International Journal of Mathematical Archive-3(12), 2012, 4816-4823
IJMA Available online through www.ijma.info ISSN 2229 – 5046

CURVATURE TENSORS EQUIPPED WITH AN INTEGRATED CONTACT METRIC STRUCTURE MANIFOLD

SHALINI SINGH*
Department of Applied Mathematics, JSS Academy of Technical Education, Noida-201301, India

(Received on: 31-07-12; Revised & Accepted on: 22-08-12)

ABSTRACT

In the present paper, I have defined an integrated contact metric structure manifold [6] admitting semi-symmetric metric connexion [4] in M_n^* and the form of curvature tensor R of the manifold relative to this conexion has been derived. Several useful theorems and results have also been derived in this manifold.

Key words: *C^∞-manifold, integrated contact structure, integrated contact metric structure, Riemannian connexion, Semi-symmetric metric connexion.*

AMS Mathematics Subject Classification No: *53.*

1. Introduction

Let M_n be a differentiable manifold of differentiability class C^∞. Let there exist in M_n a vector valued C^∞ linear function Φ, a C^∞- vector field η and a C^∞-one form ξ such that

(1.1) $$\Phi^2(X) = a^2 X - c\xi(X)\eta$$

(1.2) $$(\overline{\eta}) = 0$$

(1.3) $$G(\overline{X},\overline{Y}) = a^2 G(X,Y) - c\,\xi(X)\xi(Y)$$

Where $\Phi(X) = \overline{X}$, a is a non-zero complex number and c is an integer.

Let us agree to say that Φ gives to M_n a differentiable structure define by algebraic equation (1.1). We shall call (Φ,η,a,c,ξ) as an integrated contact structure.

Remark 1.1: The manifold M_n equipped with an integrated contact structure (Φ,η,a,c,ξ) will be called an integrated contact structure manifold.

Remark 1.2: The C^∞-manifold M_n satisfying (1.1), (1.2) and (1.3) is called an integrated contact metric structure manifold (Φ,η,a,c,G,ξ)

Agreement 1.1: All the equations which follows will hold for arbitrary vector fields X,Y,Z......... etc.

It is easy to calculate in M_n that

(1.4) $$\xi(\eta) = \frac{a^2}{c}$$

(1.5) $$\Phi(\overline{X}) = 0$$

and

(1.6) $$G(X,\eta) \underline{\underline{def}}\ \xi(X)$$

Corresponding author: *SHALINI SINGH**
Department of Applied Mathematics, JSS Academy of Technical Education, Noida-201301, India

Remark 1.3: The integrated contact metric structure manifold $(\Phi, \eta, a, c, G, \xi)$ gives an almost norden contact metric manifold [5], Lorentzian para-contact manifold [2] or an almost Para contact Riemannian manifold [1] according as $(a^2 = -1,\ c = 1), (a^2 = 1,\ c = -1)$ or $(a^2 = 1,\ c = 1)$

Agreement 1.2: An integrated contact metric structure manifold will be denoted by M_n. In the sequel, arbitrary vector fields will be denoted by $X, Y, Z, \ldots\ldots$ etc.

If we define

(1.7) $${}'\Phi(X,Y) = G(\bar{X},Y) = G(X,\bar{Y})$$

Where ${}'\Phi$ is a tensor field of the type (0, 2) then it is easy to see that

(1.8) $${}'\Phi(X,Y) = {}'\Phi(Y,X)$$

Which shows that ${}'\Phi$ is symmetric in X and Y. Also we have

(1.9) $${}'\Phi(\bar{X},\bar{Y}) = a^2\,{}'\Phi(X,Y)$$

Definition 1.1: A C^∞-manifold M_n satisfying

(1.10) $$D_X \eta = \Phi X$$

will be denoted by M_n^* where D is the Riemannian connexion in M_n corresponding to the Riemannian metric G. It is easy to calculate in M_n^*, we have

(1.11) $$(D_X \xi)(Y) = {}'\Phi(X,Y)$$

2. Semi-symmetric metric connexion in C^∞-manifold M_n^*:

Definition 2.1: Let D be a Riemannian connexion in M_n^*. We consider a semi-symmetric metric connexion B in M_n^*

