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ON THE EDGE DEGREE WEIGHT SUM OF GRAPH PRODUCTS

Didar A. Ali*
Department of Mathematics faculty of Science University of Zakho, Zakho - Kurdistan-Region. Iraq

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ABSTRACT

Let $G=(V,E)$ be a simple graph of order p and size q, and let the weight $w(e)$ of an edge $e=uv$ of a graph G is defined by $w(e)=\deg(u)+\deg(v)$ and the weight of G is defined by $w(G)=\sum_{e\in E(G)} w(e)$, where $E(G)$ is the edge set of G. In this paper some properties of $w(G)$ are determined, also we investigate the value $w(G)$ for various type of graph products, such as, Cartesian, Tensor (or Kronecker), Strong, Composition (or Lexicographic), Skew and Converse Skew products.

Key words: *Graph Theory, Edge degree weights.*

1. INTRODUCTION

Let v be any vertex in a graph G, then *degree* of v in a graph G is the number of vertices incident with v and denoted by $\deg(v)$. Among many ways of assigning weights to the edges of a graph are those that are functions of the degrees of the vertices of a given edge $e=uv$. An indispensable for the study of this subject is the servey by Gallian [5]. The weighting $w(e)$ was studied by Kattimani, Ducasse, Gargano, and Quintas in [3, 4, 6] as:

$$w(e)=\deg(u)+\deg(v),$$

and extended to the weight of a graph $G=(V,E)$ as:

$$w(G)=\sum_{e\in E(G)} w(e).$$

Here we take one of the most natural types of vertex labeling, that of assigning the degree of a vertex as its label, and we determined and investigate the value $w(G)$ for various type of graph products.

Definition 1.1: The *weight* $w(e)$ of an edge $e=uv$ is defined by $w(e)=\deg(u)+\deg(v)$ and the edge degree weight sum $w(G)$ of G relative to this edge weighting is $w(G)=\sum_{e\in E(G)} w(e)$.

2. SOME BASIC DEFINITION AND THEOREMS

We begin this paper by given the following Definition of graph products.

Definition 2.1: Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two graphs with order p_1, p_2 and size q_1, q_2, respectively. These products result in a graph G whose vertex set is $V(G)=V(G_1)\times V(G_2)$, where $V(G_i)$ is the vertex set of G_i. And two vertices (u_1,v_1) and (u_2,v_2) are adjacent in graph products, as:

1. The Cartesian product: $G=G_1\times G_2$ has edge set
$$E(G)=\{[u_1u_2\in E(G_1)\ and\ v_1\equiv v_2]\ or\ [u_1\equiv u_2\ and\ v_1v_2\in E(G_2)]\}.$$
2. The Tensor (or kronecker) product: $G=G_1\otimes G_2$ has edge set
$$E(G)=\{[u_1u_2\in E(G_1)\ and\ v_1v_2\in E(G_2)]\}.$$

Corresponding author: *Didar A. Ali**
Department of Mathematics faculty of Science University of Zakho, Zakho - Kurdistan-Region. Iraq

3. The Strong product: $G = G_1 \odot G_2$ has edge set

$$E(G) = \{[u_1u_2 \in E(G_1) \ and \ v_1 \equiv v_2] \ or \ [u_1 \equiv u_2 \ and \ v_1v_2 \in E(G_2)] \\ or \ [u_1u_2 \in E(G_1) \ and \ v_1v_2 \in E(G_2)]\}.$$

4. The Composition (or Lexicographic): $G = G_1[G_2]$ has edge set

$$E(G) = \{[u_1u_2 \in E(G_1)] \ or \ [u_1 \equiv u_2 \ and \ v_1v_2 \in E(G_2)]\}.$$

5. The Skew product: $G = G_1 \square G_2$ has edge set

$$E(G) = \{[u_1 \equiv u_2 \ and \ v_1v_2 \in E(G_2)] \ or \ [u_1u_2 \in E(G_1) \ and \ v_1v_2 \in E(G_2)]\}.$$

