

## ON THE EDGE DEGREE WEIGHT SUM OF GRAPH PRODUCTS

Didar A. Ali\*

Department of Mathematics faculty of Science University of Zakho, Zakho - Kurdistan-Region. Iraq

(Received on: 11-11-12; Revised & Accepted on: 16-12-12)

### ABSTRACT

Let  $G = (V, E)$  be a simple graph of order  $p$  and size  $q$ , and let the weight  $w(e)$  of an edge  $e = uv$  of a graph  $G$  is defined by  $w(e) = \deg(u) + \deg(v)$  and the weight of  $G$  is defined by  $w(G) = \sum_{e \in E(G)} w(e)$ ,

where  $E(G)$  is the edge set of  $G$ . In this paper some properties of  $w(G)$  are determined, also we investigate the value  $w(G)$  for various type of graph products, such as, Cartesian, Tensor (or Kronecker), Strong, Composition (or Lexicographic), Skew and Converse Skew products.

**Key words:** Graph Theory, Edge degree weights.

### 1. INTRODUCTION

Let  $v$  be any vertex in a graph  $G$ , then *degree* of  $v$  in a graph  $G$  is the number of vertices incident with  $v$  and denoted by  $\deg(v)$ . Among many ways of assigning weights to the edges of a graph are those that are functions of the degrees of the vertices of a given edge  $e = uv$ . An indispensable for the study of this subject is the survey by Gallian [5]. The weighting  $w(e)$  was studied by Kattimani, Ducasse, Gargano, and Quintas in [3, 4, 6] as:

$$w(e) = \deg(u) + \deg(v),$$

and extended to the weight of a graph  $G = (V, E)$  as:

$$w(G) = \sum_{e \in E(G)} w(e).$$

Here we take one of the most natural types of vertex labeling, that of assigning the degree of a vertex as its label, and we determined and investigate the value  $w(G)$  for various type of graph products.

**Definition 1.1:** The *weight*  $w(e)$  of an edge  $e = uv$  is defined by  $w(e) = \deg(u) + \deg(v)$  and the edge degree weight sum  $w(G)$  of  $G$  relative to this edge weighting is  $w(G) = \sum_{e \in E(G)} w(e)$ .

### 2. SOME BASIC DEFINITION AND THEOREMS

We begin this paper by given the following Definition of graph products.

**Definition 2.1:** Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two graphs with order  $p_1, p_2$  and size  $q_1, q_2$ , respectively. These products result in a graph  $G$  whose vertex set is  $V(G) = V(G_1) \times V(G_2)$ , where  $V(G_i)$  is the vertex set of  $G_i$ . And two vertices  $(u_1, v_1)$  and  $(u_2, v_2)$  are adjacent in graph products, as:

1. The Cartesian product:  $G = G_1 \times G_2$  has edge set

$$E(G) = \{[u_1u_2 \in E(G_1) \text{ and } v_1 \equiv v_2] \text{ or } [u_1 \equiv u_2 \text{ and } v_1v_2 \in E(G_2)]\}.$$

2. The Tensor (or kronecker) product:  $G = G_1 \otimes G_2$  has edge set

$$E(G) = \{[u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)]\}.$$

**Corresponding author: Didar A. Ali\***

Department of Mathematics faculty of Science University of Zakho, Zakho - Kurdistan-Region. Iraq

3. The Strong product:  $G = G_1 \odot G_2$  has edge set

$$E(G) = \{[u_1u_2 \in E(G_1) \text{ and } v_1 \equiv v_2] \text{ or } [u_1 \equiv u_2 \text{ and } v_1v_2 \in E(G_2)] \text{ or } [u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)]\}.$$

4. The Composition (or Lexicographic):  $G = G_1[G_2]$  has edge set

$$E(G) = \{[u_1u_2 \in E(G_1)] \text{ or } [u_1 \equiv u_2 \text{ and } v_1v_2 \in E(G_2)]\}.$$

5. The Skew product:  $G = G_1 \sqcap G_2$  has edge set

$$E(G) = \{[u_1 \equiv u_2 \text{ and } v_1v_2 \in E(G_2)] \text{ or } [u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)]\}.$$

6. The Converse Skew product:  $G = G_1 \diamond G_2$  has edge set

$$E(G) = \{[u_1u_2 \in E(G_1) \text{ and } v_1 \equiv v_2] \text{ or } [u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)]\}.$$