(2.1) $$B_X Y \underline{\underline{def}}\ D_X Y + \xi(Y)X - G(X,Y)\eta$$

The above equation is equivalent to

(2.2) $$B_X Y = D_X Y + H(X,Y)$$

where

(2.3) $$H(X,Y)\ \underline{\underline{def}}\ \xi(Y)X - G(X,Y)\eta$$

Let S be the torsion tensor of the semi-symmetric metric connexion B then (2.1) and (2.2) imply that

(2.4)a $$S(X,Y) = \xi(Y)X - \xi(X)Y$$

(2.4)b $$S(X,Y) = H(X,Y) - H(Y,X)$$

We define

(2.5)a $${}'H(X,Y,Z)\ \underline{\underline{def}}\ G(H(X,Y),Z)$$

(2.5)b $${}'S(X,Y,Z)\ \underline{\underline{def}}\ G(S(X,Y),Z)$$

Operating G on both the sides of (2.3) and using (2.5)a, we get

(2.6) $${}'H(X,Y,Z) = \xi(Y)G(X,Z) - \xi(Z)G(X,Y)$$

Operating G on both the sides of (2.4)a and using (2.5)b, we get

(2.7) $${}'S(X,Y,Z) = \xi(Y)G(X,Z) - \xi(X)G(Y,Z)$$

Operating G on both the sides of (2.4)b and using (2.5)a and (2.5)b, we get

(2.8) $$'S(X,Y,Z) = `H(X,Y,Z) - `H(Y,X,Z)$$

3. Curvature Tensor in M_n^*:

Let K be the curvature tensor corresponding to the Riemannian connexion D in M_n^* and R be the curvature tensor corresponding to the semi-symmetric metric connexion B in M_n^*.

From (2.1), we have

(3.1) $$B_Y Z = D_Y Z + \xi(Z)Y - G(Y,Z)\eta$$

Taking the covariant derivative of the above equation with respect to the connexion B along the vector field X, we get

$$B_X B_Y Z = B_X(D_Y Z) + \xi(Z)B_X Y + \{(B_X\xi)(Z) + \xi(B_X Z)\}Y$$
$$-G(B_X Y,Z)\eta - G(Y,B_X Z)\eta - (B_X G)(Y,Z)\eta - G(Y,Z)(B_X\eta)$$

Using (2.1) and then (1.6) in the above equation, we get

(3.2)a $$\begin{aligned}B_X B_Y Z = {} & D_X D_Y Z + \xi(D_Y Z)X - G(X,D_Y Z)\eta + \xi(Z)D_X Y \\ & +\xi(Z)\xi(Y)X - \xi(Z)G(X,Y)\eta + (D_X\xi)(Z)Y + \xi(D_X Z)Y \\ & -\xi(Y)G(X,Z)\eta + \xi(Z)\xi(X)Y - G(X,Z)\xi(\eta)Y - G(D_X Y,Z)\eta \\ & -G(X,Y)\xi(Z)\eta - G(Y,D_X Z)\eta - G(Y,X)\eta\xi(Z) - G(X,Z)\xi(Y)\eta \\ & -(B_X G)(Y,Z)\eta - G(Y,Z)\overline{X} - G(Y,Z)\xi(\eta)X + G(Y,Z)\xi(X)\eta\end{aligned}$$

Interchanging X and Y in the above equation, we get

(3.2) $$\begin{aligned}B_Y B_X Z = {} & D_Y D_X Z + \xi(D_X Z)Y - G(Y,D_X Z)\eta + \xi(Z)D_Y X \\ & +\xi(Z)\xi(X)Y - \xi(Z)G(Y,X)\eta + (D_Y\xi)(Z)X + \xi(D_Y Z)X \\ & -\xi(X)G(Y,Z)\eta + \xi(Z)\xi(Y)X - G(Y,Z)\xi(\eta)X - G(D_Y X,Z)\eta \\ & -G(Y,X)\xi(Z)\eta - G(X,D_Y Z)\eta - G(X,Y)\eta\xi(Z) - G(Y,Z)\xi(X)\eta \\ & -(B_Y G)(X,Z)\eta - G(X,Z)\overline{Y} - G(X,Z)\xi(\eta)Y + G(X,Z)\xi(Y)\eta\end{aligned}$$

