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ON CERTAIN CLASS OF HARMONIC FUNCTIONS DEFINED BY USING THE LINEAR OPERATOR

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ABSTRACT

Invoking the linear operator, a class of harmonic functions has been introduced. The coefficients bounds, distortion theorems, closer theorem, radii of close-to-convexity and radii of starlikeness are obtained for the same class of functions.

Key words: *Harmonic functions; Hadamard product; Starlike function; close-to-convex function; Generalized hypergeometric functions; Linear operator.*

INTRODUCTION

Let S_H denote the class of functions of the form $f(z) = h + \overline{g}$ which are harmonic univalent and sence-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f^{'}(0) - 1 = 0$. Analytic functions h and g may be expressed as

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m \ ,$$

$g(z) = \sum_{m=1}^{\infty} b_m z^m$, $(|b_1| < 1)$, and $f(z)$ is then, given by

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m}, \quad (|b_1| < 1). \tag{1.1}$$

Note that S_H reduces to the class S of normalized analytic univalent functions in U if the co-analytic part of f is identically zero. Also we denote by $S_{\overline{H}}$ the subfamily of S_H consisting the harmonic functions $f(z) = h + \overline{g}$ of the form

$$f(z) = z - \sum_{m=2}^{\infty} a_m z^m - \overline{\sum_{m=1}^{\infty} b_m z^m}, \quad (|b_1| < 1). \tag{1.2}$$

Cf. Silverman [4].

The Hadamard product of two power series

$$\phi(z) = z + \sum_{m=2}^{\infty} \phi_m z^m \tag{1.3}$$

and

$$\Psi(z) = z + \sum_{m=2}^{\infty} \Psi_m z^m \tag{1.4}$$

is defined by

$$(\phi * \Psi)(z) = z + \sum_{m=2}^{\infty} \phi_m \Psi_m z^m \tag{1.5}$$

For positive real value of $\alpha_1, \ldots \alpha_p$ and $\beta_1, \ldots \beta_q$ $(\beta_j \neq 0. -1, -2 \ldots . ; j = 1,2, \ldots q)$ the generalized hypergeometric function ${}_pF_q$ is defined by

$${}_pF_q(z) = {}_pF_q(\alpha_1, \ldots \alpha_p; \beta_1, \ldots \beta_q)$$

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$$=\sum_{m=0}^{\infty} \frac{(\alpha_1)_m \ldots\ldots (\alpha_p)_m}{(\beta_1)_m \ldots\ldots (\beta_q)_m (m)!} z^m \tag{1.6}$$

$(p \leq q+1; p, q \in \mathbb{N}_0; z \in U)$.

Recently Srivastava *et. al* [5] define the linear operator $\mathcal{L}_\lambda^{\tau,\alpha_p,\beta_q}$ as follows:

$$\mathcal{L}_\lambda^{0,\alpha_p,\beta_q}\phi(z) = \phi(z) \tag{1.7}$$
$$\mathcal{L}_\lambda^{1,\alpha_p,\beta_q}\phi(z) = (1-\lambda)H_p^q(\alpha_1,\beta_1)\phi(z) + \lambda z\left(H_p^q(\alpha_1,\beta_1)\phi(z)\right)'$$

where $H_p^q(\alpha_1,\beta_1)$ is the Dziok-Srivastava operator (see [2] and [3]).

$$\mathcal{L}_\lambda^{\tau,\alpha_p,\beta_q}\phi(z) = \mathcal{L}_\lambda^{\alpha_p,\beta_q}\left(\mathcal{L}_\lambda^{\tau-1,\alpha_p,\beta_q}\right)$$

i.e. $$\mathcal{L}_\lambda^{\tau,\alpha_p,\beta_q}\phi(z) = z + \sum_{m=2}^{\infty} \xi_m^\tau(\alpha_p,\beta_q)\phi_m z^m \tag{1.8}$$

where

$$\xi_m^\tau(\alpha_p,\beta_q) = \left[\frac{(\alpha_1)_{m-1}\ldots\ldots(\alpha_p)_{m-1}[1+\lambda(m-1)]}{(\beta_1)_{m-1}\ldots\ldots(\beta_q)_{m-1}(m-1)!}\right]^\tau. \tag{1.9}$$

We note that when $\tau = 1$ and $\lambda = 0$, then the linear operator $\mathcal{L}_\lambda^{\tau,\alpha_p,\beta_q}$ reduces to the Dziok-Srivastava operator (see [2] and [3]).