**Lemma 2.2:** Let  $G_1$  and  $G_2$  be disjoint regular graphs of order  $p_1$  and  $p_2$ , and degree  $r_1$  and  $r_2$ , respectively. Then

1. The Cartesian product:  $G = G_1 \times G_2$  is  $(r_1 + r_2)$ -regular.
2. The Tensor (or kronecker) product:  $G = G_1 \otimes G_2$  is  $r_1r_2$ -regular.
3. The Strong product:  $G = G_1 \odot G_2$  is  $(r_1 + r_2 + r_1r_2)$ -regular.
4. The Composition (or Lexicographic):  $G = G_1[G_2]$  is  $(p_2r_1 + r_2)$ -regular.
5. The Skew product:  $G = G_1 \sqcap G_2$  is  $(r_2 + r_1r_2)$ -regular.
6. The Converse Skew product:  $G = G_1 \diamond G_2$  is  $(r_1 + r_1r_2)$ -regular.

The following Theorem is fundamental notion for determining the edge degree weight  $w(G)$

**Theorem 2.3:** For any graph  $G$ ,  $w(G) = \sum_{v \in V(G)} \deg(v)^2$ .

**Proof:** By Definition 1.1,  $w(e) = \deg(u) + \deg(v)$  for edge  $e = uv$  in  $E(G)$ . Thus, each vertex  $v$  in  $V(G)$  contributes the value  $\deg(v)$  to the weights of  $\deg(v)$  edges and thus contributes  $\deg(v) \times \deg(v) = \deg(v)^2$  to  $w(G)$ . Hence

$$w(G) = \sum_{v \in V(G)} \deg(v)^2.$$

**Theorem 2.4:** If  $G$  is a regular graph of degree  $r$ , then  $w(e) = 2r$  for each edge  $e = uv$  in  $E(G)$  and  $w(G) = pr^2$ .

**Proof:** By Definition 1.1,  $w(e) = 2r - 2$ . By Theorem 2.3,  $w(G) = pr^2$ .

### 3. GENERAL RESULTS

The following Theorem determine the value of  $w(G)$  for the graph products.

**Theorem 3.1:** Let  $G_1$  and  $G_2$  be disjoint graphs of order  $p_1$  and  $p_2$ , and size  $q_1$  and  $q_2$ , respectively. Then,

1.  $w(G_1 \times G_2) = p_2w(G_1) + p_1w(G_2) + 8q_1q_2$ .

**Proof:** Let  $x \in V(G_1)$ ,  $y \in V(G_2)$ , and  $(x, y) \in V(G_1 \times G_2)$ . Then,

$$\begin{aligned}
 w(G_1 \times G_2) &= \sum_{(x,y) \in V(G_1 \times G_2)} \deg(x,y)^2 \\
 &= \sum_{(x,y) \in V(G_1 \times G_2)} (\deg(x) + \deg(y))^2 \\
 &= \sum_{(x,y) \in V(G_1 \times G_2)} (\deg(x)^2 + 2\deg(x)\deg(y) + \deg(y)^2) \\
 &= \begin{cases} \sum_{(x,y) \in V(G_1 \times G_2)} \deg(x)^2 + 2 \sum_{(x) \in V(G_1)} \deg(x) \sum_{(y) \in V(G_2)} \deg(y) \\ + \sum_{(x,y) \in V(G_1 \times G_2)} \deg(y)^2 \end{cases}
 \end{aligned}$$

Since,  $w(G) = \sum_{v \in V(G)} \deg(v)^2$ , and  $\sum_{v \in V(G)} \deg(v) = 2q$ , thus

$$\begin{aligned}
 w(G_1 \times G_2) &= p_2 w(G_1) + 2(2q_1)(2q_2) + p_1 w(G_2) \\
 &= p_2 w(G_1) + p_1 w(G_2) + 8q_1 q_2.
 \end{aligned}$$

$$2. \quad w(G_1 \otimes G_2) = w(G_1)w(G_2)$$

**Proof:** Let  $x \in V(G_1)$ ,  $y \in V(G_2)$ , and  $(x, y) \in V(G_1 \otimes G_2)$ . Then,

$$\begin{aligned}
 w(G_1 \otimes G_2) &= \sum_{(x,y) \in V(G_1 \otimes G_2)} \deg(x,y)^2 \\
 &= \sum_{(x,y) \in V(G_1 \otimes G_2)} (\deg(x)\deg(y))^2 \\
 &= \sum_{(x) \in V(G_1)} \deg(x)^2 \sum_{(y) \in V(G_2)} \deg(y)^2
 \end{aligned}$$