Replacing Y by $[X,Y]$ in (3.1) and using $[X,Y] = D_X Y - D_Y X$, we get

(3.2)c $$B_{[X,Y]}Z = D_{[X,Y]}Z + \xi(Z)(D_X Y) - \xi(Z)(D_Y X) - G(D_X Y,Z)\eta - G(D_Y X,Z)\eta$$

Subtracting (3.2)b, (3.2)c from (3.2)a and using $K(X,Y,Z) \underline{\underline{def}} D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$, we get

$$R(X,Y,Z) = K(X,Y,Z) + (D_X\xi)(Z)Y - 3\xi(Y)G(X,Z)\eta - (D_Y\xi)(Z)X$$
$$-G(Y,Z)\overline{X} + (B_Y G)(X,Z)\eta + G(X,Z)\overline{Y}$$

where

$$R(X,Y,Z) \underline{\underline{def}} B_X B_Y Z - B_Y B_X Z - B_{[X,Y]}Z$$

Agreement 3.1: We consider that the fundamental 2-form $`\Phi$ is closed in M_n^* i.e.

(3.4) $$(D_X `\Phi)(Y,Z) + (D_Y `\Phi)(Z,X) + (D_Z `\Phi)(X,Y) = 0$$

Theorem 3.1: In M_n^*, we have

(3.5) $$(D_Z `\Phi)(X,Y) = G((D_Z\Phi)X,Y) = `K(X,Y,Z,\eta)$$

(3.6) $$(D_X {}^{`}\Phi)(Y,\eta) = -G(\overline{X},\overline{Y})$$

Proof: We know that

$$ {}^{`}\Phi(Y,Z) = (D_Y\xi)(Z)$$

Taking the covariant derivative on both the sides of the above equation with respect to the vector field X , we have

(3.7)a $$(D_X {}^{`}\Phi)(Y,Z) = (D_X D_Y \xi)(Z) - {}^{`}\Phi(D_X Y, Z)$$

Interchanging X and Y in the above equation, we get

(3.7)b $$(D_Y {}^{`}\Phi)(X,Z) = (D_Y D_X \xi)(Z) - {}^{`}\Phi(D_Y X, Z)$$

Subtracting (3.7)b from (3.7)a and using $D_X Y - D_Y X = [X,Y]$, we have

$$(D_X {}^{`}\Phi)(Y,Z) - (D_Y {}^{`}\Phi)(X,Z) = (D_X D_Y \xi)(Z) - (D_Y D_X \xi)(Z) - {}^{`}\Phi([X,Y],Z)$$

Using (1.11) in the above equation, we get

$$\begin{aligned}(D_X {}^{`}\Phi)(Y,Z) - (D_Y {}^{`}\Phi)(X,Z) &= (D_X D_Y \xi)(Z) - (D_Y D_X \xi)(Z) - (D_{[X,Y]}\xi)(Z) \\ &= G(Z, K(X,Y,\eta)) \\ &= {}^{`}K(X,Y,\eta,Z) \\ &= -{}^{`}K(X,Y,Z,\eta) \\ &= -G(K(X,Y,Z),\eta) \\ &= -\xi(K(X,Y,Z))\end{aligned}$$

Using (3.4) in the above equation, we get (3.5).

We have

$$ {}^{`}\Phi(Y,\eta) = 0$$

$$\Rightarrow (D_X {}^{`}\Phi)(Y,\eta) = -{}^{`}\Phi(Y, D_X\eta)$$

Using (1.10) in the above equation, we get

$$(D_X {}^{`}\Phi)(Y,\eta) = -\ {}^{`}\Phi(Y,\overline{X})$$

Using (1.7) in the above equation, we get (3.6).

Theorem 3.2: In M_n^*, we have

(3.8) $$(D_Z {}^{`}\Phi)(X,Y) = c[\xi(X)G(Y,Z) - \xi(Y)G(X,Z)]$$

Proof: From (1.9), we have

$$ {}^{`}\Phi(\overline{X},\overline{Y}) = a^2\ {}^{`}\Phi(X,Y)$$

Taking the covariant derivative on both the sides of the above equation with respect to the vector field Z , we have

$$(D_Z {}^{`}\Phi)(\overline{X},\overline{Y}) + {}^{`}\Phi((D_Z\Phi)X,\overline{Y}) + {}^{`}\Phi(\overline{X},(D_Z\Phi)Y) = a^2 (D_Z {}^{`}\Phi)(X,Y)$$