The linear operator for harmonic function $f(z) = h + \overline{g}$ is defined by

$$\mathcal{L}_\lambda^{\tau,\alpha_p,\beta_q}f(z) = \mathcal{L}_\lambda^{\tau,\alpha_p,\beta_q}h(z) + \overline{\mathcal{L}_\lambda^{\tau,\alpha_p,\beta_q}g(z)} \tag{1.10}$$

Now we introduced certain subclasses of harmonic analytic functions with negative coefficients. Let $S_{\overline{H}}(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j)$ denote the class of functions $f(z) \in S_{\overline{H}}$ such that

$$(1-\delta)\frac{\mathcal{L}_\lambda^{\tau,\alpha_p,\beta_q}f(z)}{z} + \delta\frac{\mathcal{L}_\lambda^{\tau,\alpha_1,\alpha_p-1,\beta_q}f(z)}{z} \prec \frac{1+Az}{1+Bz} \quad (\delta \geq 0, -1 \leq A < B \leq 1) \tag{1.11}$$

For detail one can see [1].

Now we study the class $Q_{\overline{H}}\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j\right)$ by using the linear operator $\mathcal{L}_\lambda^{\tau,\alpha_p,\beta_q}f(z)$.

Definition1. We say that $f(z) \in S_{\overline{H}}$ is in the class $Q_{\overline{H}}\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j\right)$ if and only if

$$(1-\delta)\left(\mathcal{L}_\lambda^{\tau,\alpha_p,\beta_q}f(z)\right)' + \delta\left(\mathcal{L}_\lambda^{\tau,\alpha_1+1,\alpha_p-1,\beta_q}f(z)\right)' \prec \frac{1+Az}{1+Bz}, \tag{1.12}$$
$$(z \in U, \delta,\lambda \geq 0, -1 \leq A < B \leq 1, 0 < B \leq 1, \tau, p, q \in \mathbb{N}_0, p \leq q+1)$$

2. COEFFICIENT ESTIMATES

Theorem 1. Let the function $f(z)$ be defined by (1.2). Then $f(z) \in Q_{\overline{H}}\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j\right)$ if and only if

$$\sum_{m=2}^{\infty} \xi_m^\tau(\alpha_p,\beta_q)m(a_m+b_m)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^\tau\right\} \leq \varphi(A,B), \tag{2.1}$$

where $\xi_m^\tau(\alpha_p,\beta_q)$ is given by (1.9) and $\varphi(A,B) = B - A - Bb_1 - b_1$ (2.2)

Proof. Let $f(z) \in Q_{\overline{H}}\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j\right)$. Then

$$h(z) = (1-\delta)\left(\mathcal{L}_{\lambda}^{\tau,\alpha_p,\beta_q}f(z)\right)' + \delta\left(\mathcal{L}_{\lambda}^{\tau,\alpha_1+1,\alpha_{p-1},\beta_q}f(z)\right)' = \frac{1+A\omega(z)}{1+B\omega(z)}, \tag{2.3}$$

$(z \in U, \delta, \lambda \geq 0, -1 \leq A < B \leq 1, 0 < B \leq 1, \tau, p, q \in \mathbb{N}_0, p \leq q+1)$