6. The Converse Skew product: $G = G_1 \Diamond G_2$ has edge set

$$E(G) = \{[u_1u_2 \in E(G_1) \ and \ v_1 \equiv v_2] \ or \ [u_1u_2 \in E(G_1) \ and \ v_1v_2 \in E(G_2)]\}.$$

Lemma 2.2: Let G_1 and G_2 be disjoint regular graphs of order p_1 and p_2, and degree r_1 and r_2, respectively. Then

1. The Cartesian product: $G = G_1 \times G_2$ is $(r_1 + r_2)$-regular.
2. The Tensor (or kronecker) product: $G = G_1 \otimes G_2$ is r_1r_2-regular.
3. The Strong product: $G = G_1 \odot G_2$ is $(r_1 + r_2 + r_1r_2)$-regular.
4. The Composition (or Lexicographic): $G = G_1[G_2]$ is $(p_2r_1 + r_2)$-regular.
5. The Skew product: $G = G_1 \square G_2$ is $(r_2 + r_1r_2)$-regular.
6. The Converse Skew product: $G = G_1 \Diamond G_2$ is $(r_1 + r_1r_2)$-regular.

The following Theorem is fundamental notion for determining the edge degree weight $w(G)$

Theorem 2.3: For any graph G, $w(G) = \sum_{v \in V(G)} \deg(v)^2$.

Proof: By Definition 1.1, $w(e) = \deg(u) + \deg(v)$ for edge $e = uv$ in $E(G)$. Thus, each vertex v in $V(G)$ contributes the value $\deg(v)$ to the weights of $\deg(v)$ edges and thus contributes $\deg(v) \times \deg(v) = \deg(v)^2$ to $w(G)$. Hence

$$w(G) = \sum_{v \in V(G)} \deg(v)^2.$$

Theorem 2.4: If G is a regular graph of degree r, then $w(e) = 2r$ for each edge $e = uv$ in $E(G)$ and $w(G) = pr^2$.

Proof: By Definition 1.1, $w(e) = 2r - 2$. By Theorem 2.3, $w(G) = pr^2$.

3. GENERAL RESULTS

The following Theorem determine the value of $w(G)$ for the graph products.

Theorem 3.1: Let G_1 and G_2 be disjoint graphs of order p_1 and p_2, and size q_1 and q_2, respectively. Then,

1. $w(G_1 \times G_2) = p_2 w(G_1) + p_1 w(G_2) + 8q_1q_2$.

Proof: Let $x \in V(G_1)$, $y \in V(G_2)$, and $(x, y) \in V(G_1 \times G_2)$. Then,

$$w(G_1 \times G_2) = \sum_{(x,y)\in V(G_1\times G_2)} \deg(x,y)^2$$

$$= \sum_{(x,y)\in V(G_1\times G_2)} (\deg(x)+\deg(y))^2$$

$$= \sum_{(x,y)\in V(G_1\times G_2)} (\deg(x)^2 + 2\deg(x)\deg(y) + \deg(y)^2)$$

$$= \begin{cases} \sum_{(x,y)\in V(G_1\times G_2)} \deg(x)^2 + 2\sum_{(x)\in V(G_1)} \deg(x) \sum_{(y)\in V(G_2)} \deg(y) \\ \quad + \sum_{(x,y)\in V(G_1\times G_2)} \deg(y)^2 \end{cases}$$

Since, $w(G) = \sum_{v\in V(G)} \deg(v)^2$, and $\sum_{v\in V(G)} \deg(v) = 2q$, thus

$$w(G_1 \times G_2) = p_2 w(G_1) + 2(2q_1)(2q_2) + p_1 w(G_2)$$
$$= p_2 w(G_1) + p_1 w(G_2) + 8q_1 q_2 .$$