Using (1.7) in the above equation, we get

$$(D_Z {}^{`}\Phi)(\overline{X},\overline{Y}) + G(\overline{\overline{Y}},(D_Z\Phi)(X)) + G(\overline{\overline{X}},(D_Z\Phi)(Y)) = a^2 (D_Z {}^{`}\Phi)(X,Y)$$

Using (1.1) in the above equation, we get

$$\left(D_Z {}^{`}\Phi\right)\left(\overline{X},\overline{Y}\right)+a^2\left(D_Z {}^{`}\Phi\right)\left(X,Y\right)+c\xi\left(Y\right)G\left(\overline{Z},\overline{X}\right)-c\xi\left(X\right)G\left(\overline{Z},\overline{Y}\right)=0$$

Using (1.3) in the above equation, we get

$$\left(D_Z {}^{`}\Phi\right)\left(\overline{X},\overline{Y}\right)+a^2\left(D_Z {}^{`}\Phi\right)\left(X,Y\right)+a^2c\xi\left(Y\right)G\left(X,Z\right)-a^2c\xi\left(X\right)G\left(Y,Z\right)=0$$

We define

$$\mathrm{P}\left(X,Y,Z\right) \stackrel{def}{=} a^2\left(D_Z {}^{`}\Phi\right)\left(\overline{X},\overline{Y}\right)$$

$$Q\left(X,Y,Z\right) \stackrel{def}{=} a^2\left(D_Z {}^{`}\Phi\right)\left(X,Y\right)+a^2c\xi\left(Y\right)G\left(X,Z\right)-a^2c\xi\left(X\right)G\left(Y,Z\right)$$

Then $\mathrm{P}\left(X,Y,Z\right)+Q\left(X,Y,Z\right)=0$

Also in consequence of (3.6), we have

$$\mathrm{P}\left(\overline{X},\overline{Y},Z\right)=Q\left(X,Y,Z\right),\ Q\left(\overline{X},\overline{Y},Z\right)=\mathrm{P}\left(X,Y,Z\right)$$

All these equations are satisfied by $\mathrm{P}=Q=0$

i.e $a^2\left(D_Z {}^{`}\Phi\right)\left(X,Y\right)=-a^2c\,\xi\left(Y\right)G\left(X,Z\right)+a^2c\,\xi\left(X\right)G\left(Y,Z\right)$

i.e. $\left(D_Z {}^{`}\Phi\right)\left(X,Y\right)=c\left[\xi\left(X\right)G\left(Y,Z\right)-\xi\left(Y\right)G\left(X,Z\right)\right]$

This proves the theorem.

Theorem 3.3: In M_n^*, we have

(3.9) $${}^{`}K\left(X,Y,\overline{Z},U\right)-{}^{`}K\left(X,Y,Z,\overline{U}\right)+{}^{`}K\left(X,\overline{Y},Z,U\right)+{}^{`}K\left(\overline{X},Y,Z,U\right)+\left(D_\eta {}^{`}K\right)\left(X,Y,Z,U\right)=0$$

Proof: From (3.5), we have

(3.10)a $$\left(D_Y {}^{`}\Phi\right)\left(Z,U\right)={}^{`}K\left(Z,U,Y,\eta\right)$$

Taking the covariant derivative on both the sides of the above equation with respect to the vector field X and using (1.10) and (3.5), we have

$$\left(D_X D_Y {}^{`}\Phi\right)\left(Z,U\right)=\left(D_X {}^{`}K\right)\left(Z,U,Y,\eta\right)+{}^{`}K\left(Z,U,D_XY,\eta\right)+{}^{`}K\left(Z,U,Y,\overline{X}\right)$$
$$+{}^{`}K\left(D_XZ,U,Y,\eta\right)+\left(Z,D_XU,Y,\eta\right)-\left(D_XZ,U,Y,\eta\right)-\left(Z,D_XU,Y,\eta\right)$$

i.e.

