International Journal of Mathematical Archive-3(12), 2012, 4997-5005
IJMA Available online through www.ijma.info ISSN 2229 – 5046

A MATHEMATICAL THEOREM IN MAGNETOROTATORY THERMOHALINE CONVECTION IN POROUS MEDIUM

JYOTI PRAKASH[1*] & SANJAY KUMAR GUPTA[2]

*[1]*Associate Prof. Department of Mathematics & Statistics, H. P. University, Shimla, India.*
[2] Research Scholar, Department of Mathematics & Statistics, H. P. University, Shimla, India.

(Received on: 09-11-12; Revised & Accepted on: 20-12-12)

ABSTRACT

The present paper mathematically establishes that magnetorotatory thermohaline convection of the Veronis type in porous medium cannot manifest itself as oscillatory motions of growing amplitude in an initially bottom heavy configuration if the thermohaline Rayleigh number R_S, the Lewis number τ, the Prandtl number P_r, the porosity ϵ, satisfy the inequality $R_S \leq 4\pi^2\left(\frac{1}{D_a}+\frac{\tau}{E' P_r \epsilon}\right)$, where D_a the Darcy number and E' are constants which depend upon porosity of the medium. It further establishes that this result is uniformly valid for the quite general nature of the bounding surfaces. A similar characterization theorem is also proved for magnetorotatory thermohaline convection of the Stern type.

Keywords: *Thermohaline instability, Porous medium, Oscillatory motions.*

INTRODUCTION

The thermohaline convection problem has been extensively studied in the recent past on account of its interesting complexities as a double diffusive phenomenon as well as its direct relevance in many problems of practical interest in the fields of oceanography, astrophysics, limnology and chemical engineering etc. [1]. Two fundamental configurations have been studied in the context of thermohaline convection problems, one by Veronis [2], wherein the temperature gradient is destabilizing and the concentration gradient is stabilizing; and another by Stern [3], wherein the temperature gradient is stabilizing and the concentration gradient is destabilizing. The main results of Veronis and Stern for their respective configuration are that both allow the occurrence of a steady motion or an oscillatory motion of growing amplitude, provided the destabilizing temperature gradient or the concentration gradient is sufficiently large. In case of Veronis' configuration, oscillatory motions of growing amplitude are preferred mode of onset of instability whereas in case of Stern's configuration, stationary convection is the preferred mode of onset of instability and these results are independent of the initially gravitationally stable or unstable character of the two configurations. Thus thermohaline configurations of Veronis and Stern type can further be classified into the following two classes:

(i) the first class, in which thermohaline instability manifests itself when the total density field is initially bottom heavy, and
(ii) the second class, in which thermohaline instability manifests itself when the total density field is initially top heavy.

Banerjee et al [4] derived a characterization theorem for the nonexistence of oscillatory motions of growing amplitude in an initially bottom heavy configuration of Veronis type. The essence of Banrjee et al's theorem lies in that it provides a classification of the neutral or unstable thermohaline convection configuration of the Veronis and Stern types into two classes, the bottom heavy class and the top heavy class, and then strikes a distinction between them by means of characterization theorems which disallow the existence of oscillatory motions in the former class.

In recent years, many researchers have shown their keen interest in analyzing the onset of convection in a fluid layer subjected to a vertical temperature gradient in a porous medium [5,6,7,8,9]. The extension of these two important hydrodynamical theorems to the domains of convection in porous medium, due to its importance in the prediction of ground water movement in aquifers, in the energy extraction process from the geothermal reservoirs, in assessing the effectiveness of fibrous insulations, drying of foods or other natural minerals and in nuclear engineering, is very much sought after in the present context. This paper, which mathematically analyses the hydrodynamic thermohaline convection-configuration of the Veronis and the Stern types in porous medium wherein a uniform vertical magnetic field and a uniform rotation about he vertical is superimposed, may be regarded as a first step in this scheme of extended investigations.

Corresponding author: JYOTI PRAKASH[1*]
*[1]*Associate Prof. Department of Mathematics & Statistics, H. P. University, Shimla, India.*

The present paper mathematically establishes that magnetorotatory thermohaline convection of the Veronis type in porous medium cannot manifest itself as oscillatory motions of growing amplitude in an initially bottom heavy configuration if the thermohaline Rayleigh number R_S, the Lewis number τ, the Prandtl number P_r, the porosity ϵ, satisfy the inequality $R_S \leq 4\pi^2\left(\frac{1}{D_a}+\frac{\tau}{E' P_r \epsilon}\right)$, where D_a the Darcy number and E' are constants which depend upon porosity of the medium. It further establishes that this result is uniformly valid for the quite general nature of the bounding surfaces. A similar characterization theorem is also proved for magnetorotatory thermohaline convection of the Stern type.

1. FORMULATION OF THE PROBLEM

An infinite horizontal porous layer filled with a viscous fluid is statically confined between two horizontal boundaries z = 0 and z = d, maintained at constant temperatures T_0 and T_1 ($< T_0$) and solute concentrations S_0 and S_1 ($< S_0$) at the lower and upper boundaries respectively in the presence of rotation and a uniform vertical magnetic field acting parallel to the direction of gravity. It is further assumed that the saturating fluid and the porous layer are incompressible and that the porous medium is a constant porosity medium. The problem is to investigate the stability of this initial stationary state.

