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A COMMON FIXED POINT THEOREM FOR FOUR SELF MAPS ON A FUZZY METRIC SPACE CONTROLLED BY A SB-FUNCTION

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ABSTRACT

In this paper we introduce a new class of control functions called SB-functions, using this, we prove a common fixed point theorem for four self maps on a fuzzy metric space controlled by a SB- function. Our result improves the results of Aage. C.T and Salunke [1], Rajesh Kumar Mishra and Sanjay Choudhary [6] and Rajesh Shrivastava and Manavi Kohil[7].

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1. INTRODUCTION

Aage. C.T and Salunke [1], Rajesh Kumar Mishra and Sanjay Choudhary [6] and Rajesh Shrivastava and Manavi Kohil [7] proved common fixed point theorems for four self maps on a complete fuzzy metric space. In an attempt to improve the above results, we introduce a new class of functions, called SB-functions. We use them as control functions, to prove a common fixed point theorem for four self maps on a fuzzy metric space. Our result significantly improves the results of [1], [6] and [7].

Before giving the results, some preliminary definitions are given below.

Definition 1.1: (Zadeh. L.A [9]) A fuzzy set A in a nonempty set X is a function with domain X and values in$[0,1]$.

Definition 1.2: (Schweizer. B and Sklar. A [8]) A function $*: [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if $*$ satisfies the following conditions:

For $a, b, c, d \in [0,1]$

(i) $*$ is commutative and associative
(ii) $*$ is continuous
(iii) $a * 1 = a \; for \; all \; a \in [0,1]$
(iv) $a * b \leq c * d \; whenever \; a \leq c \; and \; b \leq d$

Definition 1.3: (Kramosil. I and Michelek. J [5]) A triple $(X, M, *)$ is said to be a fuzzy metric space (FM space, briefly) if X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,\infty)$ satisfies the following conditions:

For $x, y, z \in X$ and $s, t > 0$.

(i) $M(x,y,t) > 0, M(x,y,0) = 0$
(ii) $M(x,y,t) = 1 \; if \; and \; only \; if \; x = y$
(iii) $M(x,y,t) = M(y,x,t)$
(iv) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$
(v) $M(x,y,\cdot): [0,\infty) \longrightarrow [0,1]$ is continuous.

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Then M is called a fuzzy metric space on X.

The function $M(x,y,t)$ denotes the degree of nearness between x and y with respect to t.

Definition 1.4: (**George. A and Veeramani. P [2])** Let $(X, M,*)$ be a fuzzy metric space. Then,

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1 \ \forall\, t > 0$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $\lim_{n\to\infty} M\left(x_{n+p}, x_n, t\right) = 1 \ \forall\, t > 0$ and $p = 1,2, \ldots$

(iii) An FM –space in which every Cauchy sequence is convergent is said to be complete.

Definition 1.5: Let X be a nonempty set and f and g be self maps on X. A point x in X is called a coincidence point of f and g if $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g.

Definition 1.6: (Jungck. G [3]) Two self maps S and T of a fuzzy metric space $(X, M,*)$ are said to be weakly compatible pair if they commute at coincidence points, that is if $Sx = Tx$ for some $x \in X$, then $STx = TSx$.

Definition 1.7: Two self maps f and g of a nonempty set X are occasionally weakly compatible (owc), if there is a point x in X which is a coincidence point of f and g at which f and g commute. That is f and g are occasionally weakly compatible if there exists $x \in X \ni fx = gx$ and $fgx = gfx$.

A pair S and T of maps may be owc but may not be weakly compatible.

Example 1.8: Let $\mathbb{R}$ be usual metric space. Define $S, T: \mathbb{R} \to \mathbb{R}$ by $Sx = 2x$ and $Tx = x^2$ for all $x \in \mathbb{R}$. Then 0 and 2 are coincidence points of S and T. Also $ST0 = 0, TS0 = 0$ but $ST2 = 8 \neq 16 = TS2$.

Therefore S and T are occasionally weakly compatible self maps but are not weakly compatible.

