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ON SOME FIXED POINT THEOREMS IN 2–UNIFORM SPACES

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ABSTRACT

In this paper, some fixed point theorems in 2–uniform spaces are established and contraction type mappings in 2 – uniform spaces are introduced.

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Key words: *Pseudo 2 – metric, Uniformity, 2 – uniform space, Contraction type mapping.*

INTRODUCTION

In this paper we introduce contraction type mappings in 2 – uniform spaces and we present some fixed point theorems of operators in 2 – uniform spaces. These theorems generalize the results of many authors such as *Lal* and *Singh*, *Das* and *Sharma* etc.

In what follows X and $\mathbb{R}$ stand for a non-empty set and the real line respectively and $X^3 = X \times X \times X$. If A and B are any two sets then by the symbol $A \leq B$ we mean that A is contained in B.

1. PRELIMINARIES

1.1 Definition: A pseudo 2–metric for X is a mapping $\rho : X^3 \to \mathbb{R}$ such that for all a, b, c and d in X we have

(i) $\rho(a,b,c) > 0$ and $\rho(a,b,c) = 0$ if at least two of a,b,c are equal.

(ii) $\rho(a,b,c) = \rho(b,c,a) = \rho(c,a,b) = \ldots$

(iii) $\rho(a,b,c) \leq \rho(a,b,d) + \rho(a,d,c) + \rho(d,b,c)$

A set X together with a pseudo 2 – metric ρ is called a pseudo 2 – metric space (X, ρ).

1.2 Definition: If U is any subset of X^3 then $U^{-1} = \{(z,y,x) / (x,y,z) \in U\}$. We define the diagonal of X^3 to be the set $\Delta = \{(x,x,x) / x \in X\}$.

1.3 Definition: A 2 – uniformity for X is a non-void family $\mathscr{U}$ of subsets of X^3 such that

(i) each member of $\mathscr{U}$ contains Δ

(ii) if $U \in \mathscr{U}$ then $V \circ V \circ V \leq U$ for some V in $\mathscr{U}$

(iii) if U and V are two members of U then $U \bigcap V \in \mathscr{U}$

(iv) if $U \in \mathscr{U}$ and $U \leq V \leq X^3$ then $V \in \mathscr{U}$

By a 2 – uniform space we mean a non-empty set X together with a 2 – uniformity $\mathscr{U}$ on X and we denote it by $(X, \mathscr{U})$.

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1.4 Definition: If $(X,\mathscr{U})$ is 2 – uniform space then a subset $\mathscr{B}$ of $\mathscr{U}$ is called a basis for $(X,\mathscr{U})$

(i) if each member of $\mathscr{B}$ contains the diagonal Δ

(ii) $U\in\mathscr{B}$ then U^{-1} contains a member of $\mathscr{B}$

(iii) if $U\in\mathscr{B}$ then $V\circ V\circ V\le U$ for some V in $\mathscr{B}$

(iv) the intersection of two members of $\mathscr{B}$ contains a member of $\mathscr{B}$

1.5 Remark: By $V\circ V\circ V$ we mean that the composition by treating V as a relation in the order $V:X\to X\times X$, $V:X\times X\to X$ and $V:X\to X\times X$ respectively.

1.6 Definition: A 2 – uniform space $(X,\mathscr{U})$ is said to be sequentially complete if every cauchy sequence in X converges to a point in X.

1.7 Definition: If (X,ρ) is a pseudo 2–metric space and if r is a positive real number then we define $V_{(\rho,r)}=\left\{(x,y,z)\in X^3 / \rho(x,y,z)<r\right\}$.

1.8 Notation:

1. We denote $\mathcal{P}$ for the family of pseudo 2 – metrics on X generating the uniformity.

2. $\mathcal{V}$ denotes family of all sets of the form $\bigcap_{i=1}^{n} V_{(\rho_i,r_i)}$ where $\rho_i\in\mathcal{P}$ and r_i is a positive real number for $i=1,2,3,...n$ (n is not fixed).

3. If $V\in\mathcal{V}$ then $V=\bigcap_{i=1}^{n} V_{(\rho_i,r_i)}$. If α is positive then $\alpha V=\bigcap_{i=1}^{n} V_{(\rho_i,\alpha r_i)}$.

1.9 Definition: Let $\mathscr{B}$ be a basis for the 2 – uniform space $(X,\mathscr{U})$ and let f be a mapping from X into itsself.

(a) f said to be a contraction with respect to $\mathscr{B}$ if $\left(f(x),f(y),z\right)\in U$ whenever $(x,y,z)\in U\in\mathscr{B}$.