From (2.3), we get

$$\omega(z) = \frac{1-h(z)}{Bh(z)-A}$$

Therefore

$$h(z) = 1 - b_1 - \sum_{m=2}^{\infty} \xi_m^{\tau}(\alpha_p,\beta_q)m\left(a_m z^{m-1} + b_m \overline{z}^{m-1}\right)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\},$$

where $\xi_m^{\tau}(\alpha_p,\beta_q)$ is defined by(1.9) and $|\omega(z)| < 1$ implies

$$\left|\frac{b_1+\sum_{m=2}^{\infty}\xi_m^{\tau}(\alpha_p,\beta_q)m\left(a_m z^{m-1}+b_m\overline{z}^{m-1}\right)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}}{B-A-Bb_1-B\sum_{m=2}^{\infty}\xi_m^{\tau}(\alpha_p,\beta_q)m\left(a_m z^{m-1}+b_m\overline{z}^{m-1}\right)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}}\right| < 1 \tag{2.4}$$

$$Re\left\{\frac{b_1+\sum_{m=2}^{\infty}\xi_m^{\tau}(\alpha_p,\beta_q)m\left(a_m z^{m-1}+b_m\overline{z}^{m-1}\right)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}}{B-A-Bb_1-B\sum_{m=2}^{\infty}\xi_m^{\tau}(\alpha_p,\beta_q)m\left(a_m z^{m-1}+b_m\overline{z}^{m-1}\right)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}}\right\} < 1. \tag{2.5}$$

We consider real value of z and $\overline{z}$. Taking $|z| = |\overline{z}| = r$ with $0 \leq r < 1$. Then, for $r = 0$, the denominator (2.5) is positive and so it is positive for all r with $0 \leq r < 1$. Since $\omega(z)$ is analytic for $|z| < 1$. Then (2.5) gives

$$\sum_{m=2}^{\infty}\xi_m^{\tau}(\alpha_p,\beta_q)m(a_m+b_m)r^{m-1}(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\} \leq B-A-Bb_1-b_1 \tag{2.6}$$

Letting $r \longrightarrow 1$ in (2.6), we will get (2.1). Conversely, let $f(z) \in S_{\overline{H}}$ and satisfies (2.1). For $|z| = |\overline{z}| = r$, $0 \leq r < 1$, we have (2.6) by (2.1), since $r^{m-1} < 1$. So that

$$\begin{aligned}
&\left|b_1+\sum_{m=2}^{\infty}\xi_m^{\tau}(\alpha_p,\beta_q)m\left(a_m z^{m-1}+b_m\overline{z}^{m-1}\right)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}\right| \\
&\leq b_1+\sum_{m=2}^{\infty}\xi_m^{\tau}(\alpha_p,\beta_q)m(a_m+b_m)r^{m-1}\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\} \\
&\leq B-A-Bb_1-B\sum_{m=2}^{\infty}\xi_m^{\tau}(\alpha_p,\beta_q)m(a_m+b_m)r^{m-1}\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\} \\
&\leq \left|B-A-Bb_1-B\sum_{m=2}^{\infty}\xi_m^{\tau}(\alpha_p,\beta_q)m\left(a_m z^{m-1}+b_m\overline{z}^{m-1}\right)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}\right|
\end{aligned}$$

which gives (2.4) and hence follows that

$$(1-\delta)\left(\mathcal{L}_{\lambda}^{\tau,\alpha_p,\beta_q}f(z)\right)' + \delta\left(\mathcal{L}_{\lambda}^{\tau,\alpha_1+1,\alpha_{p-1},\beta_q}f(z)\right)' = \frac{1+A\omega(z)}{1+B\omega(z)}, \begin{pmatrix} z \in U, \delta, \lambda \geq 0, -1 \leq A < B \leq 1, 0 < B \leq 1, \\ \tau, p, q \in \mathbb{N}_0, p \leq q+1, |\omega(z)| < 1 \end{pmatrix}$$

3. SOME PROPERTIES OF $Q_{\overline{H}}\left(\tau; \delta; \lambda; A, B; \sum_{i=1}^{p} \alpha_i, \sum_{j=1}^{q} \beta_j\right)$