Since,  $w(G) = \sum_{v \in V(G)} \deg(v)^2$ , thus

$$w(G_1 \otimes G_2) = w(G_1)w(G_2).$$

$$3. \quad w(G_1 \odot G_2) = w(G_1)w(G_2) + p_2 w(G_1) + p_1 w(G_2) + 4q_2 w(G_1) + 4q_1 w(G_2) + 8q_1 q_2$$

**Proof:** Let  $x \in V(G_1)$ ,  $y \in V(G_2)$ , and  $(x, y) \in V(G_1 \odot G_2)$ . Then,

$$\begin{aligned}
 w(G_1 \odot G_2) &= \sum_{(x,y) \in V(G_1 \odot G_2)} \deg(x,y)^2 \\
 &= \sum_{(x,y) \in V(G_1 \odot G_2)} (\deg(x) + \deg(y) + \deg(x)\deg(y))^2 \\
 &= \begin{cases} \sum_{(x,y) \in V(G_1 \odot G_2)} [\deg(x)^2 + 2\deg(x)\deg(y) + \deg(y)^2 \\ + 2\deg(x)^2 \deg(y) + 2\deg(x)\deg(y)^2 + \deg(x)^2 \deg(y)^2] \end{cases}
 \end{aligned}$$

$$= \begin{cases} \sum_{(x,y) \in V(G_1 \odot G_2)} \deg(x)^2 + 2 \sum_{(x) \in V(G_1)} \deg(x) \sum_{(y) \in V(G_2)} \deg(y) \\ + \sum_{(x,y) \in V(G_1 \odot G_2)} \deg(y)^2 + 2 \sum_{(x) \in V(G_1)} \deg(x)^2 \sum_{(y) \in V(G_2)} \deg(y) \\ + 2 \sum_{(x) \in V(G_1)} \deg(x) \sum_{(y) \in V(G_2)} \deg(y)^2 + \sum_{(x) \in V(G_1)} \deg(x)^2 \sum_{(y) \in V(G_2)} \deg(y)^2 \end{cases}$$

Since,  $w(G) = \sum_{v \in V(G)} \deg(v)^2$ , and  $\sum_{v \in V(G)} \deg(v) = 2q$ , thus

$$\begin{aligned} w(G_1 \odot G_2) &= \begin{cases} p_2 w(G_1) + 2(2q_1)(2q_2) + p_1 w(G_2) + 2q_2 w(G_1) \\ + 2q_1 w(G_2) + w(G_1)w(G_2) \end{cases} \\ &= \begin{cases} w(G_1)w(G_2) + p_2 w(G_1) + p_1 w(G_2) \\ + 4q_2 w(G_1) + 4q_1 w(G_2) + 8q_1 q_2. \end{cases} \end{aligned}$$

$$4. \quad w(G_1[G_2]) = p_2^3 w(G_1) + p_1 w(G_2) + 8p_2 q_1 q_2$$

**Proof:** Let  $x \in V(G_1)$ ,  $y \in V(G_2)$ , and  $(x, y) \in V(G_1[G_2])$ . Then,

$$\begin{aligned} w(G_1[G_2]) &= \sum_{(x,y) \in V(G_1[G_2])} \deg(x, y)^2 \\ &= \sum_{(x,y) \in V(G_1[G_2])} (p_2 \deg(x) + \deg(y))^2 \\ &= \sum_{(x,y) \in V(G_1[G_2])} (p_2^2 \deg(x)^2 + 2p_2 \deg(x) \deg(y) + \deg(y)^2) \\ &= \begin{cases} p_2^2 \sum_{(x,y) \in V(G_1[G_2])} \deg(x)^2 + 2p_2 \sum_{(x) \in V(G_1)} \deg(x) \sum_{(y) \in V(G_2)} \deg(y) \\ + \sum_{(x,y) \in V(G_1[G_2])} \deg(y)^2 \end{cases} \end{aligned}$$

Since,  $w(G) = \sum_{v \in V(G)} \deg(v)^2$ , and  $\sum_{v \in V(G)} \deg(v) = 2q$ , thus

$$\begin{aligned} w(G_1[G_2]) &= p_2^2 p_2 w(G_1) + 2p_2(2q_1)(2q_2) + p_1 w(G_2) \\ &= p_2^3 w(G_1) + p_1 w(G_2) + 8p_2 q_1 q_2. \end{aligned}$$

$$5. \quad w(G_1 \sqcup G_2) = w(G_1)w(G_2) + p_1 w(G_2) + 4q_1 w(G_2) \text{ has edge}$$