(b) f said to be an expansion with respect to $\mathscr{B}$ if $(x,y,z)\in U$ whenever $\left(f(x),f(y),z\right)\in U\in\mathscr{B}$.

2. SOME PRELIMINARY LEMMAS

2.1 Lemma: If $V\in\mathcal{V}$ and α,β are positive then $\alpha(\beta V)=(\alpha\beta)V$.

2.2 Lemma: If $V\in\mathcal{V}$ and α,β are positive then $\alpha V\le\beta V$ whenever $\alpha\le\beta$.

2.3 Lemma: Let ρ be any pseudo 2 – metric on X and α,β be any two positive real numbers. If $(x,y,z)\in\alpha V_{(\rho,r_1)}\circ\beta V_{(\rho,r_2)}$ then $\rho(x,y,z)<\alpha r_1+\beta r_2$.

2.4 Lemma: If $V\in\mathcal{V}$ and α,β are two positive real numbers then $\alpha V\circ\beta V\le(\alpha-\beta)V$.

2.5 Lemma: Let $(x,y,z)\in X^3$. Then for every $V\in\mathcal{V}$ there exists a positive real number α such that $(x,y,z)\in\alpha V$.

2.6 Lemma: If $V\in\mathcal{V}$ then there exists a pseudo 2 – metric ρ on X such that $V=V_{(\rho,1)}$.

3. SOME FIXED POINT THEOREMS OF OPERATORS

In this section, we assume that $(X,\mathscr{U})$ is a 2 – uniform space which is also sequentially complete Hausdorff space.

3.1 Theorem: Let $\mathcal{A}=\left\{S_1,S_2,...,S_p\right\}$ and $\mathcal{B}=\left\{T_1,T_2,...,T_q\right\}$ be two sets of operators such that

(a) each S_i and T_j map X into itself

(b) $S_iS_j=S_jS_i$ for $1\le i,j\le p$ and $T_\alpha T_\beta=T_\beta T_\alpha$ for $1\le\alpha,\beta\le q$

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(c) for all $x, y \in X$, for every $a \in X$ and each $\rho \in \mathcal{P}$ and for any five members V_1, V_2, V_3, V_4, V_5 in $\mathcal{V}$, $(S(x), T(y), a) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ \alpha_4 V_4 \circ \alpha_5 V_5$ where $S \in \mathcal{A}$ and $T \in \mathcal{B}$ and each α_i is a non negative real number independent of $x, y, a, V_1, V_2, V_3, V_4$ and V_5 such that

$$0 < \frac{\alpha_1 + \alpha_3 + \alpha_5}{1 - \alpha_2 - \alpha_3}, \frac{\alpha_2 + \alpha_4 + \alpha_5}{1 - \alpha_1 - \alpha_4} < 1, 1 - \alpha_2 - \alpha_3 \neq 0, 1 - \alpha_1 - \alpha_4 \neq 0.$$

If $(x, S(x), a) \in V_1, (y, T(x), a) \in V_2, (x, T(y), a) \in V_3, (y, S(x), a) \in V_4, (x, y, a) \in V_5$ then all $S_i (1 \le i \le p)$ and $T_j (1 \le j \le q)$ have a common unique fixed point.

Proof: Clearly $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$. Suppose that $k_1 = \frac{\alpha_1 + \alpha_3 + \alpha_5}{1 - \alpha_2 - \alpha_3}$ and $k_2 = \frac{\alpha_2 + \alpha_4 + \alpha_5}{1 - \alpha_1 - \alpha_4}$. Let $V \in \mathcal{V}$ and $\rho \in \mathcal{P}$ suppose that x, y, a are any three points of X.. Put $\rho(x, S(x), a) = r_1$, $\rho(y, T(y), a) = r_2, \rho(x, T(y), a) = r_3, \rho = (y, S(x), a) = r_4$ and $\rho(x, y, a) = r_5$ and take $\varepsilon > 0$, then $(x, S(x), a) \in (r_1 + \varepsilon) V$, $(y, T(y), a) \in (r_2 + \varepsilon) V$, $(x, S(x), a) \in (r_3 + \varepsilon) V$, $(y, T(y), a) \in (r_4 + \varepsilon) V$, $(x, y, a) \in (r_5 + \varepsilon) V$.