Let the origin be taken on the lower boundary z = 0 with the positive direction of the z-axis along the vertically upward direction. Then the basic hydrodynamic equations that govern the problem are given by:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1}$$

$$\frac{1}{\epsilon}\frac{\partial u}{\partial t}+\frac{1}{\epsilon^2}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)-\frac{\mu_e}{4\pi\rho_0}\left(H_1\frac{\partial H_1}{\partial x}+H_2\frac{\partial H_1}{\partial y}+H_3\frac{\partial H_1}{\partial z}\right)$$
$$=-\frac{\partial}{\partial x}\left(\frac{p}{\rho_0}+\frac{\mu_e|H|^2}{8\pi\rho_0}-\frac{1}{2}|\Omega\times r|^2\right)+\frac{2}{\epsilon}(q\times\Omega)_x-\frac{\nu}{k_1}u \tag{2}$$

$$\frac{1}{\epsilon}\frac{\partial v}{\partial t}+\frac{1}{\epsilon^2}\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)-\frac{\mu_e}{4\pi\rho_0}\left(H_1\frac{\partial H_2}{\partial x}+H_2\frac{\partial H_2}{\partial y}+H_3\frac{\partial H_2}{\partial z}\right)$$
$$=-\frac{\partial}{\partial y}\left(\frac{p}{\rho_0}+\frac{\mu_e|H|^2}{8\pi\rho_0}-\frac{1}{2}|\Omega\times r|^2\right)+\frac{2}{\epsilon}(q\times\Omega)_y-\frac{\nu}{k_1}v \tag{3}$$

$$\frac{1}{\epsilon}\frac{\partial w}{\partial t}+\frac{1}{\epsilon^2}\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)-\frac{\mu_e}{4\pi\rho_0}\left(H_1\frac{\partial H_3}{\partial x}+H_2\frac{\partial H_3}{\partial y}+H_3\frac{\partial H_3}{\partial z}\right)$$
$$=-\frac{\partial}{\partial z}\left(\frac{p}{\rho_0}+\frac{\mu_e|H|^2}{8\pi\rho_0}-\frac{1}{2}|\Omega\times r|^2\right)+\frac{2}{\epsilon}(q\times\Omega)_z-\frac{\nu}{k_1}w-\frac{\rho}{\rho_0}g \tag{4}$$

$$E\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\kappa_T\nabla^2 T \tag{5}$$

$$E'\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x}+v\frac{\partial S}{\partial y}+w\frac{\partial S}{\partial z}=\kappa_S\nabla^2 S \tag{6}$$

Equations of Magnetic Induction

$$\epsilon\frac{\partial H_1}{\partial t}+u\frac{\partial H_1}{\partial x}+v\frac{\partial H_1}{\partial y}+w\frac{\partial H_1}{\partial z}=H_1\frac{\partial u}{\partial x}+H_2\frac{\partial u}{\partial y}+H_3\frac{\partial u}{\partial z}+\epsilon\eta\nabla^2 H_1 \tag{7}$$

$$\epsilon\frac{\partial H_2}{\partial t}+u\frac{\partial H_2}{\partial x}+v\frac{\partial H_2}{\partial y}+w\frac{\partial H_2}{\partial z}=H_1\frac{\partial v}{\partial x}+H_2\frac{\partial v}{\partial y}+H_3\frac{\partial v}{\partial z}+\epsilon\eta\nabla^2 H_2 \tag{8}$$

$$\epsilon\frac{\partial H_3}{\partial t}+u\frac{\partial H_3}{\partial x}+v\frac{\partial H_3}{\partial y}+w\frac{\partial H_3}{\partial z}=H_1\frac{\partial w}{\partial x}+H_2\frac{\partial w}{\partial y}+H_3\frac{\partial w}{\partial z}+\epsilon\eta\nabla^2 H_3 \tag{9}$$

Equation of Solenoidal character of the Magnetic Field

$$\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y}+\frac{\partial H_3}{\partial z}=0 \tag{10}$$

Equation of State

$$\rho=\rho_0[1+\alpha(T_0-T)-\gamma(S_0-S)], \tag{11}$$

where u, v, w are the components of velocity in the x, y, z-directions respectively, H_1, H_2, H_3 are components of magnetic field **H** in x, y, z-directions and $\frac{p}{\rho_0}+\frac{\mu_e|H|^2}{8\pi\rho_0}-\frac{1}{2}|\Omega\times r|^2$ is the modified magnetorotatory hydrodynamic pressure. Further t, ρ, T, S, ϵ, k_1, μ_e, ν, κ_T, κ_S and η, are, respectively, the time, the density, the temperature, the concentration, the porosity of the porous medium, the permeability of the porous medium, the magnetic permeability, the kinematic viscosity, the thermal diffusivity, the mass diffusivity and the resistivity; and α and γ are respectively the coefficients of volume expansion due to temperature and concentration variation . Here $E = \epsilon + (1-\epsilon)\frac{\rho_s C_s}{\rho_0 C_f}$ is a constant and E' is also a constant analogous to E but corresponding to concentration rather than heat, where ρ_s, C_s and ρ_0, C_f stand for density and heat capacity of the solid (porous matrix) material and fluid respectively. The suffix '0' denotes the values of the various parameters at some suitably chosen reference temperature T_0 and concentration S_0.

