International Journal of Mathematical Archive-3(12), 2012, 4898-4902
Available online through www.ijma.info ISSN 2229 – 5046

AN INEQUALITY FOR CERTAIN ANALYTIC FUNCTIONS DEFINED BY A NEW MULTIPLIER TRANSFORMATION

S R SWAMY[1]*, ROHAN R[1] AND NIRMALA J[2]

[1]Department of Computer Science and Engineering, R V College of Engineering, Mysore Road, Bangalore - 560 059, India

[2]Department of Mathematics, Maharani's Science College for women, Palace road, Bangalore-560 001, India

(Received on: 16-11-12; Revised & Accepted on: 19-12-12)

ABSTRACT

In this paper, the authors investigate an interesting differential inequality of certain analytic functions, defined by a new multiplier transformation in the open unit disc $U = \{z : |z| < 1\}$.

2000 Mathematical Subject Classification: *30C45.*

Key words and Phrases: *Multivalent function, Starlike function, Convex function, Multiplier transformation.*

1. INTRODUCTION

Let $A_p(n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, (p, n \in N = \{1,2,3...\}). \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. In particular, we set $A_p(1) = A_p$, $A_1(n) = A(n)$ and $A_1(1) = A_1 = A(1) = A$, a well-known class of normalized analytic functions in U.

For $0 \le \rho < p$, $p, n \in N$, we denote by $S_p^*(n, \rho)$ and $K_p(n, \rho)$ the subclasses of $A_p(n)$ consisting of all analytic functions which are, respectively, n-p-valent starlike of order ρ and n-p-valent convex of order ρ. Note that $S_1^*(1, \rho) = S^*(\rho)$ and $K_1(1, \rho) = K(\rho)$ are, respectively, the usual classes of univalent starlike functions of order ρ and univalent convex functions of order ρ, $0 \le \rho < 1$. We also write $S^*(0) = S^*$ and $K(0) = K$, which are the classes of univalent starlike (w.r.t origin) and univalent convex functions in U.

Definition 1.1. ([18, 19]). Let $p, n \in N$, $m \in N_0 = N \cup \{0\}$, $\beta \ge 0$ and α a real number with $\alpha + p\beta > 0$. Then we define the operator $I_{p,\alpha,\beta}^m : A_p(n) \to A_p(n)$, by

$$I_{p,\alpha,\beta}^0 f(z) = f(z), (f \in A_p(n)),$$

$$I_{p,\alpha,\beta}^1 f(z) = \frac{\alpha f(z) + \beta z f'(z)}{\alpha + p\beta},$$

$$\ldots,$$

$$I_{p,\alpha,\beta}^m f(z) = I_{p,\alpha,\beta}(I_{p,\alpha,\beta}^{m-1} f(z)).$$

Remark 1.2. We observe that $I_{p,\alpha,\beta}^m$ is a linear operator and for $f(z)$ given by (1.1), we have

Corresponding author: *S R SWAMY[1]*, [1]Department of Computer Science and Engineering, R V College of Engineering, Mysore Road, Bangalore - 560 059, India.*

$$I^m_{p,\alpha,\beta} f(z) = z^p + \sum_{k=p+n}^{\infty} \left(\frac{\alpha + k\beta}{\alpha + p\beta} \right)^m a_k z^k. \tag{1.2}$$

It follows from (1.2) that

$$I^m_{p,\alpha,0} f(z) = f(z), \tag{1.3}$$

$$(\alpha + p\beta) I^{m+1}_{p,\alpha,\beta} f(z) = \alpha I^m_{p,\alpha,\beta} f(z) + \beta z (I^m_{p,\alpha,\beta} f(z))', \tag{1.4}$$

We note that

- $I^m_{1,\alpha,\beta} f(z) = I^m_{\alpha,\beta} f(z)$ (See [17]).
- $I^m_{p,\alpha,1} f(z) = I^m_p(\alpha) f(z), \alpha > -p$ (See [1], [14] and [16]).
- $I^m_{p,l+p-p\beta,\beta} f(z) = I^m_p(\beta, l) f(z), l > -p, \beta \ge 0$ (See Catas [6]).
- $I^m_{p,0,\beta} f(z) = D^m_p f(z)$ (See [4], [9] and [12]).