Theorem 2. $Q_{\overline{H}}\left(\tau; \delta; \lambda; A, B; \alpha_1+1, \sum_{i=2}^{p} \alpha_i, \sum_{j=1}^{q} \beta_j\right) \subset Q_{\overline{H}}\left(\tau; \delta; \lambda; A, B; \sum_{i=1}^{p} \alpha_i, \sum_{j=1}^{q} \beta_j\right)$

$$\left(\begin{array}{c} z \in U, \delta, \lambda \geq 0, -1 \leq A < B \leq 1, 0 < B \leq 1, \\ \tau, p, q \in \mathbb{N}_0, p \leq q+1 \end{array}\right)$$

Proof. By Theorem 1, we have $\sum_{m=2}^{\infty} \xi_m^\tau(\alpha_p, \beta_q) m(a_m+b_m)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^\tau\right\}$

$$\leq \sum_{m=2}^{\infty} \xi_m^\tau(\alpha_p, \beta_q) m(a_m+b_m)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^\tau\right\}\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^\tau$$

$$= \sum_{m=2}^{\infty} \xi_m^\tau(\alpha_1+1, \alpha_{p-1}, \beta_q) m(a_m+b_m)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^\tau\right\}$$

$$\leq \varphi(A, B)$$

i.e.

$$\sum_{m=2}^{\infty} \xi_m^\tau(\alpha_p, \beta_q) m(a_m+b_m)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^\tau\right\} \leq \varphi(A, B),$$

where $\xi_m^\tau(\alpha_p, \beta_q)$ *and* $\varphi(A, B)$ is defined by (1.9) and (2.2) respectively. The theorem is completely proved.

Theorem 3. $Q_{\overline{H}}\left(\tau; \delta_2; \lambda; A, B; \alpha_1+1, \sum_{i=2}^{p} \alpha_i, \sum_{j=1}^{q} \beta_j\right) \subset Q_{\overline{H}}\left(\tau; \delta_1; \lambda; A, B; \sum_{i=1}^{p} \alpha_i, \sum_{j=1}^{q} \beta_j\right)$

$$\left(\begin{array}{c} z \in U, \delta, \lambda \geq 0, -1 \leq A < B \leq 1, 0 < B \leq 1, \\ \tau, p, q \in \mathbb{N}_0, p \leq q+1 \end{array}\right)$$

Proof. By Theorem 1, we have

$$\sum_{m=2}^{\infty} \xi_m^\tau(\alpha_p, \beta_q) m(a_m+b_m)(1+B)\left\{1-\delta_1+\delta_1\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^\tau\right\}$$

$$\leq \sum_{m=2}^{\infty} \xi_m^\tau(\alpha_p, \beta_q) m(a_m+b_m)(1+B)\left\{1-\delta_2+\delta_2\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^\tau\right\} \leq \varphi(A, B),$$

where $\varphi(A, B)$ is defined by (2.2), for

$f(z) \in Q_{\overline{H}}\left(\tau; \delta_2; \lambda; A, B; \alpha_1+1, \sum_{i=2}^{p} \alpha_i, \sum_{j=1}^{q} \beta_j\right)$.

Hence $f(z) \in Q_{\overline{H}}\left(\tau; \delta_1; \lambda; A, B; \alpha_1+1, \sum_{i=2}^{p} \alpha_i, \sum_{j=1}^{q} \beta_j\right)$.

The theorem is completely proved.

4. DISTORTION THEOREMS

Theorem 4. Let the function $f(z)$ defined by (1.2) be in the class $Q_{\overline{H}}\left(\tau; \delta; \lambda; A, B; \sum_{i=1}^{p} \alpha_i, \sum_{j=1}^{q} \beta_j\right)$.

