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HIGHLY ACCURATE METHOD FOR SOLVING FIRST ORDER HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION IN THREE SPACE DIMENSIONS

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ABSTRACT

In this paper, we shall develop a new approach to an implicit method for solving the first-order hyperbolic partial differential equation in three space dimensions. The suggested method gives highly accurate result. The stability condition and the advantages of the considered method compared with the classical methods as Lax-Wendroff method are discussed.

Keywords: *Pade` approximation, Restrictive Pade` approximation, finite difference and hyperbolic partial differential equations.*

1. INTRODUCTION

Consider the first-order hyperbolic partial differential equation in three space dimensions

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}+c\frac{\partial u}{\partial z}=0, \qquad 0\le x,y,z\le 1, \quad t\ge 0 \tag{1}$$

where a, b and c are real constants, and u(x, y, z, t) satisfy the initial and boundary conditions:

$$\left.\begin{array}{llll} u(x,y,z,0)=f(x,y,z), & 0\le x,y,z\le 1 & & \\ u(0,y,z,t)=g_0(y,z,t), & u(1,y,z,t)=g_1(y,z,t), & t\ge 0, & 0\le y,z\le 1 \\ u(x,0,z,t)=g_2(x,z,t), & u(x,1,z,t)=g_3(x,z,t), & t\ge 0, & 0\le x,z\le 1 \\ u(x,y,0,t)=g_2(x,y,t), & u(x,y,1,t)=g_3(x,y,t), & t\ge 0, & 0\le x,y\le 1 \end{array}\right\} \tag{2}$$

The explicit finite difference method, which is most widely used, is the Lax-Wendroff method [1] is given by:

$$\begin{aligned} u^{n+1}_{i,j,k} &= u^n_{i,j,k} - 0.5\,p[u^n_{i+1,j,k} - u^n_{i-1,j,k} + u^n_{i,j+1,k} - u^n_{i,j-1,k} + u^n_{i,j,k+1} - u^n_{i,j,k-1} \\ &\quad + 0.5p^2[u^n_{i+1,j,k} - 6u^n_{i,j,k} + u^n_{i-1,j,k} + u^n_{i,j+1,k} + u^n_{i,j-1,k} + u^n_{i,j,k+1} + u^n_{i,j,k-1}] \\ &\quad + 0.25p^2[u^n_{i+1,j,k+1} - u^n_{i-1,j,k+1} - u^n_{i+1,j,k-1} + u^n_{i-1,j,k-1} + u^n_{i+1,j+1,k} - u^n_{i-1,j+1,k} \\ &\quad - u^n_{i+1,j-1,k} - u^n_{i-1,j-1,k} + u^n_{i,j+1,k+1} - u^n_{i,j-1,k+1} - u^n_{i,j+1,k-1} + u^n_{i,j-1,k-1}], \; p=\frac{k}{h} \end{aligned} \tag{3}$$

In this paper we define an implicit method for solving the first-order hyperbolic partial differential equation in three space dimensions produces very high accuracy compared with the other classical method, i.e. the numerical solution produced by the considered method is almost identical to the exact solution. We use the restrictive Pade` approximation as done in [2],[3],[8]and[5] to approximate the exponential function.

2. RESTRICTIVE PADE` APPROXIMATION FOR THE FIRST-ORDER HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION IN THREE SPACE DIMENSIONS

Consider first-order hyperbolic partial differential equation (1). The exact solution of grid representation of equations(1) is:

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$$u_{i,j,k}^{n+1} = \exp\left(k\frac{\partial}{\partial t}\right)u_{i,j,k}^{n} = \exp\left(-k\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\right)\right)u_{i,j,k}^{n} ,$$

then the approximate solution of grid representation of equation (1) can take the form:

$$u_{i,j,k}^{n+1} = \exp(-r(aD_x + bD_y + cD_z))u_{i,j,k,}^{n}, \quad r = \frac{k}{2h} , \tag{4}$$

Where,

$D_x u_{i,j,k}^n \cong (u^n{}_{i+1,j,k} - u^n{}_{i-1,j,k})$, $D_y u_{i,j,k}^n \cong (u^n{}_{i,j+1,k} - u^n{}_{i,j-1,k})$ *and* $D_z u_{i,j,k}^n \cong (u^n{}_{i,j,k+1} - u^n{}_{i,j,k+1})$