**Proof:** Let  $x \in V(G_1)$ ,  $y \in V(G_2)$ , and  $(x, y) \in V(G_1 \sqcup G_2)$ . Then,

$$\begin{aligned} w(G_1 \sqcup G_2) &= \sum_{(x,y) \in V(G_1 \sqcup G_2)} \deg(x, y)^2 \\ &= \sum_{(x,y) \in V(G_1 \sqcup G_2)} (\deg(y) + \deg(x) \deg(y))^2 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{(x,y) \in V(G_1 \square G_2)} (\deg(y)^2 + 2\deg(x)\deg(y)^2 + \deg(x)^2 \deg(y)^2) \\
 &= \begin{cases} \sum_{(x,y) \in V(G_1 \square G_2)} \deg(y)^2 + 2 \sum_{(x) \in V(G_1)} \deg(x) \sum_{(y) \in V(G_2)} \deg(y)^2 \\ + \sum_{(x) \in V(G_1)} \deg(x)^2 \sum_{(y) \in V(G_2)} \deg(y)^2 \end{cases}
 \end{aligned}$$

Since,  $w(G) = \sum_{v \in V(G)} \deg(v)^2$ , and  $\sum_{v \in V(G)} \deg(v) = 2q$ , thus

$$\begin{aligned}
 w(G_1 \square G_2) &= p_1 w(G_2) + 2 \sum_{(x) \in V(G_1)} \deg(x) w(G_2) + w(G_1) w(G_2) \\
 &= w(G_1) w(G_2) + p_1 w(G_2) + 4q_1 w(G_2).
 \end{aligned}$$

$$6. \quad w(G_1 \diamond G_2) = w(G_1) w(G_2) + p_2 w(G_1) + 4q_2 w(G_1).$$

**Proof:** Let  $x \in V(G_1)$ ,  $y \in V(G_2)$ , and  $(x, y) \in V(G_1 \diamond G_2)$ . Then,

$$\begin{aligned}
 w(G_1 \diamond G_2) &= \sum_{(x,y) \in V(G_1 \diamond G_2)} \deg(x, y)^2 \\
 &= \sum_{(x,y) \in V(G_1 \diamond G_2)} (\deg(x) + \deg(y))^2 \\
 &= \sum_{(x,y) \in V(G_1 \diamond G_2)} (\deg(x)^2 + 2\deg(x)\deg(y) + \deg(y)^2) \\
 &= \begin{cases} \sum_{(x,y) \in V(G_1 \diamond G_2)} \deg(x)^2 + 2 \sum_{(x) \in V(G_1)} \deg(x) \sum_{(y) \in V(G_2)} \deg(y) \\ + \sum_{(x) \in V(G_1)} \deg(x)^2 \sum_{(y) \in V(G_2)} \deg(y)^2 \end{cases}
 \end{aligned}$$

Since,  $w(G) = \sum_{v \in V(G)} \deg(v)^2$ , and  $\sum_{v \in V(G)} \deg(v) = 2q$ , thus

$$\begin{aligned}
 w(G_1 \diamond G_2) &= p_2 w(G_1) + 2w(G_1) 2q_2 + w(G_1) w(G_2) \\
 &= w(G_1) w(G_2) + p_2 w(G_1) + 4q_2 w(G_1).
 \end{aligned}$$

**Corollary 3.2:** Let  $G_1$  and  $G_2$  be disjoint regular graphs of order  $p_1$  and  $p_2$ , and degree  $r_1$  and  $r_2$ , respectively. Then

1.  $w(G_1 \times G_2) = p_1 p_2 (r_1 + r_2)^2$ .
2.  $w(G_1 \otimes G_2) = p_1 p_2 r_1^2 r_2^2$ .
3.  $w(G_1 \odot G_2) = p_1 p_2 (r_1 + r_2 + r_1 r_2)^2$ .
4.  $w(G_1[G_2]) = p_1 p_2 (p_2 r_1 + r_2)^2$ .
5.  $w(G_1 \square G_2) = p_1 p_2 (r_1 r_2 + r_2)^2$ .
6.  $w(G_1 \diamond G_2) = p_1 p_2 (r_1 r_2 + r_1)^2$ .