2. $w(G_1 \otimes G_2) = w(G_1) w(G_2)$

Proof: Let $x \in V(G_1)$, $y \in V(G_2)$, and $(x,y) \in V(G_1 \otimes G_2)$. Then,

$$w(G_1 \otimes G_2) = \sum_{(x,y)\in V(G_1\otimes G_2)} \deg(x,y)^2$$

$$= \sum_{(x,y)\in V(G_1\otimes G_2)} (\deg(x)\deg(y))^2$$

$$= \sum_{(x)\in V(G_1)} \deg(x)^2 \sum_{(y)\in V(G_2)} \deg(y)^2$$

Since, $w(G) = \sum_{v\in V(G)} \deg(v)^2$, thus

$$w(G_1 \otimes G_2) = w(G_1) w(G_2).$$

3. $w(G_1 \odot G_2) = w(G_1)w(G_2) + p_2 w(G_1) + p_1 w(G_2) + 4q_2 w(G_1) + 4q_1 w(G_2) + 8q_1 q_2$

Proof: Let $x \in V(G_1)$, $y \in V(G_2)$, and $(x,y) \in V(G_1 \odot G_2)$. Then,

$$w(G_1 \odot G_2) = \sum_{(x,y)\in V(G_1\odot G_2)} \deg(x,y)^2$$

$$= \sum_{(x,y)\in V(G_1\odot G_2)} (\deg(x)+\deg(y)+\deg(x)\deg(y))^2$$

$$= \begin{cases} \sum_{(x,y)\in V(G_1\odot G_2)} [\deg(x)^2 + 2\deg(x)\deg(y) + \deg(y)^2 \\ +2\deg(x)^2\deg(y) + 2\deg(x)\deg(y)^2 + \deg(x)^2\deg(y)^2] \end{cases}$$

$$= \begin{cases} \sum_{(x,y)\in V(G_1\odot G_2)} \deg(x)^2 + 2\sum_{(x)\in V(G_1)} \deg(x) \sum_{(y)\in V(G_2)} \deg(y) \\ + \sum_{(x,y)\in V(G_1\odot G_2)} \deg(y)^2 + 2\sum_{(x)\in V(G_1)} \deg(x)^2 \sum_{(y)\in V(G_2)} \deg(y) \\ +2\sum_{(x)\in V(G_1)} \deg(x) \sum_{(y)\in V(G_2)} \deg(y)^2 + \sum_{(x)\in V(G_1)} \deg(x)^2 \sum_{(y)\in V(G_2)} \deg(y)^2 \end{cases}$$

Since, $w(G) = \sum_{v\in V(G)} \deg(v)^2$, and $\sum_{v\in V(G)} \deg(v) = 2q$, thus

$$w(G_1 \odot G_2) = \begin{cases} p_2 w(G_1) + 2(2q_1)(2q_2) + p_1 w(G_2) + 2q_2 w(G_1) \\ +2q_1 w(G_2) + w(G_1)w(G_2) \end{cases}$$

$$= \begin{cases} w(G_1)w(G_2) + p_2 w(G_1) + p_1 w(G_2) \\ +4q_2 w(G_1) + 4q_1 w(G_2) + 8q_1 q_2. \end{cases}$$

4. $w(G_1[G_2]) = p_2{}^3 w(G_1) + p_1 w(G_2) + 8p_2 q_1 q_2$

Proof: Let $x \in V(G_1)$, $y \in V(G_2)$, and $(x,y) \in V(G_1[G_2])$. Then,

$$w(G_1[G_2]) = \sum_{(x,y)\in V(G_1[G_2])} \deg(x,y)^2$$

$$= \sum_{(x,y)\in V(G_1[G_2])} (p_2 \deg(x) + \deg(y))^2$$

$$= \sum_{(x,y)\in V(G_1[G_2])} (p_2^2 \deg(x)^2 + 2p_2 \deg(x)\deg(y) + \deg(y)^2)$$

$$= \begin{cases} p_2^2 \sum_{(x,y)\in V(G_1[G_2])} \deg(x)^2 + 2p_2 \sum_{(x)\in V(G_1)} \deg(x) \sum_{(y)\in V(G_2)} \deg(y) \\ + \sum_{(x,y)\in V(G_1[G_2])} \deg(y)^2 \end{cases}$$