Then for $|z| = |\overline{z}| = r < 1$, we have

$$r + b_1 r - \frac{\varphi(A,B)\, r^2}{2\xi_2^\tau(\alpha_p,\beta_q)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+1}{\alpha_1}\right)^\tau\right\}} \leq |f(z)|$$

$$\leq r + b_1 r - \frac{\varphi(A,B)\, r^2}{2\xi_2^\tau(\alpha_p,\beta_q)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+1}{\alpha_1}\right)^\tau\right\}}. \quad (4.1)$$

and

$$1+b_1-\frac{r\varphi(A,B)}{\xi_2^\tau(\alpha_p,\beta_q)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+1}{\alpha_1}\right)^\tau\right\}}\le|f'(z)|$$

$$\le 1+b_1+\frac{r\varphi(A,B)}{\xi_2^\tau(\alpha_p,\beta_q)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+1}{\alpha_1}\right)^\tau\right\}}. \tag{4.2}$$

Proof. Since $\xi_m^\tau(\alpha_p,\beta_q)m\left\{1-\delta_1+\delta_1\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^\tau\right\}$ is an increasing function of $m(m\ge 2)$ and $f(z)\in Q_{\overline{H}}\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j\right)$.

By theorem 1, we have

$$2\xi_2^\tau(\alpha_p,\beta_q)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+1}{\alpha_1}\right)^\tau\right\}\sum_{m=2}^{\infty}(a_m+b_m)$$

$$\le\sum_{m=2}^{\infty}\xi_m^\tau(\alpha_p,\beta_q)m(a_m+b_m)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^\tau\right\}\le\varphi(A,B) \tag{4.3}$$

i.e.

$$\sum_{m=2}^{\infty}(a_m+b_m)\le\frac{\varphi(A,B)}{2\xi_2^\tau(\alpha_p,\beta_q)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+1}{\alpha_1}\right)^\tau\right\}}, \tag{4.4}$$

where $\xi_2^\tau(\alpha_p,\beta_q)=\left[\frac{\alpha_1\ldots\alpha_p}{\beta_1\ldots\beta_q}(1-\lambda)\right]^\tau.$ (4.5)

$$|f(z)|\le r+b_1r+\sum_{m=2}^{\infty}(a_m+b_m)r^m$$

$$\le r+b_1r+\frac{\varphi(A,B)\sum_{m=2}^{\infty}r^m}{2\xi_2^\tau(\alpha_p,\beta_q)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+1}{\alpha_1}\right)^\tau\right\}}$$

$$\le r+b_1r+\frac{\varphi(A,B)\,r^2}{2\xi_2^\tau(\alpha_p,\beta_q)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+1}{\alpha_1}\right)^\tau\right\}}$$

and

$$|f(z)|\ge r+b_1r-\frac{\varphi(A,B)\,r^2}{2\xi_2^\tau(\alpha_p,\beta_q)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+1}{\alpha_1}\right)^\tau\right\}}.$$

Hence (4.1) follows.

Also by theorem 1, we have

$$\xi_2^\tau(\alpha_p,\beta_q)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+1}{\alpha_1}\right)^\tau\right\}\sum_{m=2}^{\infty}(a_m+b_m)m\le\varphi(A,B) \tag{4.6}$$

Thus

$$|f'(z)|\le 1+b_1+\sum_{m=2}^{\infty}(a_m+b_m)mr^{m-1}$$

$$\le 1+b_1+\frac{r\varphi(A,B)}{\xi_2^\tau(\alpha_p,\beta_q)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+1}{\alpha_1}\right)^\tau\right\}}$$

and

$$|f'(z)|\ge 1+b_1-\frac{r\varphi(A,B)}{\xi_2^\tau(\alpha_p,\beta_q)(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+1}{\alpha_1}\right)^\tau\right\}}.$$

This proves the theorem completely.

