International Journal of Mathematical Archive-3(12), 2012, 4836-4839
IJMA Available online through www.ijma.info ISSN 2229 – 5046

ON GENERALIZED BERNOULLI POLYNOMIALS OF NEGATIVE ORDER CONNECTING WITH FRACTIONAL INTEGRALS AND FRACTIONAL DIFFERENCES

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(Received on: 18-11-12; Revised & Accepted on: 28-12-12)

ABSTRACT

In this paper, some identities on generalized Bernoulli polynomials of negative order connecting with Riemann-Liouville fractional integrals and Grünwald-Letnikov fractional differences are studied.

AMS Subject Classification: *11B68; 26A33; 41A60.*

Keywords: *Bernoulli polynomials and numbers; Stirling numbers of second kind, fractional integral; fractional difference.*

1. INTRODUCTION

The generalized Bernoulli polynomials have important applications both in analytic theory of numbers and numerical analysis. Usually, Bernoulli polynomial $B_n^{(\alpha)}(x)$ of (real and complex) of order α is defined by the following generating function [3, 4].

$$\frac{t^\alpha e^{xt}}{(e^t-1)^\alpha} = \sum_{k=0}^{\infty} B_k^{(\alpha)}(x)\frac{t^k}{k!}. \tag{1}$$

The Riemann-Liouville right and left handed integral of f of fractional order α [2, 5] defined on $-\infty \le 0 \le x < \infty$ and $-\infty < 0 \le x \le \infty$ respectively as follows:

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-u)^{\alpha-1} f(u)du. \tag{2}$$

and

$$_xI^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_x^0 (u-x)^{\alpha-1} f(u)du. \tag{3}$$

The Grünwald-Letnikov right and left handed fractional difference of order α of g [2, 5] are defined as follows:

$$\Delta_x^\alpha g(x) = \sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k} g(x-k). \tag{4}$$

and

$$_x\Delta^\alpha g(x) = \sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k} g(x+k). \tag{5}$$

Recently in Ref [3], some properties and applications of generalized Bernoulli polynomials of positive order connecting with Riemann-Liouville fractional integral and Grünwald-Letnikov fractional difference are studied. In the present work, some properties and applications of generalized Bernoulli polynomials of negative order connecting with fractional integrals and fractional differences are studied.

2. MAIN RESULTS

Theorem 2.1. Let $\alpha > 0$ and $k \in W$. Then

$$\Delta_x^\alpha I_x^\alpha((x+\alpha)^k) = B_k^{(-\alpha)}(x) \tag{6}$$

Proof. Replacing α by $-\alpha$ in (1), then

$$\frac{(e^t-1)^\alpha e^{xt}}{t^\alpha} = \sum_{k=0}^{\infty} B_k^{(-\alpha)}(x)\frac{t^k}{k!}. \tag{7}$$

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In Ref [3], it is well known that

$$\Delta_x^\alpha I_x^\alpha(e^{xt}) = e^{xt}\left(\frac{1-e^{-t}}{t}\right)^\alpha. \tag{8}$$

Using (8) in (7), then

$$\Delta_x^\alpha I_x^\alpha(e^{xt})e^{\alpha t} = \sum_{k=0}^{\infty} B_k^{(-\alpha)}(x)\frac{t^k}{k!}.$$

Since α is constant, after simplification, gives

$$\Delta_x^\alpha I_x^\alpha\left(e^{(x+\alpha)t}\right) = \sum_{k=0}^{\infty} B_k^{(-\alpha)}(x)\frac{t^k}{k!}.$$

Expanding the exponential function on the left hand side of the above equation

$$\sum_{k=0}^{\infty} \Delta_x^\alpha I_x^\alpha((x+\alpha)^k)\frac{t^k}{k!} = \sum_{k=0}^{\infty} B_k^{(-\alpha)}(x)\frac{t^k}{k!}.$$

Then, comparing coefficients of t^k, $k = 0,1,2, ...,$ gives (6).

Theorem 2.2. Let $\alpha > 0$ and $k \in W$. Then

$$ {}_x\Delta^\alpha {}_xI^\alpha(x^k) = B_k^{(-\alpha)}(x) \tag{9}$$

Proof. Replacing t by $-t$ in (7), then

$$\frac{(1-e^{-t})^\alpha e^{-xt}}{t^\alpha} = \sum_{k=0}^{\infty} B_k^{(-\alpha)}(x)(-1)^k\frac{t^k}{k!}. \tag{10}$$

In Ref [3], it is well known that

$$ {}_x\Delta^\alpha {}_xI^\alpha(e^{-xt}) = e^{-xt}\left(\frac{1-e^{-t}}{t}\right)^\alpha \tag{11}$$

Using (11) in (10), then

$$ {}_x\Delta^\alpha {}_xI^\alpha(e^{-xt}) = \sum_{k=0}^{\infty} B_k^{(-\alpha)}(x)(-1)^k\frac{t^k}{k!}.$$

Expanding the exponential function on the left hand side of the above equation and then comparing coefficient of $t^k, k = 0,1,2, ...,$ gives (9).