(3.10)b $$\left(D_X D_Y {}^{`}\Phi\right)\left(Z,U\right)=\left(D_X {}^{`}K\right)\left(Z,U,Y,\eta\right)+{}^{`}K\left(Z,U,D_XY,\eta\right)+{}^{`}K\left(Z,U,Y,\overline{X}\right)$$

Interchanging X and Y in the above equation, we get

(3.10)c $$\left(D_Y D_X {}^{`}\Phi\right)\left(Z,U\right)=\left(D_Y {}^{`}K\right)\left(Z,U,X,\eta\right)+{}^{`}K\left(Z,U,D_YX,\eta\right)+{}^{`}K\left(Z,U,X,\overline{Y}\right)$$

Replacing Y by $\left[X,Y\right]$ in (3.10)a, we have

(3.10)d $$\left(D_{[X,Y]} {}^{`}\Phi\right)\left(Z,U\right)={}^{`}K\left(Z,U,\left[X,Y\right],\eta\right)$$

Subtracting (3.10)c and (3.10)d from (3.10)b, we have

$$\left(D_X D_Y {}^{`}\Phi\right)\left(Z,U\right)-\left(D_Y D_X {}^{`}\Phi\right)\left(Z,U\right)-\left(D_{[X,Y]} {}^{`}\Phi\right)\left(Z,U\right)={}^{`}K\left(X,Y,\overline{Z},U\right)-{}^{`}K\left(X,Y,Z,\overline{U}\right)$$

i.e.

$$
\begin{aligned}
{}'K\left(X,Y,\bar{Z},U\right)-{}'K\left(X,Y,Z,\bar{U}\right)=\left(D_X{}'K\right)\left(Z,U,Y,\eta\right) &+{}'K\left(Z,U,D_XY,\eta\right)\\
&+{}'K\left(Z,U,Y,\bar{X}\right)-\left(D_Y{}'K\right)\left(Z,U,X,\eta\right)-{}'K\left(Z,U,D_YX,\eta\right)-{}'K\left(Z,U,X,\bar{Y}\right)\\
&-{}'K\left(Z,U,[X,Y],\eta\right)\ {}'K\left(X,Y,\bar{Z},U\right)-{}'K\left(X,Y,Z,\bar{U}\right)=\left(D_X{}'K\right)\left(Z,U,Y,\eta\right)\\
&+{}'K\left(Z,U,Y,\bar{X}\right)-\left(D_Y{}'K\right)\left(Z,U,X,\eta\right)-{}'K\left(Z,U,X,\bar{Y}\right)
\end{aligned}
$$

i.e.

$$
\begin{aligned}
{}'K\left(X,Y,\bar{Z},U\right)-{}'K\left(X,Y,Z,\bar{U}\right)-{}'K\left(Z,U,Y,\bar{X}\right)+{}'K\left(Z,U,X,\bar{Y}\right)\\
=\left(D_X{}'K\right)\left(Z,U,Y,\eta\right)-\left(D_Y{}'K\right)\left(Z,U,X,\eta\right)\\
=\left(D_X{}'K\right)\left(Y,\eta,Z,U\right)-\left(D_Y{}'K\right)\left(\eta,X,Z,U\right)
\end{aligned}
$$

Using Bianchi's second identity [3] in the above equation, we get (3.9).

Corollary 3.1: In M_n^*, we have

(3.11)a
$$
\begin{aligned}
a^2\,{}'K\left(X,Y,Z,U\right)+{}'K\left(X,Y,\bar{Z},\bar{U}\right)&=-c\,\xi\left(U\right){}'K\left(X,Y,Z,\eta\right)\\
&-{}'K\left(X,\bar{Y},Z,\bar{U}\right)-{}'K\left(\bar{X},Y,Z,\bar{U}\right)-\left(D_\eta{}'K\right)\left(X,Y,Z,\bar{U}\right)
\end{aligned}
$$

(3.11)b
$$
\begin{aligned}
{}'K\left(\bar{X},Y,\bar{Z},U\right)-{}'K\left(\bar{X},Y,Z,\bar{U}\right)+{}'K\left(\bar{X},\bar{Y},Z,U\right)+a^2\,{}'K\left(X,Y,Z,U\right)\\
-c\,\xi\left(X\right){}'K\left(\eta,Y,Z,U\right)+\left(D_\eta{}'K\right)\left(X,\bar{Y},Z,U\right)=0
\end{aligned}
$$

(3.11)c
$$
\begin{aligned}
{}'K\left(X,\bar{Y},\bar{Z},U\right)-{}'K\left(X,\bar{Y},Z,\bar{U}\right)+a^2\,{}'K\left(X,Y,Z,U\right)\\
-c\,\xi\left(Y\right){}'K\left(X,\eta,Z,U\right)+{}'K\left(\bar{X},\bar{Y},Z,U\right)+\left(D_\eta{}'K\right)\left(X,\bar{Y},Z,U\right)=0
\end{aligned}
$$

Proof: Barring U on both the sides of (3.9) and using (1.1), we get (3.11)a.