**Proof:** All prove strictly holds by Theorem 2.2, and Theorem 2.4.

**Corollary 3.2:** Let  $G_1$  and  $G_2$  be disjoint regular graphs of the same order  $p$  and same degree  $r$ , then

1.  $w(G_1 \times G_2) = (2pr)^2$ .
2.  $w(G_1 \otimes G_2) = p^2 r^4$ .
3.  $w(G_1 \odot G_2) = p^2(r^2 + 2r)^2$ .
4.  $w(G_1[G_2]) = p^2(rp + r)^2$ .
5.  $w(G_1 \sqcup G_2) = p^2(r^2 + r)^2$ .
6.  $w(G_1 \diamond G_2) = p^2(r^2 + r)^2$ .

#### 4. RELATION BETWEEN THE WEIGHTING OF GRAPH PRODUCTS

The concept of the relational product gives us a good tool to investigate properties of the graph product. Here, we investigate relations between the weights of two products from six products.

**Remark 4.1:** Clearly, if  $G_1$  and  $G_2$  have the same degree sequence. Then  $w(G_1) = w(G_2)$ .

**Theorem 4.2:** Let  $G_1$  and  $G_2$  be two connected graphs, then the following relation are valid.

1.  $w(G_1 \times G_2) = w(G_1 \otimes G_2)$  iff  $G_1 \cong G_2 \cong C_n$  ( $n$ : odd).
2.  $w(G_1 \times G_2) = w(G_1 \odot G_2)$  iff  $G_1 = K_1$  or  $G_2 = K_1$ .
3.  $w(G_1 \times G_2) = w(G_1[G_2])$  iff  $G_1 = K_1$  or  $G_2 = K_1$ .
4.  $w(G_1 \times G_2) = w(G_1 \sqcup G_2)$  iff  $G_1 = K_1$  or  $G_1$  is bipartite and  $G_2 = K_2$ .
5.  $w(G_1 \times G_2) = w(G_1 \diamond G_2)$  iff  $G_2 = K_1$  or  $G_1 = K_2$  and  $G_2$  is bipartite.
6.  $w(G_1 \otimes G_2) = w(G_1 \sqcup G_2)$  iff  $G_2 = K_1$ .
7.  $w(G_1 \otimes G_2) = w(G_1 \diamond G_2)$  iff  $G_1 = K_1$ .
8.  $w(G_1 \odot G_2) = w(G_1[G_2])$  iff  $G_2$  is a trivial graph or a complete graph.
9.  $w(G_1 \odot G_2) = w(G_1 \sqcup G_2)$  iff  $G_1 = K_1$ .
10.  $w(G_1 \odot G_2) = w(G_1 \diamond G_2)$  iff  $G_2 = K_1$ .
11.  $w(G_1[G_2]) = w(G_1 \sqcup G_2)$  iff  $G_1 = K_1$ .
12.  $w(G_1 \odot G_2) = w(G_1 \diamond G_2)$  iff  $G_2 = K_1$ .
13.  $w(G_1 \sqcup G_2) = w(G_1 \diamond G_2)$  iff  $G_1 = G_2$ .

**Proof:** Since in each case the two products are isomorphic [7, 8], And use Remark 4.1, then, all cases are valid.

#### REFERENCES

- [1] L.W. Beineke, and R.J. Wilson; *Selected Topics in Graph Theory*, Academic Press, Inc., London, (1978).
- [2] F. Buckley and F. Harary; *Distance in Graphs*, Addison–Wesley, New York (1990).
- [3] E.G. DuCasse, M.L. Gargano, M.B. Kattimani, and L.V. Quintas; The edge degree weight sum of a graph, *Graph Theory Notes of New York*, **LVI**: 6, The Mathematical Association of America, 38–43 (2009).
- [4] E.G. DuCasse and L.V. Quintas; Edge degree weight generalizations, *Graph Theory Notes of New York*, **LXI**:4, The Mathematical Association of America, 25–30 (2011).
- [5] J.A. Gallian; A dynamic survey of graph labeling, *Electronic Journal of Combinatorics*, **14**, #DS6 (2007).
- [6] M.B. Kattimani; A note on edge degree weighted sums of a graph, *Graph Theory Notes of New York*, **LV**:3, NewYork Academy of Sciences, 25–26 (2008).
- [7] D.J. Miller; The categorical product of graphs, *Canad. J. Math.*, Vol..20, 1511-1521 (1968).
- [8] Y. Shibata; and Y. Kikuchi; Graph product based on the distance in graphs, *IEICE Trans. Fundamentals*, Vol. E83-A, No.3, 459-464 (2000).

Source of support: Nil, Conflict of interest: None Declared