The basic state is assumed to be quiescent state and is given by

$$\left.\begin{aligned} (u,v,w) &\equiv (0,0,0)\\ p &\equiv p(z)\\ T &\equiv T(z)\\ S &\equiv S(z)\\ (H_1,H_2,H_3) &\equiv (0,0,H)\\ \rho &\equiv \rho(z) \end{aligned}\right\} \tag{12}$$

Thus the basic state solution on the basis of the basic state is given by

$$\left.\begin{aligned} (u,v,w) &= (0,0,0)\\ \frac{p}{\rho_0} + \frac{\mu_e |H|^2}{8\pi\rho_0} - \frac{1}{2}\,|\Omega\times r|^2 &= P = P_0 - g\rho_0\left(z + \frac{\alpha\beta z^2}{2} - \frac{\gamma\delta z^2}{2}\right)\\ T &= T_0 - \beta z\\ S &= S_0 - \delta z\\ (H_1,H_2,H_3) &= (0,0,H)\\ \rho &= \rho_0\left[1 + \alpha(T_0 - T) - \gamma(S_0 - S)\right]\\ &= \rho_0\left[1 + \alpha\beta z - \gamma\delta z\right] \end{aligned}\right\} \tag{13}$$

where H is a constant and P_0 represents the pressure at the lower boundary z = 0, and $\beta = \frac{T_0 - T_1}{d}$ and $\delta = \frac{S_0 - S_1}{d}$ are respectively the maintained temperature and concentration gradients.

The initial stationary state is now slightly perturbed so that the perturbed state is given by

$$\left.\begin{aligned} (u,v,w)_{PS} &= (0+u', 0+v', 0+w')\\ (p)_{PS} &= P + P'\\ (T)_{PS} &= T_0 - \beta z + \theta' = T + \theta'\\ (S)_{PS} &= S_0 - \delta z + \phi' = S + \phi'\\ (H_1,H_2,H_3) &= (0+h_x', 0+h_y', H+h_z')\\ (\rho)_{PS} &= \rho_0\left[1 + \alpha(T_0 - T - \theta') - \gamma(S_0 - S - \phi')\right] \end{aligned}\right\} \tag{14}$$

where u', v', w', P', θ', ϕ' denote, respectively, the perturbations in three components of velocity, pressure, temperature and concentration and are assumed to be small around the basic state. Then the linearized perturbation equations are given by

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \tag{15}$$

$$\frac{1}{\epsilon}\frac{\partial u'}{\partial t} - \frac{\mu_e H}{4\pi\rho_0}\frac{\partial h_x'}{\partial z} = -\frac{\partial P'}{\partial x} + \frac{2}{\epsilon}\Omega v' - \frac{\nu}{k_1}u', \tag{16}$$

$$\frac{1}{\epsilon}\frac{\partial v'}{\partial t} - \frac{\mu_e H}{4\pi\rho_0}\frac{\partial h_y'}{\partial z} = -\frac{\partial P'}{\partial y} - \frac{2}{\epsilon}\Omega u' - \frac{\nu}{k_1}v', \tag{17}$$

$$\frac{1}{\epsilon}\frac{\partial w'}{\partial t} - \frac{\mu_e H}{4\pi\rho_0}\frac{\partial h_z'}{\partial z} = -\frac{\partial P'}{\partial z} + g\alpha\theta' - g\gamma\phi' - \frac{\nu}{k_1}w', \tag{18}$$

$$E\frac{\partial \theta'}{\partial t} - \beta w' = \kappa_T \nabla^2\theta', \tag{19}$$

$$E'\frac{\partial \phi'}{\partial t} - \delta w' = \kappa_S \nabla^2\phi', \tag{20}$$

$$\epsilon \frac{\partial h_x'}{\partial t} = H \frac{\partial u'}{\partial z} + \epsilon\eta\nabla^2 h_x', \tag{21}$$

$$\epsilon \frac{\partial h_y'}{\partial t} = H \frac{\partial v'}{\partial z} + \epsilon\eta\nabla^2 h_y', \tag{22}$$

$$\epsilon \frac{\partial h_z'}{\partial t} = H \frac{\partial w'}{\partial z} + \epsilon\eta\nabla^2 h_z' \tag{23}$$

and

$$\frac{\partial h_x'}{\partial x} + \frac{\partial h_y'}{\partial y} + \frac{\partial h_z'}{\partial z} = 0 . \tag{24}$$

Now we analyze the perturbations u', v', w', P', θ', ϕ', h_x', h_y' and h_z' into two-dimensional periodic waves. We assume, to all quantities describing the perturbation, a dependence on x, y, and t of the form

$$F'(x,y,z,t) = F''(z)\exp[i(k_x x + k_y y) + nt] \tag{25}$$

where k_x and k_y are the wave numbers along the x- and y- directions, respectively, and $k = \sqrt{(k_x^2 + k_y^2)}$ is the resultant wave number. Following the normal mode analysis, equations (15) – (24), thus, becomes

$$ik_x u'' + ik_y v'' + \frac{dw''}{dz} = 0, \tag{26}$$

$$\frac{1}{\epsilon} nu'' - \frac{\mu_e H}{4\pi\rho_0}\frac{dh_x''}{dz} = -i k_x P'' + \frac{2}{\epsilon}\Omega v'' - \frac{\nu}{k_1} u'', \tag{27}$$

$$\frac{1}{\epsilon} nv'' - \frac{\mu_e H}{4\pi\rho_0}\frac{dh_y''}{dz} = -i k_y P'' - \frac{2}{\epsilon}\Omega u'' - \frac{\nu}{k_1} v'', \tag{28}$$

$$\frac{1}{\epsilon} nw'' - \frac{\mu_e H}{4\pi\rho_0}\frac{dh_z''}{dz} = -\frac{dP''}{dz} + g\alpha\theta'' - g\gamma\phi'' - \frac{\nu}{k_1} w'', \tag{29}$$