Barring X on both the sides of (3.9) and using (1.1), we get (3.11)b.

Barring Y on both the sides of (3.9) and using (1.1), we get (3.11)c.

Theorem 3.4: In M_n^*, we have

(3.12)a
$$Ric\left(\bar{Y},Z\right)+a^2Ric\left(Y,Z\right)=c\xi\left(Y\right)Ric\left(\eta,Z\right)-\left(D_\eta Ric\right)\left(\bar{Y},Z\right)$$

(3.12)b
$$Ric\left(\bar{Y},Z\right)+a^2Ric\left(Y,Z\right)=c\xi\left(Z\right)Ric\left(\eta,Y\right)-\left(D_\eta Ric\right)\left(Y,\bar{Z}\right)$$

Proof: Equation (3.9) can be written as

$$
\begin{aligned}
&G\left(K\left(X,Y,\bar{Z}\right),U\right)-G\left(\overline{K\left(X,Y,Z\right)},U\right)+G\left(K\left(X,\bar{Y},Z\right),U\right)\\
&+G\left(K\left(\bar{X},Y,Z\right),U\right)+G\left(\left(D_\eta K\right)\left(X,Y,Z\right),U\right)=0
\end{aligned}
$$

Which is equivalent to

$$K\left(X,Y,\bar{Z}\right)-\overline{K\left(X,Y,Z\right)}+K\left(X,\bar{Y},Z\right)+K\left(\bar{X},Y,Z\right)+\left(D_\eta K\right)\left(X,Y,Z\right)=0$$

Contracting the above equation with respect to the vector field X, we get

(3.13)
$$Ric\left(Y,\bar{Z}\right)+Ric\left(\bar{Y},Z\right)+\left(D_\eta Ric\right)\left(Y,Z\right)=0$$

Barring Y in the above equation and using (1.1), we get (3.12)a.

Barring Z in the above equation and using (1.1), we get (3.12)b.

Corollary 3.2: The scalar curvature of M_n^*-manifold is constant along the vector field η.

Proof: Equation (3.13) can be written as

$$G\left(R\overline{Y},Z\right)+G\left(RY,\overline{Z}\right)+G\left(\left(D_\eta R\right)\left(Y\right),Z\right)=0$$

Where R is the Ricci tensor of the type $(1,1)$.

Using (1.7) in the above equation, we get

$$G\left(R\overline{Y},Z\right)+G\left(\overline{RY},Z\right)+G\left(\left(D_\eta R\right)\left(Y\right),Z\right)=0$$

which is equivalent to

$$R\overline{Y}+\overline{RY}+\left(D_\eta R\right)\left(Y\right)=0$$

Contracting the above equation with respect to the vector field X, we get

$$\left(D_\eta r\right)=0\Rightarrow \eta r=0$$

This proves the theorem.

Theorem 3.5: In M_n^*, we have

(3.14)a $$`K\left(X,Y,\overline{Z},U\right)-`K\left(X,Y,Z,\overline{U}\right)=c\left[G\left(U,Y\right)`\Phi\left(X,Z\right)-G\left(Y,Z\right)`\Phi\left(X,U\right)-G\left(U,X\right)`\Phi\left(Y,Z\right)+G\left(X,Z\right)`\Phi\left(Y,U\right)\right]$$

(3.14)b $$`K\left(X,Y,\overline{Z},\overline{U}\right)-a^2\ `K\left(X,Y,Z,U\right)=c\left[`\Phi\left(Y,U\right)`\Phi\left(X,Z\right)-`\Phi\left(Y,Z\right)`\Phi\left(X,U\right)\right]-a^2\left[G\left(Y,Z\right)G\left(X,U\right)-G\left(X,Z\right)G\left(Y,U\right)\right]$$

