forall t > 0 . \end{aligned}$$

Hence $gu = u$.

Thus u is a common fixed point of $f, g \ and \ h$.

UNIQUENESS

Suppose that v is a common fixed point of $f, g \ and \ h$. Then

$$\begin{aligned} M(u,v,t) &= M(fu,gv,t) \\ &\geq M(hv,hv,\varphi(t)) \\ &= M(u,v,\varphi(t)) \\ &\geq M(u,v,t) \quad \forall t > 0 . \end{aligned}$$

Hence $u = v$ and so common fixed point of $f, g \ and \ h$ is unique.

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