Remark 1.3 i) $I^m_p(\alpha) f(z)$ was considered in [1],[14] and [16], for $\alpha \ge 0$ and $I^m_p(\beta, l) f(z)$ was defined in [6] for $l \ge 0, \beta \ge 0$, ii) $I^m_p(l) f(z) = I^m_p(1, l) f(z), l > -p$, iii) $I^m_p(\beta, 0) f(z) = D^m_p(\beta) f(z)$, $\beta \ge 0,$ was mentioned in Aouf et.al. [3], iv) $D^m_1(\beta), \beta \ge 0,$ was introduced by Al-Oboudi [2],v) $D^m_1(1) f(z) = D^m f(z)$ was defined by Salagean [13] and was considered for $m \ge 0$ in [5] , vi) $I^m_1(\alpha) f(z), \alpha \ge 0,$ was investigated in [7] and [8] and vii) $I^m_1(1)$ was due to Uralegaddi and Somanatha[20].

Definition 1.4. A function $f \in A_p(n)$ is said to be in the class $S^m_p(n, \alpha, \beta, \rho)$ for all z in U if it satisfies

$$\mathrm{Re}\left(\frac{I^{m+1}_{p,\alpha,\beta} f(z)}{I^m_{p,\alpha,\beta} f(z)} \right) > \frac{\rho}{p}, \tag{1.3}$$

where $0 \le \rho < p,\ p, n \in N,\ m \in N_0 = N \cup \{0\}$, $\beta \ge 0$ and α a real number with $\alpha + p\beta > 0$.

We note that $S^m_p(1, \alpha, 1, \rho)$ is the class considered in [15] for $\alpha \ge 0.$ we also note that

$$S^0_p(n,0,1,\rho) = S^*_p(n,\rho) \text{ and } S^1_p(n,0,1,\rho) = K_p(n,\rho).$$

The basic tool in proving our result is the following lemma.

Lemma 1.5 [10, 11]). Let Ω be a set in the complex plane C. Suppose that the function $\Psi : C^2 \times U \to C$ satisfies the condition $\Psi(ix_2, y_1; z) \notin \Omega$ for all $z \in U$ and for all real x_2 and y_1 such that

$$y_1 \le \frac{n}{2}(1 + x_2^2).$$

If p (z) = 1 + $c_n z^n$ + … is analytic in E and for $z \in$ E, $\psi(p(z), zp'(z); z) \subset \Omega,$ then $\mathrm{Re}(p(z)) > 0$ in U .

2. MAIN RESULTS

Theorem 2.1. Let $\lambda \ge 0, \gamma \le 1, 0 \le \rho < p$ be real numbers such that $\gamma \le \lambda$ and $\gamma\left(1 - \frac{\rho}{p}\right) < \frac{1}{2}$. Let $\beta \ge 0, \alpha$ a real number such that $\alpha + p\beta > 0$ and let

$$M(p,n,\lambda,\gamma,\alpha,\beta,\rho) = \frac{(1-\lambda)(\rho/p) + \lambda(\rho/p)^2 - \dfrac{n\lambda\beta(1-(\rho/p))}{2(\alpha+p\beta)}}{1 - \gamma(1-(\rho/p))}.$$

If $f \in A_p(n)$ satisfies the condition

$$\mathrm{Re}\left(\frac{(1-\lambda)I^{m+1}_{p,\alpha,\beta}f(z)+\lambda I^{m+2}_{p,\alpha,\beta}f(z)}{(1-\gamma)I^{m}_{p,\alpha,\beta}f(z)+\gamma I^{m+1}_{p,\alpha,\beta}f(z)}\right) > M(p,n,\lambda,\gamma,\alpha,\beta,\rho), \tag{2.1}$$

then

$$\mathrm{Re}\left(\frac{I^{m+1}_{p,\alpha,\beta}f(z)}{I^{m}_{p,\alpha,\beta}f(z)}\right) > \frac{\rho}{p}$$

i.e., $f(z) \in S^m_p(n,\alpha,\beta,\rho)$.

Proof. Since ,$0 \le \rho < p$, let us write $\tau = \dfrac{\rho}{p}$.Thus, we have ,$0 \le \tau < 1$. Now we define

$$\frac{I^{m+1}_{p,\alpha,\beta}f(z)}{I^{m}_{p,\alpha,\beta}f(z)} = \tau + (1-\tau)p(z), z \in U. \tag{2.2}$$

Therefore $p(z)$ is analytic in U and $p(0)=1$. Differentiating (2.2) logarithmically and using (1.4), we obtain

$$\frac{I^{m+2}_{p,\alpha,\beta}f(z)}{I^{m+1}_{p,\alpha,\beta}f(z)} = [\tau+(1-\tau)p(z)] + \frac{(1-\tau)\beta zp'(z)}{(\alpha+p\beta)[\tau+(1-\tau)p(z)]}. \tag{2.3}$$