$$En\theta'' - \beta w'' = \kappa_T \left(\frac{d^2}{dz^2} - k^2\right)\theta'' \tag{30}$$

$$E' n\phi'' - \delta w'' = \kappa_S \left(\frac{d^2}{dz^2} - k^2\right)\phi'' \tag{31}$$

$$n\epsilon h_x'' = H\frac{du''}{dz} + \epsilon\eta\left(\frac{d^2}{dz^2} - k^2\right)h_x'' \tag{32}$$

$$n\epsilon h_y'' = H\frac{dv''}{dz} + \epsilon\eta\left(\frac{d^2}{dz^2} - k^2\right)h_y'' \tag{33}$$

$$n\epsilon h_z'' = H\frac{dw''}{dz} + \epsilon\eta\left(\frac{d^2}{dz^2} - k^2\right)h_z'', \tag{34}$$

and

$$ik_x h_x'' + ik_y h_y'' + \frac{dh_z''}{dz} = 0, \tag{35}$$

where $\frac{\partial}{\partial t} = n$, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = -k^2$ and $\nabla^2 = \frac{d^2}{dz^2} - k^2$. (36)

Now multiplying equations (27) and (28) by ik_x and ik_y respectively, adding the resulting equations and using equations (26) and (35), we obtain

$$-\left(\frac{n}{\epsilon} + \frac{\nu}{k_1}\right)\frac{dw''}{dz} + \frac{\mu_e H}{4\pi\rho_0}\frac{d^2 h_z''}{dz^2} = k^2 P'' + \frac{2}{\epsilon}\Omega\zeta''. \tag{37}$$

where $\zeta'' = i(k_x v'' - k_y u'')$ is the z – component of vorticity.

Now eliminating P″ between (29) and (37), we get

$$\left(\frac{n}{\epsilon} + \frac{\nu}{k_1}\right)\left(\frac{d^2}{dz^2} - k^2\right)w'' = \frac{\mu_e H}{4\pi\rho_0}\frac{d}{dz}\left(\frac{d^2}{dz^2} - k^2\right)h_z'' - k^2(g\alpha\theta'' - g\gamma\phi'') - \frac{2}{\epsilon}\Omega\frac{d\zeta''}{dz} \tag{38}$$

In order to obtain an equation governing ζ'', multiplying equations (27) and (28) by ik_y and ik_x respectively and subtracting the resulting equations, we obtain

$$\left(\frac{n}{\epsilon}+\frac{\nu}{k_1}\right)\zeta'' = \frac{\mu_e H}{4\pi\rho_0}\frac{d\xi''}{dz} + \frac{2}{\epsilon}\Omega\frac{dw''}{dz} \tag{39}$$

where $\xi'' = i(k_x h_y'' - k_y h_x'')$ is the z – component of current density.

Similarly, we obtain an equation governing ξ'' we multiply equations (32) and (33) by ik_y and ik_x respectively and subtract the resulting equations, we get

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\eta}\right)\xi'' = -\frac{H}{\epsilon\,\eta}\frac{d\zeta''}{dz} \tag{40}$$

Also equations (30), (31) and (34) can be written as

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{E\,n}{\kappa_T}\right)\theta'' = -\frac{\beta}{\kappa_T}w'' \tag{41}$$

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{E'\,n}{\kappa_S}\right)\phi'' = -\frac{\delta}{\kappa_S}w'' \tag{42}$$

and $$\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\eta}\right)h_z'' = -\frac{H}{\epsilon\,\eta}\frac{dw''}{dz} \tag{43}$$

Now using the following non-dimensional parameters

a = kd, $z_* = \frac{z}{d}$, $\tau_* = \frac{\kappa_S}{\kappa_T}$, $P_{r_*} = \frac{\nu}{\kappa_T}$, $p_{2_*} = \frac{\nu}{\eta}$, $D_{a_*} = \frac{k_1}{d^2}$, $D_* = d\frac{d}{dz}$, $\sigma_* = \frac{nd^2}{\nu}$, $R_* = \frac{g\alpha\beta d^4}{\kappa_T\,\nu}$, $R_{S_*} = \frac{g\gamma\delta d^4}{\kappa_T\,\nu}$, $Q_* = \frac{\mu_e H^2 d^2}{4\pi\rho_0\nu\,\eta}$, $T_{a_*} = \frac{4\Omega^2 d^4}{\nu^2}$, $w_* = \frac{\beta d^2}{\kappa_T}w''$, $\theta_* = \theta''$, $\phi_* = \frac{\beta}{\delta}\phi''$, $\zeta_* = \frac{\beta\nu d}{2\Omega\kappa_T}\zeta''$, $\xi_* = \frac{\beta\,\nu\,\eta}{2\Omega\kappa_T H}\xi''$ and $h_{z_*} = \frac{\eta\beta d}{H\kappa_T}h_z''$.

we can write equations (38) – (43) in the following non-dimensional form(dropping the asterisks for simplicity)

$$\left(\frac{\sigma}{\epsilon}+\frac{1}{D_a}\right)(D^2-a^2)w = -R\,a^2\theta + R_S a^2\phi + Q\,D(D^2-a^2)h_z - \frac{1}{\epsilon}T_a D\zeta \tag{44}$$

$$(D^2 - a^2 - E\,\sigma P_r)\theta = -w\ , \tag{45}$$

$$\left(D^2 - a^2 - \frac{E'\sigma P_r}{\tau}\right)\phi = -\frac{1}{\tau}w\ , \tag{46}$$

$$(D^2 - a^2 - \sigma\,p_2)h_z = -\frac{1}{\epsilon}Dw\ , \tag{47}$$

$$\left(\frac{\sigma}{\epsilon}+\frac{1}{D_a}\right)\zeta = QD\xi + \frac{1}{\epsilon}Dw\ , \tag{48}$$

and $$(D^2 - a^2 - \sigma\,p_2)\xi = -\frac{1}{\epsilon}D\zeta\ . \tag{49}$$

The equations (44) – (49) are to be solved by using the following boundary conditions:

$w = \theta = \phi = Dw = h_z = \zeta = D\xi = 0$ at z = 0 and at z = 1, (50)
(when both the boundaries are rigid and perfectly conducting)

$w = \theta = \phi = D^2w = h_z = D\zeta = D\xi = 0$ at z = 0 and at z = 1. (51)
(when both the boundaries are free and perfectly conducting)

where z is the vertical co-ordinate such that $0 \le z \le 1$, D is the differentiation w.r.t. z, a^2 is square of the wave number, $p_1 > 0$ the Prandtl number, $\tau > 0$ is the Lewis number, $R > 0$ is the Rayleigh number, $R_S > 0$ is the thermohaline Rayleigh number, $\sigma = \sigma_r + i\sigma_i$ is the complex growth rate which is complex constant in general and as a consequence the dependent variables $w(z) = w_r(z) + iw_i(z)$, $\theta(z) = \theta_r(z) + i\theta_i(z)$ and $\phi(z) = \phi_r(z) + i\phi_i(z)$ are complex valued functions of the real variable z such that $w_r(z)$, $w_i(z)$, $\theta_r(z)$, $\theta_i(z)$, $\phi_r(z)$ and $\phi_i(z)$ are real valued functions of the real variable z.

2. MATHEMATICAL ANALYSIS

Theorem1: If $R > 0$, $R_S > 0$, $Q > 0$, $\frac{\tau p_2}{\pi^2 E' P_r} \leq 1$, and $\frac{\tau D_a}{E' P_r \epsilon} \leq 1$, $p_r \geq 0$, $p_i \neq 0$ and $R_S \leq 4\pi^2\left(\frac{1}{D_a} + \frac{\tau}{E'\epsilon P_r}\right)$, then a necessary condition for the existence of nontrivial solution $(w, \theta, \phi, h_z, \sigma)$ of Eqs. (44) – (49) with boundary conditions (50) or (51) is that $R_S < R$.

Proof: Multiplying Eq. (44) by w* (the superscript * here denotes the complex conjugation) and integrating the resulting equation over vertical range of z, we obtain

$$\left(\frac{1}{D_a} + \frac{\sigma}{\epsilon}\right)\int_0^1 w^* (D^2 - a^2) w \, dz = -Ra^2 \int_0^1 w^* \theta \, dz + R_S a^2 \int_0^1 w^* \phi \, dz + Q\int_0^1 w^* D(D^2 - a^2) h_z dz - \frac{T_a}{\epsilon}\int_0^1 w^* D\zeta \, dz \tag{52}$$

Making use of equations (45) – (49) and the fact that w (0) = 0 = w (1), we can write

$$-R a^2 \int_0^1 w^* \theta \, dz = Ra^2 \int_0^1 \theta (D^2 - a^2 - E\sigma^* p_1)\theta^* dz \tag{53}$$

$$R_S a^2 \int_0^1 w^* \phi \, dz = -R_S a^2 \tau \int_0^1 \phi \left(D^2 - a^2 - \frac{E'\sigma^* P_r}{\tau}\right)\phi^* dz \tag{54}$$

$$Q\int_0^1 w^* D(D^2 - a^2) h_z \, dz = Q\epsilon \int_0^1 (D^2 - a^2) h_z (D^2 - a^2 - \sigma^* p_2) h_z^* \, dz \tag{55}$$

$$-\frac{T_a}{\epsilon}\int_0^1 w^* D\zeta \, dz = T_a\left(\frac{1}{D_a} + \frac{\sigma^*}{\epsilon}\right)\int_0^1 \zeta^* \zeta \, dz - T_a Q\epsilon \int_0^1 \xi^* (D^2 - a^2 - \sigma p_2)\xi \, dz \tag{56}$$

Combining equations (52) – (56), we get

$$\begin{aligned}\left(\frac{1}{D_a} + \frac{\sigma}{\epsilon}\right)\int_0^1 w^* (D^2 - a^2) w \, dz = {} & Ra^2 \int_0^1 \theta (D^2 - a^2 - E\sigma^* p_1)\theta^* dz \\ & - R_S a^2 \tau \int_0^1 \phi \left(D^2 - a^2 - \frac{E'\sigma^* P_r}{\tau}\right)\phi^* dz \\ & + Q\epsilon \int_0^1 (D^2 - a^2) h_z (D^2 - a^2 - \sigma^* p_2) h_z^* \, dz \\ & + T_a \left(\frac{1}{D_a} + \frac{\sigma^*}{\epsilon}\right)\int_0^1 \zeta^* \zeta \, dz - T_a Q\epsilon \int_0^1 \xi^* (D^2 - a^2 - \sigma p_2)\xi \, dz\end{aligned} \tag{57}$$