5. CLOSURE THEOREM

Let the functions $f_k(z)$ be defined, for $k=1,2,\ldots,\nu$ by

$$f_k(z)=z-\sum_{m=2}^{\infty}a_{m,k}z^m-\sum_{m=1}^{\infty}b_{m,k}z^m \tag{5.1}$$

Theorem 5. Let the function $f_k(z)$ defined by (5.1) be in the classes $Q_{\overline{H}}\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j\right)$. Then the function $h(z)$ defined by

$$h(z)=z-\sum_{m=2}^{\infty}a_{m,k}z^m-\overline{\sum_{m=1}^{\infty}b_{m,k}z^m} \tag{5.2}$$

is in the class $Q_{\overline{H}}\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j\right)$,

where

$$A=\min_{1\le k\le\nu}\{A_i\} \text{ and } B=\min_{1\le k\le\nu}\{B_i\}. \tag{5.3}$$

Proof. Since $f_k(z)\in Q_{\overline{H}}\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j\right)$ for $k=1,2,\dots,\nu$, by Theorem 1 we have

$$\sum_{m=2}^{\infty}\xi_m^{\tau}(\alpha_p,\beta_q)m(a_{m,k}+b_{m,k})(1+B_k)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}\le\varphi(A_k,B_k)$$

Hence

$$\begin{aligned}\sum_{m=2}^{\infty}\xi_m^{\tau}(\alpha_p,\beta_q)m(1+B_k)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}\left\{\frac{1}{\nu}\sum_{k=1}^{\nu}(a_{m,k}+b_{m,k})\right\}\\ =\frac{1}{\nu}\sum_{k=1}^{\nu}\left[\sum_{m=2}^{\infty}\xi_m^{\tau}(\alpha_p,\beta_q)m(a_{m,k}+b_{m,k})(1+B_k)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}\right]\\ \le\frac{1}{\nu}\sum_{k=1}^{\nu}B_k-A_k-B_kb_1-b_1\le B-A-Bb_1-b_1\end{aligned} \tag{5.4}$$

Thus

$$\sum_{m=2}^{\infty}\xi_m^{\tau}(\alpha_p,\beta_q)m(1+B_k)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}\left\{\frac{1}{\nu}\sum_{k=1}^{\nu}(a_{m,k}+b_{m,k})\right\}\le B-A-Bb_1-b_1 \tag{5.5}$$

The theorem is proved completely.

Theorem 6. Let the functions $f_k(z)(k=1,2,\dots,\nu)$ defined by (5.1) be in the class $Q_{\overline{H}}\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j\right)$.

Then the function $h(z)$ defined by

$$h(z)=\sum_{k=1}^{\nu}d_kf_k(z) \tag{5.6}$$

is also in the class $Q_{\overline{H}}\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j\right)$, where $\sum_{k=1}^{\nu}d_k=1$ (5.7)

Proof. By (5.6) and (5.1), we have

$$h(z)=z-\sum_{m=2}^{\infty}\left(\sum_{k=1}^{\nu}d_ka_{m,k}\right)z^m-\overline{\sum_{m=1}^{\infty}\left(\sum_{k=1}^{\nu}d_kb_{m,k}\right)z^m} \tag{5.8}$$

Since $f_k(z)\in Q_{\overline{H}}\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j\right)$. Then we have

$$\sum_{m=2}^{\infty} \xi_m^{\tau}(\alpha_p,\beta_q)m(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}\sum_{k=1}^{v} d_k(a_{m,k}+b_{m,k})$$

$$=\sum_{i=1}^{v} d_k\left[\sum_{m=2}^{\infty} \xi_m^{\tau}(\alpha_p,\beta_q)m(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}(a_{m,k}+b_{m,k})\right]$$

$$\leq \sum_{i=1}^{v} d_k\varphi(A,B)=\varphi(A,B)$$

Thus $h(z)$ is in the class $Q_{\overline{H}}\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j\right)$, where $\varphi(A,B)$ is defined by (2.2).The theorem is completely proved.