A simple calculation yields

$$\left(\frac{(1-\lambda)I^{m+1}_{p,\alpha,\beta}f(z)+\lambda I^{m+2}_{p,\alpha,\beta}f(z)}{(1-\gamma)I^{m}_{p,\alpha,\beta}f(z)+\gamma I^{m+1}_{p,\alpha,\beta}f(z)}\right) = \psi(p(z), zp'(z); z),$$

where

$$\psi(p(z), zp'(z); z) = \frac{(1-\lambda)[\tau+(1-\tau)p(z)] + \lambda[\tau+(1-\tau)p(z)]^2 + \dfrac{(1-\tau)\lambda\beta}{(\alpha+p\beta)}zp'(z)}{(1-\gamma)+\gamma[\tau+(1-\tau)p(z)]}. \tag{2.4}$$

Let $p(z) = u_1 + iu_2$ and $zp'(z) = v_1 + iv_2$, where u_1, u_2, v_1, v_2 are real numbers with $v_1 \le -\dfrac{n}{2}(1+u_2^2)$. Then we have

$$\mathrm{Re}(\psi(iu_2, v_1; z)) \le \frac{S + Ru_2^2}{(1-\gamma+\gamma\tau)^2 + \gamma^2(1-\tau)^2u_2^2} = \phi(u_2) \le \max\phi(u_2), \tag{2.5}$$

where,

$$S = \left[(1-\lambda)\tau + \lambda\tau^2 - \frac{n\lambda\beta(1-\tau)}{2(\alpha+p\beta)}\right](1-\gamma+\gamma\tau)$$

and

$$R = (1-\tau)^2(\gamma-\lambda+\lambda\gamma\tau) - (1-\gamma+\gamma\tau)\frac{n\lambda\beta(1-\tau)}{2(\alpha+p\beta)}.$$

It can be easily verified that $\phi'(u_2) = 0$ implies $u_2 = 0$ and $\phi''(0) < 0$, under the given conditions. Therefore,

$$\max\phi(u_2) = \phi(0) = M(p,n,\lambda,\gamma,\alpha,\beta,\rho). \tag{2.6}$$

Let

$$\Omega = \{w : \mathrm{Re}(w) > M(p,n,\lambda,\gamma,\alpha,\beta,\rho)\}.$$

Then from (2.1) and (2.4), we have $\psi(p(z), zp'(z); z) \in \Omega$ for all $z \in U$, but $\psi(iu_2, v_1; z) \notin \Omega$, in view of (2.5) and (2.6). Therefore, by Lemma 2.1 and (2.2), we conclude that

$$\operatorname{Re}\left(\frac{I_{p,\alpha,\beta}^{m+1} f(z)}{I_{p,\alpha,\beta}^{m} f(z)}\right) > \frac{\rho}{p}.$$

Remark 2.2. Taking $\beta = n = 1$ in Theorem 2.1, we obtain Theorem 2.2 of Singh et.al. [15] (Considered for $\alpha \geq 0$). Our result hold true for $\alpha > -p$.

$\alpha = 0$ in Theorem 2.1 yields

Corollary 2.3. Let $\lambda \geq 0, \gamma \leq 1, 0 \leq \rho < p$ be real numbers such that $\gamma \leq \lambda$ and $\gamma\left(1-\frac{\rho}{p}\right) < \frac{1}{2}$.

Let

$$T(p,n,\lambda,\gamma,\rho) = \frac{(1-\lambda)(\rho/p) + \lambda(\rho/p)^2 - \frac{n\lambda(1-(\rho/p))}{2}}{1-\gamma(1-(\rho/p))}.$$

If $f \in A_p(n)$ satisfies the condition

$$\operatorname{Re}\left(\frac{(1-\lambda)D_p^{m+1} f(z) + \lambda D_p^{m+2} f(z)}{(1-\gamma)D_p^{m} f(z) + \gamma D_p^{m+1} f(z)}\right) > T(p,n,\lambda,\gamma,\rho),$$

then

$$\operatorname{Re}\left(\frac{D_p^{m+1} f(z)}{D_p^{m} f(z)}\right) > \frac{\rho}{p}.$$

Remark 2.4. In the case when $p = n = 1$, Corollary 2.3 reduces to Corollary 3.1 of Singh *et. al* [15].