Since, $w(G) = \sum_{v\in V(G)} \deg(v)^2$, and $\sum_{v\in V(G)} \deg(v) = 2q$, thus

$$w(G_1[G_2]) = p_2^2 p_2 w(G_1) + 2p_2(2q_1)(2q_2) + p_1 w(G_2)$$

$$= p_2^3 w(G_1) + p_1 w(G_2) + 8p_2 q_1 q_2.$$

5. $w(G_1 \square G_2) = w(G_1)w(G_2) + p_1 w(G_2) + 4q_1 w(G_2)$ has edge

Proof: Let $x \in V(G_1)$, $y \in V(G_2)$, and $(x,y) \in V(G_1 \square G_2)$. Then,

$$w(G_1 \square G_2) = \sum_{(x,y)\in V(G_1\square G_2)} \deg(x,y)^2$$

$$= \sum_{(x,y)\in V(G_1\square G_2)} (\deg(y) + \deg(x)\deg(y))^2$$

$$= \sum_{(x,y)\in V(G_1\Box G_2)} (\deg(y)^2 + 2\deg(x)\deg(y)^2 + \deg(x)^2\deg(y)^2)$$

$$= \begin{cases} \displaystyle\sum_{(x,y)\in V(G_1\Box G_2)} \deg(y)^2 + 2\sum_{(x)\in V(G_1)} \deg(x) \sum_{(y)\in V(G_2)} \deg(y)^2 \\ \displaystyle + \sum_{(x)\in V(G_1)} \deg(x)^2 \sum_{(y)\in V(G_2)} \deg(y)^2 \end{cases}$$

Since, $w(G) = \sum_{v\in V(G)} \deg(v)^2$, and $\sum_{v\in V(G)} \deg(v) = 2q$, thus

$$w(G_1\Box G_2) = p_1 w(G_2) + 2\ 2q_1)w(G_2) + w(G_1)w(G_2)$$
$$= w(G_1)w(G_2) + p_1 w(G_2) + 4q_1 w(G_2)\,.$$

6. $w(G_1\Diamond G_2) = w(G_1)w(G_2) + p_2 w(G_1) + 4q_2 w(G_1)$.

Proof: Let $x\in V(G_1)$, $y\in V(G_2)$, and $(x,y)\in V(G_1\Diamond G_2)$. Then,

$$w(G_1\Diamond G_2) = \sum_{(x,y)\in V(G_1\Box G_2)} \deg(x,y)^2$$

$$= \sum_{(x,y)\in V(G_1\Diamond G_2)} (\deg(x) + \deg(x)\deg(y))^2$$

$$= \sum_{(x,y)\in V(G_1\Diamond G_2)} (\deg(x)^2 + 2\deg(x)^2\deg(y) + \deg(x)^2\deg(y)^2)$$

$$= \begin{cases} \displaystyle\sum_{(x,y)\in V(G_1\Diamond G_2)} \deg(x)^2 + 2\sum_{(x)\in V(G_1)} \deg(x)^2 \sum_{(y)\in V(G_2)} \deg(y) \\ \displaystyle + \sum_{(x)\in V(G_1)} \deg(x)^2 \sum_{(y)\in V(G_2)} \deg(y)^2 \end{cases}$$

Since, $w(G) = \sum_{v\in V(G)} \deg(v)^2$, and $\sum_{v\in V(G)} \deg(v) = 2q$, thus

$$w(G_1\Diamond G_2) = p_2 w(G_1) + 2w(G_1)2q_2 + w(G_1)w(G_2)$$
$$= w(G_1)w(G_2) + p_2 w(G_1) + 4q_2 w(G_1)\,.$$