6. RADII OF CLOSE-TO-CONVEXITY, STARLIKENESS AND CONVEXITY

Theorem7. Let the function $f(z)$ defined by (1.2) be in the class $Q_{\overline{H}}\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j\right)$. Then $f(z)$ is close-to-convex of order $\rho(0\leq\rho<1)$ in $|z|<r_1\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j,\rho\right)$,

where $$r_1\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j,\rho\right)=\inf_{m}\left[\frac{\xi_m^{\tau}(\alpha_p,\beta_q)(1-\rho)(1+B)\left(1-\frac{b_1}{(1-\rho)}\right)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}}{\varphi(A,B)}\right]^{1/m-1} \quad (6.1)$$

Proof. It is sufficient to prove that

$$|f'(z)-1|\leq 1-\rho$$

$$|f'(z)-1|\leq\left|\sum_{m=2}^{\infty} a_m m z^{m-1}+\overline{\sum_{m=1}^{\infty} b_m m z^{m-1}}\right| \quad (6.2)$$

Hence

$$\sum_{m=2}^{\infty} a_m\left(\frac{m}{1-\rho}\right)|z|^{m-1}+\sum_{m=1}^{\infty} b_m\left(\frac{m}{1-\rho}\right)|\overline{z}|^{m-1}\leq 1 \quad (6.3)$$

By Theorem 1, we have

$$\frac{\sum_{m=2}^{\infty}\xi_m^{\tau}(\alpha_p,\beta_q)m(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}(a_m+b_m)}{\varphi(A,B)}\leq 1 \quad (6.4)$$

$$\frac{\left(\frac{m}{1-\rho}\right)|z|^{m-1}}{1-\left(\frac{b_1}{1-\rho}\right)}\leq\frac{\xi_m^{\tau}(\alpha_p,\beta_q)m(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}}{\varphi(A,B)}$$

$$|z|=\left[\frac{\xi_m^{\tau}(\alpha_p,\beta_q)(1-\rho)(1+B)\left(1-\frac{b_1}{(1-\rho)}\right)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}}{\varphi(A,B)}\right]^{1/m-1}\quad(m\geq 2).$$

The Theorem is completely proved.

Theorem 8. Let the function $f(z)$ defined by (1.2) be in the class $Q_{\overline{H}}\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j\right)$. Then $f(z)$ is starlike of order $\rho(0\le\rho<1)$ in $|z|<r_2\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j,\rho\right)$,

where $r_2\left(\tau;\delta;\lambda;A,B;\sum_{i=1}^{p}\alpha_i,\sum_{j=1}^{q}\beta_j,\rho\right)=\underset{m}{inf}\left[\frac{\xi_m^{\tau}(\alpha_p,\beta_q)(1-\rho)m(1+B)(1-b_1)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}}{\varphi(A,B)(m-\rho)}\right]^{1/m-1}$ (6.5)

Proof. It is sufficient to prove that

$$\left|\frac{zf'(z)}{f(z)}-1\right|\le 1-\rho$$

$$\left|\frac{zf'(z)}{f(z)}-1\right|\le\left|\sum_{m=2}^{\infty}a_m m z^{m-1}+\overline{\sum_{m=1}^{\infty}b_m m z^{m-1}}\right| \tag{6.6}$$

Hence

$$\left|\frac{zf'(z)}{f(z)}-1\right|\le\frac{\sum_{m=2}^{\infty}a_m(m-1)|z|^{m-1}+\sum_{m=1}^{\infty}b_m(m-1)|\overline{z}|^{m-1}}{1-\sum_{m=2}^{\infty}a_m|z|^{m-1}b_m|\overline{z}|^{m-1}} \tag{6.7}$$

Thus

$$\frac{(m-\rho)}{(1-\rho)(1-b_1)}(a_m+b_m)|z|^{m-1}\le 1$$

By using (6.4) and (6.8)

$$\frac{(m-\rho)}{(1-\rho)(1-b_1)}|z|^{m-1}\le\frac{\xi_m^{\tau}(\alpha_p,\beta_q)m(1+B)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}}{\varphi(A,B)}$$

$$|z|\le\left[\frac{\xi_m^{\tau}(\alpha_p,\beta_q)m(1+B)(1-\rho)(1-b_1)\left\{1-\delta+\delta\left(\frac{\alpha_1+m-1}{\alpha_1}\right)^{\tau}\right\}}{\varphi(A,B)(m-\rho)}\right]^{1/m-1}$$

The theorem is completely proved.

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