Taking $\alpha = l + p - p\beta, l > -p$, in Theorem 2.1, we obtain

Corollary 2.5. Let $\lambda \geq 0, \gamma \leq 1, 0 \leq \rho < p$ be real numbers such that $\gamma \leq \lambda$ and $\gamma\left(1-\frac{\rho}{p}\right) < \frac{1}{2}$. Let $\beta \geq 0$ and $l > -p$ and let

$$\mathrm{T}(p,n,\lambda,\gamma,l,\beta,\rho) = \frac{(1-\lambda)(\rho/p) + \lambda(\rho/p)^2 - \frac{n\lambda\beta(1-(\rho/p))}{2(l+p)}}{1-\gamma(1-(\rho/p))}.$$

If $f \in A_p(n)$ satisfies the condition

$$\operatorname{Re}\left(\frac{(1-\lambda)I_p^{m+1}(l,\beta) f(z) + \lambda I_p^{m+2}(l,\beta) f(z)}{(1-\gamma)I_p^{m}(l,\beta) f(z) + \gamma I_p^{m+1}(l,\beta) f(z)}\right) > \mathrm{T}(p,n,\lambda,\gamma,l,\beta,\rho),$$

then

$$\operatorname{Re}\left(\frac{I_p^{m+1}(l,\beta) f(z)}{I_p^{m}(l,\beta) f(z)}\right) > \frac{\rho}{p}.$$

REFERENCES

[1] R. Aghalary, R. M. Ali , S. B. Joshi and V. Ravichandran, Inequalities for functions defined by certain linear operator, Int. j. Math. Sci., **4**, no.2 (2005), 267-274.

[2] F. M. Al-Oboudi, On univalent functions defined by a generalized Salagean operator, Int. J. Math.Math. Sci., 27(2004), 1429 - 1436.

[3] M. K. Aouf, R. M. El-Ashwah and S. M. El-Deeb, Some inequalities for certain p-valent functions involving extended multiplier transformations, Proc. Pak. Acad. Sci.,**46**(4)(2009), 217-221.

[4] M. K. Aouf and A. O. Mostafa, On a subclasses of n-p-valent prestarlike, functions, Comput. Math. Appl., **55**(2008), 851-861.

[5] S. S. Bhoosnurmath and S. R. Swamy, On certain classes of analytic functions, Soochow J. Math., 20, no.1, (1994), 1 - 9.

[6] A. Catas, On certain class of p-valent functions defined by new multiplier transformations, Proceedings book of the international symposium on geometric function theory and applications, August, 20-24, 2007, TC Isambul Kultur Univ., Turkey, 241 - 250.

[7] N. E. Cho and H. M. Srivastava , Argument estimates of certain analytic functions defined by a class of multiplier transformations, Math. Comput. Modeling, 37(1-2) (2003), 39 - 49.

[8] N. E. Cho and T. H. Kim, Multiplier transformations and strongly Close-to-Convex functions, Bull. Korean Math. Soc., 40(3) (2003), 399 - 410.

[9] M. Kamali and H. Orhan, On a subclass of certain starlike functions with negative coefficients, Bull. Korean Math. Soc., **41**(2004), 53-71.

[10] S. S. Miller and P. T. Mocanu, Differential Subordinations and inequalities in the complex plane , J. Diff. Eqns.,, 67(1987), 199 –211.

[11] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Text Books in Pure and Applied Mathematics (N.225), Marcel Dekker, New York and Besel, 2000.

[12] H. Orhan and H. Kiziltunc, A generalization on subfamily of p-valent functions with negative coefficients, Appl. Math. Comput. **155**(2004)521-530.

[13] G. St. Salagean, Subclasses of univalent functions, Proc. Fifth Rou. Fin. Semin. Buch. Complex Anal., Lect. notes in Math., Springer –Verlag, Berlin, 1013(1983), 362 - 372.

[14] S. Shivaprasad Kumar, H. C. Taneja and V. Ravichandran, Classes of Multivalent functions defined by Dziok-Srivastava linear operator and multiplier transformation, Kyungpook Math. J., **46**(2006), no.1, 97-109.

[15] S. Singh, S. Gupta and S. Singh, On starlikeness and convexity of analytic functions satisfying a differential inequality, J. Ineq. Pure. Appl. Math. , 9, no.3, (2008), Article 81, 7 pp.

[16] H. M. Srivastava, K. B. Suchitra, A. Stephen and S. Sivasubramanian, Inclusion and neighborhood properties of certain subclasses of multivalent functions of complex order, JIPAM, **7**,Issue (2006), article 7,1-8

[17] S. R. Swamy, Inclusion properties of certain subclasses of analytic functions, Int. Math. Forum, **7,** no.36, (2012), 1751 – 1760.

[18] S. R. Swamy, Inclusion properties of certain subclasses of analytic functions defined by a generalized multiplier transformation, Int. J. Math. Anal., **6**, no. 32, (2012), 1553 – 1564.

[19] S. R. Swamy, Some properties of analytic functions defined by a new generalized multiplier transformation, J. Math, Comput, Sci. **2**, no.3, (2012), 759-767.

[20] B. A. Uralegaddi and C. Somanatha, Certain classes of univalent functions, Current topics in analytic function theory, World Sci. Publishing, River Edge, N. Y., (1992), 371 – 374.

Source of support: Nil, Conflict of interest: None Declared