Proof: We have

(3.15)a $$`K\left(X,Y,\overline{Z},U\right)-`K\left(X,Y,Z,\overline{U}\right)+`K\left(X,\overline{Y},Z,U\right)+`K\left(\overline{X},Y,Z,U\right)=\left(D_X`K\right)\left(Y,\eta,Z,U\right)-\left(D_Y`K\right)\left(X,\eta,Z,U\right)$$

From (3.5) and (3.8), we have

(3.15)b $$`K\left(X,Y,Z,\eta\right)=c\left[\xi\left(X\right)G\left(Y,Z\right)-\xi\left(Y\right)G\left(Z,X\right)\right]$$

The above equation can be written as

$$`K\left(Y,\eta,Z,U\right)=c\left[\xi\left(Z\right)G\left(U,Y\right)-\xi\left(U\right)G\left(Y,Z\right)\right]$$

Taking the covariant derivative of the above equation with respect to the vector field X, we have

$$\left(D_X`K\right)\left(Y,\eta,Z,U\right)+`K\left(Y,\overline{X},Z,U\right)=c\left[\left(D_X\xi\right)\left(Z\right)G\left(U,Y\right)-\left(D_X\xi\right)\left(U\right)G\left(Y,Z\right)\right]$$

Using (1.11) in the above equation, we get

(3.16)a $$\left(D_X`K\right)\left(Y,\eta,Z,U\right)=`K\left(\overline{X},Y,Z,U\right)+c\left[G\left(U,Y\right)`\Phi\left(X,Z\right)-G\left(Y,Z\right)`\Phi\left(X,U\right)\right]$$

Interchanging X and Y in the above equation, we get

(3.16)b $$\left(D_Y`K\right)\left(X,\eta,Z,U\right)=-`K\left(X,\overline{Y},Z,U\right)+c\left[G\left(U,X\right)`\Phi\left(Y,Z\right)-G\left(X,Z\right)`\Phi\left(Y,U\right)\right]$$

Subtracting (3.16)b from (3.16)a, we have

(3.17) $$\left(D_X`K\right)\left(Y,\eta,Z,U\right)-\left(D_Y`K\right)\left(X,\eta,Z,U\right)=`K\left(\overline{X},Y,Z,U\right)+K\left(X,\overline{Y},Z,U\right)+c\left[G\left(U,Y\right)`\Phi\left(X,Z\right)-G\left(Y,Z\right)`\Phi\left(X,U\right)-G\left(U,X\right)`\Phi\left(Y,Z\right)+G\left(X,Z\right)`\Phi\left(Y,U\right)\right]$$

From (3.15)a and (3.17), we get (3.14)a.

Barring U on both the sides of (3.14)a and using (1.1) and (1.7), we get

$$
\begin{aligned}
`K\left(X,Y,\bar{Z},\bar{U}\right) - a^2\, `K\left(X,Y,Z,U\right) + c\,\xi\left(U\right)`K\left(X,Y,Z,\eta\right) \\
= c\left[`\Phi\left(Y,U\right)`\Phi\left(X,Z\right) - `\Phi\left(Y,Z\right)`\Phi\left(X,U\right)\right. \\
\left. - a^2\,G\left(X,U\right)G\left(Y,Z\right) + a^2\,G\left(X,Z\right)G\left(Y,U\right)\right]
\end{aligned}
$$

Using (3.15)b in the above equation, we get (3.14)b.

REFERENCES

[1] Adati T. and Matsumoto K., On almost paracontact Riemannian manifold, T.R.U. Maths., 13(2) (1977), 22-39.

[2] Matsumoto K., On Lorentzian Paracontact manifolds, Bull. Yamogata Univ. Nat. Sci., 12(1989), 151-156.

[3] Mishra R. S., Structure on a differentiable manifold and their application, Chandrma Prakashan, Allahabad, India. (1984).

[4] Sharfuddin A. and Hussain S. I., Semi-symmetric metric connections in almost contact manifolds. Tensor, N. S., 30, (1976), 133-139.

[5] Singh S. D. and Singh D., Tensor of the type (0, 4) in an almost Norden contact metric manifold, Acta Cincia Indica, India, 18 M (1) (1997), 11-16.

[6] Singh Shalini, Certain affine connexions on an integrated contact metric structure manifold, Adv. Theor. Appl. Math., 4 (1-2) (2009), 11-19.

Source of support: Nil, Conflict of interest: None Declared