The restrictive Pade` approximation [1/1] can take the form:

$$RPA[1/1]_{\exp(-r(aD_x+bD_y+cD_z))}(r) = \left(1 + (\varepsilon_{i,j.k} + \frac{1}{2}(aD_x + bD_y + cD_z))r\right)^{-1}\left(1 + (\varepsilon_{i,j,k} - \frac{1}{2}(aD_x + bD_y + cD_z))\, r\right)$$

Then we can approximate equation (4) as:

$$u_{i,j,k}^{n+1} = \left(1 + (\varepsilon_{i,j.k} + \frac{1}{2}(aD_x + bD_y + cD_z))\, r\right)^{-1}\left(1 + (\varepsilon_{i,j,k} - \frac{1}{2}(aD_x + bD_y + cD_z))\, r\right)u_{i,j,k}^{n} \tag{5}$$

which can take the equivalent scalar form:

$$\begin{aligned}&(1 + r\varepsilon_{i,j,k})u_{i,j,k}^{n+1} + 0.5\, r_1(u_{i+1,j,k}^{n+1} - u_{i-1,j,k}^{n+1}) + 0.5r_2(u_{i,j+1,k}^{n+1} - u_{i,j-1,k}^{n+1}) + 0.5r_3(u_{i,j,k+1}^{n+1} - u_{i,j,k-1}^{n+1})\\ &= (1 + r\varepsilon_{i,j,k})u_{i,j,k}^{n} - 0.5\, r_1(u_{i+1,j,k}^{n} - u_{i-1,j,k}^{n}) - 0.5r_2(u_{i,j+1,k}^{n} - u_{i,j-1,k}^{n}) + -0.5r_2(u_{i,j,k+1}^{n} - u_{i,j,k-1}^{n}),\end{aligned} \tag{6}$$

where , $r_1 = ar, r_2 = br, \ r_3 = cr$. To determine the restrictive parameters $\varepsilon_{i,j,k}$ we must have the exact solution at the first level , this enables the value of u(x, y, z, t) at the grid point.

3. THE STABILITY ANALYSIS

A Von Neumann stability analysis must considered the finite difference equations (6). This is accomplished by substituting the Fourier components of $u_{i,j,k}^n$ *as* $u_{i,j,k}^n = U^n e^{I\alpha hi} e^{I\beta hj} e^{I\gamma hk}$, *where* $I = \sqrt{-1}$, U^n is the amplitude at time level n, and α, β, γ are the wave numbers in the x, y, z directions respectively. If a phase angles $\theta = \alpha h, \ \phi = \beta h, \ \psi = \gamma h$ are defined, then $u_{i,j,k}^n = U^n e^{I\theta i} e^{I\phi j} e^{I\psi k}$. The amplification factor is:

$$G = \frac{(1 + r\varepsilon_{i,j}) + I(r_1\sin\theta + r_2\sin\phi + r_3\sin\psi)}{(1 + r\varepsilon_{i,j}) - I(r_1\sin\theta + r_2\sin\phi + r_3\sin\psi)}, \quad \text{i. e.} \quad |G| = 1, \ \forall \varepsilon, r, r_1, r_2, r_3 .$$

Consequently the considered method is unconditionally stable.

4. NUMERICAL RESULTS

We present some numerical examples to compare the considered method (6) with Lax-Wendroff method (3), and we consider two cases. We apply our method on the examples 1and 2 such that the exact solution is given at the first level to determine the restrictive parameters $\varepsilon_{i,j,k}$, and hence we use it for another levels for calculation. In general the exact solution at the first level is unknown, so we can use the Lax-Wendroff method by equation (3), to evaluate the solutions at the first time level by large number of very small time step length k to determine the restrictive parameters $\varepsilon_{i,j,k}$, then we can use large time step length k to evaluate the solution at another levels.