Lemma 1.9: (Jungck. G and Rhoades .B.E [4]) Let X be a nonempty set and f and g are owc self maps on X. If f and g have unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g.

Aage. C.T and Salunke [1] proved the following theorem.

Theorem 1.10: (Aage. C.T and Salunke [1], Theorem 3.1) Let $(X, M,*)$ be a complete fuzzy metric space and A, B, S and T be self maps of X. Let the pairs (A, S) and (B, T) be owc. Suppose there exists $u \in (0,1)$ such that

$$M(Ax, By, ut) \geq min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\} \tag{1.10.1}$$
for $x, y \in X$ and for $t > 0$.

Then there exists a unique point $w \in X$ such that $w = Aw = Sw$ and a unique point $z \in X$ such that $z = Bz = Tz$. Moreover $z = w$, so that there is a unique common fixed point of A, B, S and T.

The following theorem follows immediately from Theorem 1.10

Theorem 1.11: (Aage. C.T and Salunke [1], Theorem 3.2) Let $(X, M,*)$ be a complete fuzzy metric space and A, B, S and T be self maps of X. Let the pairs (A, S) and (B, T) be owc. Suppose there exists $u \in (0,1)$ such that

$M(Ax, By, ut) \geq \psi(min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\})$ for $x, y \in X$ and for $t > 0$, where $\psi: [0,1] \to [0,1]$ is such that $\psi(t) > t$ for $t \in (0,1)$.

Then there exists a unique common fixed point of A, B, S and T.

Rajesh Kumar Mishra and Sanjay Choudhary [6] proved the following theorem.

Theorem 1.12: (Rajesh Kumar Mishra and Sanjay Choudhary [6], Theorem 2.1) Let $(X, M,*)$ be a complete fuzzy metric space and A, B, S and T be self maps of X. Let the pairs (A, S) and (B, T) be owc. Suppose there exists $u \in (0,1)$ such that

$$M(Ax, By, ut) \geq min\left\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \left[\frac{M(Ax,Ty,t)+M(By,Sx,t)}{2}\right]\right\} \tag{1.12.1}$$
for $x, y \in X$ and for $t > 0$.

Then there exists a unique point $w \in X$ such that $w = Aw = Sw$ and a unique point $z \in X$ such that $z = Bz = Tz$. Moreover $z = w$, so that there is a unique common fixed point of A, B, S and T.

The following Theorem follows immediately from Theorem 1.12

Theorem 1.13: (Rajesh Kumar Mishra and Sanjay Choudhary [6], Theorem 2.2) Let $(X, M,*)$ be a complete fuzzy metric space and A, B, S and T be self maps of X. Let the pairs (A, S) and (B, T) be OWC. Suppose there exists $u \in (0,1)$ such that

$$M(Ax, By, ut) \geq \psi\left(min\left\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \left[\frac{M(Ax,Ty,t)+M(By,Sx,t)}{2}\right]\right\}\right)\text{r } x, y \in X \text{ and for } t > 0$$

where $\psi: [0,1] \to [0,1]$ is such that $\psi(t) > t$ for $t \in (0,1)$.

Then there exists a unique common fixed point of A, B, S and T.

Rajesh Shrivastava and Manavi Kohil [7] proved the following Theorem.

Theorem 1.14: (Rajesh Shrivastava and Manavi Kohil [7], Theorem 2.1) Let $(X, M,*)$ be a complete fuzzy metric space and A, B, S and T be self maps of X. Let the pairs (A, S) and (B, T) be OWC. Suppose there exists $u \in (0,1)$ such that

$$M(Ax, By, ut) \geq min\left\{M(Sx, Ty, t), M(By, Ty, t), \left[\frac{M(Ax,Ty,t)+M(By,Sx,t)}{2}\right]\right\} \tag{1.14.1}$$

for $x, y \in X$ and for $t > 0$.

Then there exists a unique point $w \in X$ such that $w = Aw = Sw$ and a unique point $z \in X$ such that $z = Bz = Tz$. Moreover $z = w$, so that there is a unique common fixed point of A, B, S and T.