Corollary 3.2: Let G_1 and G_2 be disjoint regular graphs of order p_1 and p_2, and degree r_1 and r_2, respectively. Then

1. $w(G_1\times G_2) = p_1 p_2 (r_1 + r_2)^2$.
2. $w(G_1\otimes G_2) = p_1 p_2 r_1^2 r_2^2$.
3. $w(G_1\odot G_2) = p_1 p_2 (r_1 + r_2 + r_1 r_2)^2$.
4. $w(G_1[G_2]) = p_1 p_2 (p_2 r_1 + r_2)^2$.
5. $w(G_1\Box G_2) = p_1 p_2 (r_1 r_2 + r_2)^2$.
6. $w(G_1\Diamond G_2) = p_1 p_2 (r_1 r_2 + r_1)^2$.

Proof: All prove strictly holds by Theorem 2.2, and Theorem 2.4.

Corollary 3.2: Let G_1 and G_2 be disjoint regular graphs of the same order p and same degree r, then

1. $w(G_1 \times G_2) = (2pr)^2$.
2. $w(G_1 \otimes G_2) = p^2 r^4$.
3. $w(G_1 \odot G_2) = p^2 (r^2 + 2r)^2$.
4. $w(G_1[G_2]) = p^2 (rp + r)^2$.
5. $w(G_1 \Box G_2) = p^2 (r^2 + r)^2$.
6. $w(G_1 \Diamond G_2) = p^2 (r^2 + r)^2$.

4. RELATION BETWEEN THE WEIGHTING OF GRAPH PRODUCTS

The concept of the relational product gives us a good tool to investigate properties of the graph product. Here, we investigate relations between the weights of two products from six products.

Remark 4.1: Clearly, if G_1 and G_2 have the same degree sequence. Then $w(G_1) = w(G_2)$.

Theorem 4.2: Let G_1 and G_2 be two connected graphs, then the following relation are valid.

1. $w(G_1 \times G_2) = w(G_1 \otimes G_2)$ iff $G_1 \cong G_2 \cong C_n \, (n : odd)$.
2. $w(G_1 \times G_2) = w(G_1 \odot G_2)$ iff $G_1 = K_1$ or $G_2 = K_1$.
3. $w(G_1 \times G_2) = w(G_1[G_2])$ iff $G_1 = K_1$ or $G_2 = K_1$.
4. $w(G_1 \times G_2) = w(G_1 \Box G_2)$ iff $G_1 = K_1$ or G_1 is bipartite and $G_2 = K_2$.
5. $w(G_1 \times G_2) = w(G_1 \Diamond G_2)$ iff $G_2 = K_1$ or $G_1 = K_2$ and G_2 is bipartite.
6. $w(G_1 \otimes G_2) = w(G_1 \Box G_2)$ iff $G_2 = K_1$.
7. $w(G_1 \otimes G_2) = w(G_1 \Diamond G_2)$ iff $G_1 = K_1$.
8. $w(G_1 \odot G_2) = w(G_1[G_2])$ iff G_2 is a trivial graph or a complete graph.
9. $w(G_1 \odot G_2) = w(G_1 \Box G_2)$ iff $G_1 = K_1$.
10. $w(G_1 \odot G_2) = w(G_1 \Diamond G_2)$ iff $G_2 = K_1$.
11. $w(G_1[G_2]) = w(G_1 \Box G_2)$ iff $G_1 = K_1$.
12. $w(G_1 \odot G_2) = w(G_1 \Diamond G_2)$ iff $G_2 = K_1$.
13. $w(G_1 \Box G_2) = w(G_1 \Diamond G_2)$ iff $G_1 = G_2$.

Proof: Since in each case the two products are isomorphic [7, 8], And use Remark 4.1, then, all cases are valid.

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