Integrating the various terms of Eq. (57), by parts, for an appropriate number of times and making use of either of the boundary conditions (50) or (51), it follows that

$$\begin{aligned}\left(\frac{1}{D_a} + \frac{\sigma}{\epsilon}\right)\int_0^1 (|Dw|^2 + a^2|w|^2)\, dz = {} & Ra^2 \int_0^1 (|D\theta|^2 + a^2|\theta|^2 + EP_r\sigma^*|\theta|^2)\, dz \\ & - R_S a^2 \tau \int_0^1 \left(|D\phi|^2 + a^2|\phi|^2 + \frac{E'\sigma^* P_r}{\tau}|\phi|^2\right) dz \\ & - Q\epsilon\left[\int_0^1 |(D^2 - a^2) h_z|^2 \, dz + p_2\sigma^* \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\, dz\right] \\ & - T_a \left(\frac{1}{D_a} + \frac{\sigma^*}{\epsilon}\right)\int_0^1 |\zeta|^2 dz - T_a Q\epsilon \int_0^1 (|D\xi|^2 + a^2|\xi|^2 + \sigma p_2 |\xi|^2)\, dz\end{aligned} \tag{58}$$

Equating the real and imaginary parts of both sides of equation (58) and cancelling σ_i ($\neq 0$) throughout from the imaginary part, we get

$$\begin{aligned}\left(\frac{1}{D_a} + \frac{\sigma_r}{\epsilon}\right)\int_0^1 (|Dw|^2 + a^2|w|^2)\, dz = {} & Ra^2 \int_0^1 (|D\theta|^2 + a^2|\theta|^2 + EP_r\sigma_r|\theta|^2)\, dz \\ & - R_S a^2 \tau \int_0^1 \left(|D\phi|^2 + a^2|\phi|^2 + \frac{E'\sigma_r P_r}{\tau}|\phi|^2\right) dz \\ & - Q\epsilon\left[\int_0^1 |(D^2 - a^2) h_z|^2 \, dz + p_2\sigma_r \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\, dz\right] \\ & - T_a \left(\frac{1}{D_a} + \frac{\sigma_r}{\epsilon}\right)\int_0^1 |\zeta|^2 dz - T_a Q\epsilon \int_0^1 (|D\xi|^2 + a^2|\xi|^2 + \sigma_r p_2 |\xi|^2)\, dz\end{aligned} \tag{59}$$

and

$$\begin{aligned}\frac{1}{\epsilon}\int_0^1 (|Dw|^2 + a^2|w|^2)\, dz = {} & -R a^2 E P_r \int_0^1 |\theta|^2 dz + R_S a^2 E' P_r \int_0^1 |\phi|^2 dz \\ & + Q\epsilon p_2 \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\, dz + \frac{T_a}{\epsilon}\int_0^1 |\zeta|^2 dz - T_a Q\epsilon p_2 \int_0^1 |\xi|^2 dz\end{aligned} \tag{60}$$

We write equation (59) in the alternative form

$$\left(\frac{1}{D_a}+\frac{\sigma_r}{\epsilon}\right)\int_0^1(|Dw|^2+a^2|w|^2)dz = Ra^2\int_0^1(|D\theta|^2+a^2|\theta|^2)\,dz - R_S a^2\tau\int_0^1(|D\phi|^2+a^2|\phi|^2)dz$$
$$-Q\epsilon\int_0^1|(D^2-a^2)\,h_z|^2\,dz-\frac{T_a}{D_a}\int_0^1|\zeta|^2\,dz - T_aQ\epsilon\int_0^1(|D\xi|^2+a^2|\xi|^2)\,dz$$
$$+\sigma_r\left[Ra^2\,E\,P_r\int_0^1|\theta|^2\,dz - R_S a^2\,E'P_r\int_0^1|\phi|^2\,dz - Q\epsilon p_2\int_0^1(|Dh_z|^2+a^2|h_z|^2)\,dz\right.$$
$$\left.-\frac{T_a}{\epsilon}\int_0^1|\zeta|^2\,dz - T_aQ\epsilon p_2\int_0^1|\xi|^2\,dz\right] \tag{61}$$

and derive the validity of the theorem from the resulting inequality obtained by replacing each one of the terms of this equation by its appropriate estimate.

We first note that since w, θ, ϕ and h_z satisfy $w(0) = 0 = w(1)$, $\theta(0) = 0 = \theta(1)$, $\phi(0) = 0 = \phi(1)$ and $h_z(0) = 0 = h_z(1)$, we have by the Rayleigh- Ritz inequality [10].

$$\int_0^1|Dw|^2\,dz \geq \pi^2\int_0^1|w|^2\,dz \tag{62}$$

$$\int_0^1|D\theta|^2\,dz \geq \pi^2\int_0^1|\theta|^2\,dz \tag{63}$$

$$\int_0^1|D\phi|^2\,dz \geq \pi^2\int_0^1|\phi|^2\,dz \tag{64}$$

$$\int_0^1|Dh_z|^2\,dz \geq \pi^2\int_0^1|h_z|^2\,dz \tag{65}$$

Utilizing inequality (62), we have

$$\int_0^1(|Dw|^2+a^2|w|^2)\,dz \geq (\pi^2+a^2)\int_0^1|w|^2\,dz. \tag{66}$$

Since $\sigma_r \geq 0$, we have

$$\frac{\sigma_r}{\epsilon}\int_0^1(|Dw|^2+a^2|w|^2)\,dz \geq 0 \tag{67}$$

Multiplying equation (45) by θ^* throughout and integrating the various terms on the left hand side of the resulting equation, by parts, for an appropriate number of times by making use of the boundary conditions on θ, namely, $\theta(0) = 0 = \theta(1)$, we have from the real part of the final equation

$$\begin{aligned}\int_0^1(|D\theta|^2+a^2|\theta|^2)dz+\sigma_r\,E\,P_r\int_0^1|\theta|^2\,dz &= \text{Real part of }\int_0^1 w\,\theta^* dz\\ &\leq \left|\int_0^1 w\;\theta^* dz\right|\\ &\leq \int_0^1|w\theta^*|\,dz\\ &\leq \int_0^1|w|\,|\theta^*|\,dz\\ &\leq \int_0^1|w|\,|\theta|\,dz\\ &\leq \left[\int_0^1|w|^2\,dz\right]^{1/2}\left[\int_0^1|\theta|^2\,dz\right]^{1/2},\end{aligned}$$