The following Theorem follows immediately from Theorem 1.14

Theorem 1.15: (Rajesh Shrivastava and Manavi Kohil [7], Theorem 2.2) Let $(X, M,*)$ be a complete fuzzy metric space and A, B, S and T be self maps of X. Let the pairs (A, S) and (B, T) be OWC. Suppose there exists $u \in (0,1)$ such that

$$M(Ax, By, ut) \geq \psi\left(min\left\{M(Sx, Ty, t), M(By, Ty, t), \left[\frac{M(Ax,Ty,t)+M(By,Sx,t)}{2}\right]\right\}\right) x, y \in X \text{ and for } t > 0$$

where $\psi: [0,1] \to [0,1]$ is such that $\psi(t) > t$ for $t \in (0,1)$.

Then there exists a unique common fixed point of A, B, S and T.

2. MAIN RESULTS

In this section, we introduce a special class Ψ of self maps on $[0,1]$, show that the class Ψ is non empty, and making use of members of this class as control functions, prove a common fixed point theorem (Theorem 2.3) for four self maps on a fuzzy metric space.

Further we show that Theorem 1.10 (Aage. C.T and Salunke [1], Theorem 3.1), Theorem 1.12 (Rajesh Kumar Mishra and Sanjay Choudhary [6], Theorem 2.1) and Theorem 1.14 (Rajesh Shrivastava and Manavi Kohil [7], Theorem 2.1) follow as corollaries of our result. We start with

Definition 2.1: A function $\psi: [0,1] \to [0,1]$ is said to be a SB- function if (i) $\psi(t) < t$ and (ii) $\psi^n(t) \to 1$ as $t \to 1$ and $n \to \infty$.

Notation: Let $\Psi = \{\psi/\psi: [0,1] \to [0,1]\ is\ a\ SB - function\}$.

The following example shows that $\Psi \neq \emptyset$

Example 2.2: Let $\alpha_n = 1 - \frac{1}{n}$ for $n = 2,3, ...$
For a given $n \geq 2$, let $\beta_{nk} = \alpha_n + \frac{1}{n(n+1)2^k}, k = 0,1,2, ...$

Thus $\beta_{n0} = \alpha_{n+1}$, β_{n1} is the midpoint of $[\alpha_n, \alpha_{n+1}]$, and in general β_{nk+1} is the midpoint of $[\alpha_n, \beta_{nk}]$. Let $t \in (0,1)$.

Then there exists *n* such that $\alpha_n < t \le \alpha_{n+1}$

Further, there exists *k* such that $\beta_{nk+1} < t \le \beta_{nk}$ (2.2.1)

Now, define $\psi(t) = \beta_{nk+1}$, if *t* satisfies (2.2.1) so that $\psi(t) < t$.

Also define $\psi(0) = 0$ and $\psi(1) = 1$.

We observe that, if *t* satisfies (2.2.1), then $\alpha_n < \psi^m(t)$ for $m = 1,2, ...$

Since $\alpha_n \to 1$ as $n \to \infty$, follows that $\psi^m(t) > \alpha_n = 1 - \frac{1}{n}$

Now, given $\varepsilon > 0$, choose *N* such that $\alpha_N < 1 - \varepsilon < \alpha_{N+1}$ and $T = \alpha_N$

Suppose $T < t < 1$.

Then there exists $n \ge N$ such that $\alpha_n < t \le \alpha_{n+1}$.

Further by (2.2.1) there exists $k \ge 1$ such that $\beta_{nk+1} < t \le \beta_{nk}$

Hence $\psi^m(t) \ge \alpha_n$ for $m = 1,2, ...$

Therefore $\psi^m(t) > 1 - \varepsilon$

$\psi^m(t) \to 1$ as $t \to 1$ and $m \to \infty$.

Consequently $\psi \in \Psi$.

Note: The function ψ described in the above example satisfies something more. Namely,

Given $\varepsilon > 0, \exists\ N \textit{ such that } t > \alpha_N \Rightarrow \psi^m(t) > 1 - \varepsilon$ for $m = 1,2, ...$

Now, we state and prove our main result.