Then we have

$$(S(x), T(y), a) \in \alpha_1 (r_1 + \varepsilon) V \circ \alpha_2 (r_2 + \varepsilon) V \circ \alpha_3 (r_3 + \varepsilon) V \circ \alpha_3 (r_3 + \varepsilon) V \circ \alpha_4 (r_4 + \varepsilon) V \circ \alpha_5 (r_5 + \varepsilon)$$

$$\begin{aligned} \Rightarrow \rho(S(x), T(y), a) &\le \alpha_1 (r_1 + \varepsilon) + \alpha_2 (r_2 + \varepsilon) + \alpha_3 (r_3 + \varepsilon) + \alpha_4 (r_4 + \varepsilon) + \alpha_5 (r_5 + \varepsilon) \\ &= \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 + \alpha_4 r_4 + \alpha_5 r_5 + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\rho(S(x), T(y), a) \le \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 + \alpha_4 r_4 + \alpha_5 r_5$.

Fix $x_0 \notin X$. Construct a sequence $\{x_n\}$ in X such that $x_{2n+1} = S(x_{2n})$ and $x_{2n+2} = T(x_{2n+1})$ where $n = 0, 1, 2, 3, \ldots$.

Clearly $\{x_n\}$ is a Cauchy sequence in X and hence there exists a point u in X such that $u = \lim_{n \to \infty} x_n$.

Then

$$\begin{aligned} \rho(u, S(u), a) &\le \rho(u, S(u), x_{2n}) + \rho(u, x_{2n}, a) + \rho(x_{2n}, S(u), a) \\ &= \rho(u, S(u), x_{2n}) + \rho(u, x_{2n}, a) + \rho(T(x_{2n-1}), S(u), a) \\ &\le \rho(u, S(u), x_{2n}) + \rho(u, x_{2n}, a) + \alpha_1 \rho(u, S(u), a) + \alpha_2 \rho(x_{2n-1}, T(x_{2n-1}), a) \\ &\quad + \alpha_3 \rho(u, x_{2n-1}, a) + \alpha_4 \rho(x_{2n-1}, S(u), a) + \alpha_5 \rho(x_{2n}, a) \end{aligned}$$

Letting $n \to \infty$, we get $(1 - \alpha_1 - \alpha_4) \rho(u, S(u), a) \le 0 \Rightarrow \rho(u, S(u), a) = 0 \Rightarrow u = S(u)$
$\Rightarrow u$ is a fixed point of S.

Similarly u is a fixed point of T. Furthermoer u is unique common fixed point of S and T.

3.2 Theorem: Suppose that $S : X \to X$ and $T : X \to X$ are two operators such that

(a) $ST = TS$ (b) For all x, y, z_1, z_2 in X, for each $\rho \in \mathcal{P}$ and for any six members $V_1, V_2, V_3, V_4, V_5, V_6$ in $\mathcal{V}$, $(S(x), T(y), a) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ \alpha_4 V_4 \circ \alpha_5 V_5 \circ \alpha_6 V_6$.

If $(x, S^k(z_1), a) \in V_1, (y, T^k(z_2), a) \in V_2, (x, S^k(z_2), a) \in V_3, (y, S^k(z_2), a) \in V_4$, $(S^k(z_1), T^k(z_2), a) \in V_5$ and $(x, y, a) \in V_6$ where k is a positive integer and $\sum_{i=1}^{6} \alpha_i > 1$ then S and T have a unique common fixed point in X.

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3.3 Theorem: Suppose that $S : X \to X$ and $T : X \to X$ are two operators such that
(a) $ST = TS$ (b) For all x, y, z_1, z_2 in X, for each $\rho \in \mathcal{P}$ and for any five members V_1, V_2, V_3, V_4, V_5 in $\mathcal{V}$, $\left(S(x), T(y), a\right) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ \alpha_4 V_4 \circ \alpha_5 V_5$.

If $\left(x, S^k(z_1), a\right) \in V_1, \left(y, T^k(z_2), a\right) \in V_2, \left(x, S^k(z_2), a\right) \in V_3, \left(y, S^k(z_2), a\right) \in V_4, \left(x, y, a\right) \in V_5$ where k is a positive integer and $\sum_{i=1}^{5} \alpha_i > 1$ then S and T have a unique common fixed point in X.

3.4 Theorem: Suppose that $S : X \to X$ and $T : X \to X$ are two operators such that
(a) $ST = TS$ (b) For all x, y, z_1, z_2, z_3, z_4 in X, for each $\rho \in \mathcal{P}$ and for any four members V_1, V_2, V_3, V_4 in $\mathcal{V}$, $\left(S(x), T(y), a\right) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ \alpha_4 V_4$.

If $\left(x, S^k(z_1), a\right) \in V_1, \left(y, T^k(z_2), a\right) \in V_2, \left(S(z_1), S^k(z_3), a\right) \in V_3, \left(T(y), S^k(z_4), a\right) \in V_4$ where k is a positive integer and $\sum_{i=1}^{4} \alpha_i > 1$ then S and T have a unique common fixed point in X.

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