(Using Cauchy- Schwartz inequality)

and combining this inequality with the inequality (63) and the fact that $\sigma_r \geq 0$, we get

$$(\pi^2+a^2)\int_0^1|\theta|^2\,dz \leq \left[\int_0^1|w|^2\,dz\right]^{1/2}\left[\int_0^1|\theta|^2\,dz\right]^{1/2}$$

which implies that

$$\left[\int_0^1|\theta|^2\,dz\right]^{1/2} \leq \frac{1}{(\pi^2+a^2)}\left[\int_0^1|w|^2\,dz\right]^{1/2},$$

and thus $\int_0^1(|D\theta|^2+a^2|\theta|^2)dz \leq \frac{1}{(\pi^2+a^2)}\int_0^1|w|^2\,dz$. (68)

Further, since $h_z(0) = 0 = h_z(1)$, we have

$$\int_0^1|Dh_z|^2dz = -\int_0^1 h_z^* D^2 h_z\,dz \leq \left|-\int_0^1 h_z^* D^2 h_z\,dz\right| \leq \int_0^1|h_z^* D^2 h_z|dz$$

$$\leq \int_0^1 |h_z^*||D^2 h_z| dz \leq \int_0^1 |h_z||D^2 h_z| dz \leq \left[\int_0^1 |h_z|^2 \, dz\right]^{1/2} \left[\int_0^1 |D^2 h_z|^2 \, dz\right]^{1/2}$$

(Using Schwartz inequality)

$$\leq \frac{1}{\pi}\left[\int_0^1 |Dh_z|^2 \, dz\right]^{1/2} \left[\int_0^1 |D^2 h_z|^2 \, dz\right]^{1/2},$$

(Using inequality (65))

so that we have

$$\int_0^1 |D^2 h_z|^2 \, dz \geq \pi^2 \int_0^1 |Dh_z|^2 dz \,, \tag{69}$$

and thus we can write

$$Q\epsilon \int_0^1 |(D^2 - a^2) h_z|^2 \, dz = Q\epsilon \int_0^1 (|D^2 h_z|^2 + 2a^2 |Dh_z|^2 + a^4 |h_z|^2) \, dz$$

$$\geq Q\epsilon \left[\pi^2 \int_0^1 |D h_z|^2 dz + a^2 \int_0^1 |Dh_z|^2 dz + a^2 \int_0^1 |Dh_z|^2 dz + a^4 \int_0^1 |h_z|^2 dz \right]$$

(Using inequality (69))

$$\geq Q\epsilon \left[(\pi^2 + a^2) \int_0^1 |D h_z|^2 dz + a^2 (\pi^2 + a^2) \int_0^1 |h_z|^2 dz \right]$$

(Using inequality (65))

$$\geq Q\epsilon \, (\pi^2 + a^2) \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) \, dz$$

$$\geq Q\epsilon \, \pi^2 \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) \, dz \tag{70}$$

Utilizing inequality (64), we have

$$\int_0^1 (|D\phi|^2 + a^2 |\phi|^2) \, dz \geq (\pi^2 + a^2) \int_0^1 |\phi|^2 dz \geq \int_0^1 |\phi|^2 dz \tag{71}$$

From equation (60), we have

$$\int_0^1 |\phi|^2 dz \geq \frac{1}{R_S a^2 E' P_r} \left[\frac{1}{\epsilon} \int_0^1 (|Dw|^2 + a^2 |w|^2) dz - Q\epsilon p_2 \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) dz - \frac{T_a}{\epsilon} \int_0^1 |\zeta|^2 dz\right] \tag{72}$$

Combining inequalities (71) and (72), we have

$$\int_0^1 (|D\phi|^2 + a^2 |\phi|^2) \, dz \geq \frac{1}{R_S a^2 E' P_r} \left[\frac{1}{\epsilon} \int_0^1 (|Dw|^2 + a^2 |w|^2) dz - Q\epsilon p_2 \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) dz - \frac{T_a}{\epsilon} \int_0^1 |\zeta|^2 dz\right] \tag{73}$$

Using inequalities (62) in inequality (73), we obtain

$$\int_0^1 (|D\phi|^2 + a^2 |\phi|^2) \, dz \geq \frac{(\pi^2 + a^2)}{R_S a^2 E' P_r \epsilon} \int_0^1 |w|^2 dz - \frac{Q\epsilon p_2}{R_S a^2 E' P_r} \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) dz - \frac{T_a}{R_S a^2 E' P_r \epsilon} \int_0^1 |\zeta|^2 dz \tag{74}$$

From equation (60) and the fact that $\sigma_r \geq 0$, we have

$$\sigma_r \Big[R a^2 E P_r \int_0^1 |\theta|^2 \, dz - R_S a^2 E' P_r \int_0^1 |\phi|^2 \, dz - Q\epsilon p_2 \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) \, dz - \frac{T_a}{\epsilon} \int_0^1 |\zeta|^2 \, dz - T_a Q\epsilon p_2 \int_0^1 |\xi|^2 \, dz\Big] \leq 0 \tag{75}$$