Theorem 2.3: Let $(X, M,*)$ be a fuzzy metric space and A, B, S and T be self maps of *X*. Let the pairs (A, S) and (B, T) be OWC. Suppose there exists $\psi \in \Psi$ and $u \in (0,1)$ such that

$$M(Ax, By, ut) \ge \psi(min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\}) \quad (2.3.1)$$

for $x, y \in X$ and for $t > 0$.

Then A, B, S and T have a unique common fixed point in *X*.

Proof: Since (A, S) and (B, T) are owc self maps on *X*, there exist points $x, y \in X$ such that $Ax = Sx, By = Ty, ASx = SAx$ and $BTy = TBy$.

Now we prove that $Ax = By$.

From (2.3.1), we have

$$\begin{aligned} M(Ax, By, ut) &\ge \psi(min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\}) \\ &= \psi(min\{M(Ax, By, t), M(Ax, Ax, t), M(By, By, t), M(Ax, By, t), M(By, Ax, t)\}) \\ &= \psi(min\{M(Ax, By, t), 1,1, M(Ax, By, t), M(Ax, By, t)\}) \\ &= \psi\big(M(Ax, By, t)\big) \end{aligned}$$

Therefore $M(Ax, By, ut) \ge \psi\big(M(Ax, By, t)\big)$ for $t > 0$.

Therefore $M(Ax, By, t) \ge \psi\left(M\left(Ax, By, \frac{t}{u}\right)\right)) \ge \psi^2\left(M\left(Ax, By, \frac{t}{u^2}\right)\right) \ge \cdots \ge \psi^n\left(M\left(Ax, By, \frac{t}{u^n}\right)\right)$

Since $\psi \in \Psi$, $\psi^n\left(M\left(Ax, By, \frac{t}{u^n}\right)\right) \to 1$ as $n \to \infty$.

Therefore $M(Ax, By, t) \geq 1$

Therefore $Ax = By$.

Thus $Ax = Sx = By = Ty$.

Suppose that there exists another point $z \in X \ni Az = Sz$.

Then from (2.3.1), we have $Az = Sz = By = Ty$.

Therefore $Ax = Az$ and $Sx = Sz$

Therefore $w = Ax = Sx$ is the unique point of coincidence of A and S.

Therefore by Lemma 1.9, w is the only common fixed point of A and S.

Since $Ax = Sx = By = Ty$, w is the only common fixed point of B and T.

Thus A, B, S and T have a unique common fixed point in X.

Note: In this theorem, completeness of the fuzzy metric space is not used.

Now we show that Theorem 1.10, Theorem 1.12 and Theorem 1.14 follows from Theorem 2.3.

Corollary 2.4: (Aage. C.T and Salunke [1], Theorem 3.1) Let $(X, M, *)$ be a complete fuzzy metric space and A, B, S and T be self maps of X. Let the pairs (A, S) and (B, T) be OWC. Suppose there exists $u \in (0,1)$ such that

$$M(Ax, By, ut) \geq min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\} \quad (1.10.1)$$
for $x, y \in X$ and for $t > 0$.

Then there exists a unique point $w \in X$ such that $w = Aw = Sw$ and a unique point $z \in X$ such that $z = Bz = Tz$. Moreover $z = w$, so that there is a unique common fixed point of A, B, S and T.

Proof: If (1.10.1) is satisfied, then for any $\psi \in \Psi$, $u \in (0,1)$, $x, y \in X$ and $t > 0$

$$M(Ax, By, ut) \geq min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\}$$

$$> \psi(min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\})$$
so that (2.3.1) is satisfied.

Consequently A, B, S and T have a unique common fixed point in X.

Note: Since Theorem 1.12 is corollary to Theorem 1.10 and Theorem 1.14 is corollary to Theorem 1.12, so Theorem 1.10, Theorem 1.12 and Theorem 1.14 follows from Theorem 2.3.

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