Theorem 2.3. Let f be a continuously differentiable function, $\alpha > 0$. Then

$$\Delta_x^\alpha I_x^\alpha f(x+\alpha) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}B_k^{(-\alpha)}(x) = {}_x\Delta^\alpha {}_xI^\alpha f(x) \tag{12}$$

Proof. Expanding $f(x+\alpha)$ by Taylor series at $x+\alpha = 0$, then

$$f(x+\alpha) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}(x+\alpha)^k \tag{13}$$

Applying Riemann-Liouville right hand fractional integral and Grunwald-Letnikov right hand fractional difference on both sides of (13), then

$$\Delta_x^\alpha I_x^\alpha f(x+\alpha) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\,\Delta_x^\alpha I_x^\alpha((x+\alpha)^k)$$

Using Theorem 2.1, yields

$$\Delta_x^\alpha I_x^\alpha f(x+\alpha) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} B_k^{(-\alpha)}(x) \tag{14}$$

Similarly, expanding $f(x)$ by Taylor series at $x = 0$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \tag{15}$$

Applying Riemann-Liouville left hand fractional integral and Grunwald-Letnikov left hand fractional difference on both sides of (15), then

$$_x\Delta^\alpha {}_xI^\alpha f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} {}_x\Delta^\alpha {}_xI^\alpha (x^k)$$

Using Theorem 2.2, yields

$$_x\Delta^\alpha {}_xI^\alpha f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} B_k^{(-\alpha)}(x) \tag{16}$$

Combining (14) and (16), completes the theorem.

Corollary 2.4. Let f be a continuously differentiable function, $m \in N$. then

$$\Delta_0^m I_x^m f(x+m) = m! \sum_{k=0}^{\infty} f^{(k)}(0) \frac{S(m,m+k)}{(m+k)!} = {}_0\Delta^m {}_xI^m f(x). \tag{17}$$

Where $S(m, m+k)$ is Striling numbers of second of order m[1].

Proof. Let $\alpha = m, m \in N$ and $x = 0$ in (12). Then

$$\Delta_0^m I_x^m f(x+m) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} B_k^{(-m)}(0) = {}_0\Delta^m {}_xI^m f(x). \tag{18}$$

But $B_k^{(-m)}(0) = B_k^{(-m)}$, where $B_k^{(-m)}$ is k^{th} Bernoulli number of order $-m$.

Also, $B_k^{(-m)} = S(m, m+k) \frac{k!m!}{(m+k)!}$. (19)

The equation (19) can be easily found by comparing coefficients of powers of t of the following generating function of $B_k^{(-m)}$ and Stirling number of second kind $S(m, m+k)$[1]
.

$$\frac{(e^t-1)^m}{t^m} = \sum_{k=0}^{\infty} B_k^{(-m)} \frac{t^k}{k!}.$$

and

$$(e^t-1)^m = m! \sum_{k=m}^{\infty} S(m,k) \frac{t^k}{k!}.$$

Using (19) in (18), completes the corollary.

Theorem 2.5. Let $m \in N$ and $n \in W$. Then

$$\frac{\Delta^m B_{n+m}^{(m)}(m)}{m!(n+m)!} = \sum_{k=0}^{n} \frac{B_{n-k}^{(m)}}{(n-k)!} \frac{S(m,m+k)}{(m+k)!}. \tag{20}$$

Proof. Let $f(x) = B_n^{(m)}(x)$ in (17). Then

$$\Delta_0^m I_x^m B_n^{(m)}(x+m) = m!\, n! \sum_{k=0}^{n} \frac{B_{n-k}^{(m)}}{(n-k)!} \frac{S(m,m+k)}{(m+k)!}. \tag{21}$$

Using (2), gives

$$I_x^m B_n^{(m)}(x+m) = \frac{1}{\Gamma(m)} \int_0^x (x-u)^{m-1} B_n^{(m)}(u+m)du.$$

Using integral by parts and applying Δ_0^m, it is easily find that

$$\Delta_0^m I_x^m B_n^{(m)}(x+m) = \frac{\Delta^m B_n^{(m)}(m)}{(n+1)(n+2)\ldots(n+m)}. \tag{22}$$

Comparing (22) and (21), gives (20). This completes the theorem.

3. CONCLUSION

Some properties and applications of negative order Bernoulli polynomials connecting with Riemann-Lioville fractional integral and Grünwald-Letnikov fractional differences have been developed in this paper. The expansion of Riemann-Lioville fractional integrals and Grünwald-Letnikov fractional differences of a function can be expressed in terms of negative order Bernoulli polynomials.

REFERENCES

[1] I.S. Gradshteyn & I.M. Ryzhik, *Tables of Integrals, Series and Products, 6 Ed*, Academic Press, USA (2000).

[2] A. A. Kilbas, H. M. Srivastava & J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, USA (2006).

[3] R.K. Muthumalai, *Some properties and applications of generalized Bernoulli polynomials,* Int. Jour. Appl. Math, Vol 25 (1), (2012), 83-93.

[4] N. E. Norlund, *Vorlesungen über Differenzenrechnung*, Springer, Berlin (1924).

[5] I. Podlubny, *Fractional differential equations*, Academic Press, USA (1999).

Source of support: Nil, Conflict of interest: None Declared