Now, if permissible, let $R_S \geq R$, then from equation (61) and inequalities (66), (68), (74) and (75), we get

$$\frac{(\pi^2 + a^2)}{D_a} \int_0^1 |w|^2 dz + \frac{\tau (\pi^2 + a^2)}{E' P_r \epsilon} \int_0^1 |w|^2 dz - \frac{\tau Q\epsilon p_2}{E' P_r} \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) \, dz - \frac{T_a \tau}{E' P_r \epsilon} \int_0^1 |\zeta|^2 \, dz - \frac{R_S a^2}{(\pi^2 + a^2)} \int_0^1 |w|^2 dz + Q\epsilon \pi^2 \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) \, dz + \frac{T_a}{D_a} \int_0^1 |\zeta|^2 \, dz \leq 0, \tag{76}$$

which implies that

$$\left[\frac{(\pi^2 + a^2)^2}{a^2} \left(\frac{1}{D_a} + \frac{\tau}{E' p_1 \epsilon}\right) - R_S\right] \int_0^1 |w|^2 dz + \frac{Q\epsilon \pi^2 (\pi^2 + a^2)}{a^2} \left(1 - \frac{\tau p_2}{\pi^2 E' P_r}\right) \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) \, dz + \frac{T_a (\pi^2 + a^2)}{a^2 D_a} \left(1 - \frac{\tau D_a}{E' P_r \epsilon}\right) \int_0^1 |\zeta|^2 \, dz \leq 0. \tag{77}$$

Using the assumption, $\frac{\tau D_a}{E' P_r \epsilon} \leq 1$ and $\frac{\tau p_2}{\pi^2 E' P_r} \leq 1$, we have from inequality (77) that

$$R_S > 4\pi^2 \left(\frac{1}{D_a} + \frac{\tau}{E' P_r \epsilon}\right), \tag{78}$$

Since, minimum value of $\frac{(\pi^2 + a^2)^2}{a^2}$ is $4\pi^2$ (for $a^2 = \pi^2$). Hence if $R_S \leq 4\pi^2 \left(\frac{1}{D_a} + \frac{\tau}{E' P_r \epsilon}\right)$, then we must have $R_S < R$, and this completes the proof of the theorem.

Theorem 1 can be stated in an equivalent form as ‘magnetorotatory thermohaline convection of the Veronis type in porous medium cannot manifest itself as oscillatory motions of growing amplitude in an initially bottom heavy configuration if R_S, τ, P_r, ϵ, D_a and E' satisfy the inequality

$$R_S \leq 4\pi^2 \left(\frac{1}{D_a} + \frac{\tau}{E' P_r \epsilon}\right).$$

Further, this result is uniformly valid for any combination of rigid and free perfectly conducting boundaries.

A similar theorem can be proved for magnetorotatory thermohaline convection of Stern [3] type in the porous medium as follows:

Theorem 2: If $R < 0$, $R_S < 0$, $Q > 0$, $\frac{p_2}{\pi^2 E P_r} \leq 1$, , $\frac{D_a}{E P_r \epsilon} \leq 1$ and $\sigma_r \geq 0$, $\sigma_i \neq 0$ and $|R| \leq 4\pi^2 \tau \left(\frac{1}{D_a} + \frac{1}{E P_r \epsilon}\right)$ then we must have $|R| < |R_S|$.

Proof: Replacing R and R_S by $-|R|$ and $-|R_S|$, respectively, in equations (44) and proceeding exactly as in Theorem 1, we get the desired result.

Theorem 2 can be stated in an equivalent form as ‘Magnetorotatory thermohaline convection of Stern type cannot manifest itself as oscillatory motions of growing amplitude in an initially bottom heavy configuration if $|R|$, τ, ϵ, P_r, D_a and E satisfy the inequality

$$|R| \leq 4\pi^2 \tau \left(\frac{1}{D_a} + \frac{1}{E P_r \epsilon}\right).$$

REFERENCES

[1] J. S. Turner, Double-diffusive phenomina, Ann. Rev. Fluid Mech. **6**(1974) 37 – 54.
[2] G. Veronis, On finite amplitude instability in thermohaline convection, J. Mar. Res. **23**(1965), 1 – 17.
[3] M. E. Stern, The salt fountain and thermohaline convection, Tellus **12**(1960), 172 – 175.
[4] M. B. Banerjee, J. R. Gupta and Jyoti Prakash, On thermohaline convection of the Veronis type, J. Math. Anal. Appl. **179**(1993), 327 – 334.
[5] E. R. Lapwood, Convection of a fluid in a porous medium, *Proc. Camb. Phil. Soc.***44** (1948), 508 – 521.
[6] R. A. Wooding, Rayleigh instability of a thermal boundary layer in flow through a porous medium, J. *Fluid Mech.* **9** (1960), 183 – 192.
[7] J. W. Tounton and F. N. Lightfoot, Thermohaline instability and salt fingers in a porous medium, *The Phys. Fluids*, **15**(5) (1972), 748 – 753.
[8] D. A. Nield and A. Bejan, *Convection in porous Media*, Springer (2006).
[9] B. Straughan, *Stability and wave motion in porous media*, Springer (2008).
[10] M. H. Schultz, *Spline Analysis*, Prentice Hall, Englewood Cliffs, NJ, (1973).

Source of support: Nil, Conflict of interest: None Declared