Example1:

$u_t + u_x + u_y + u_z = 0, with\, u(x, y, z, 0) = \exp(x + y + z),\ u(0, y, z, t) = \exp(y + z - 3t),$

$u(1, y, z, t) = \exp(1 + y + z - 3t), u(x, 0, z, t) = \exp(x + z - 3t),\ u(x, 1, z, t) = \exp(x + 1 + z - 3t),$

$u(x, y, 0, t) = \exp(x + y - 3t),\ u(x, y, 1, t) = \exp(x + y + 1 - 3t)$

its exact solution is given by : $u(x, y, z, t) = \exp(x + y + z - 3t)$

Example2:

$u_t + u_x + u_y + u_z = 0$,

with $u(x,y,z,0) = \exp(-x-y-z)$, $u(0,y,z,t) = \exp(-y-z+3t)$, $u(1,y,z,t) = \exp(-1-y-z+3t)$,

$u(x,0,z,t) = \exp(-x-z+3t)$, $u(x,1,z,t) = \exp(-x-1-z+3t)$,

$u(x,y,0,t) = \exp(-x-y+3t)$, $u(x,y,1,t) = \exp(-x-y-1+3t)$

its exact solution is given by: $u(x,y,z,t) = \exp(-x-y-z+3t)$.

t	(x, y, z)	Lax-Wendroff method	The considered method
		A. E.	A. E.
0.1	(0.2,0.2,0.2)	1.8×10^{-3}	2.3×10^{-14}
	(0.4,0.4,0.4)	4.3×10^{-3}	2.2×10^{-15}
	(0.6,0.6,0.6)	7.7×10^{-3}	3.5×10^{-14}
	(0.8,0.8,0.8)	2.1×10^{-2}	1.4×10^{-14}
0.2	(0.2,0.2,0.2)	2.0×10^{-3}	1.4×10^{-14}
	(0.4,0.4,0.4)	6.4×10^{-3}	6.0×10^{-14}
	(0.6,0.6,0.6)	1.1×10^{-2}	3.9×10^{-14}
	(0.8,0.8,0.8)	4.8×10^{-2}	1.8×10^{-14}
0.5	(0.2,0.2,0.2)	9.3×10^{-4}	4.6×10^{-15}
	(0.4,0.4,0.4)	5.8×10^{-3}	3.2×10^{-15}
	(0.6,0.6,0.6)	5.7×10^{-3}	3.7×10^{-15}
	(0.8,0.8,0.8)	1.3×10^{-1}	4.4×10^{-15}
1.0	(0.2,0.2,0.2)	3.4×10^{-3}	3.3×10^{-15}
	(0.4,0.4,0.4)	7.5×10^{-3}	1.1×10^{-15}
	(0.6,0.6,0.6)	1.1×10^{-2}	3.1×10^{-15}
	(0.8,0.8,0.8)	6.8×10^{-2}	1.1×10^{-16}

Table (1)

Comparison of the absolute errors (A.E.) between Lax-Wendroff and the considered method for h=0.2 and k=0.01, for example 1.

t	(x, y, z)	Lax-Wendroff method	The considered method
		A. E.	A. E.
0.5	(0.2,0.2,0.2)	2.2×10^{-3}	3.5×10^{-15}
	(0.4,0.4,0.4)	3.1×10^{-3}	0.0
	(0.6,0.6,0.6)	2.3×10^{-3}	4.4×10^{-16}
	(0.8,0.8,0.8)	3.1×10^{-3}	2.8×10^{-15}
1.0	(0.2,0.2,0.2)	9.9×10^{-3}	4.4×10^{-14}
	(0.4,0.4,0.4)	1.4×10^{-2}	4.2×10^{-14}
	(0.6,0.6,0.6)	1.1×10^{-2}	1.7×10^{-15}
	(0.8,0.8,0.8)	2.4×10^{-2}	1.4×10^{-14}
2.5	(0.2,0.2,0.2)	**8.9×10^{-1}**	5.6×10^{-12}
	(0.4,0.4,0.4)	**1.29**	2.0×10^{-12}
	(0.6,0.6,0.6)	**1.02**	1.7×10^{-12}
	(0.8,0.8,0.8)	**2.23**	3.1×10^{-12}
5.0	(0.2,0.2,0.2)	**1625.33**	3.0×10^{-9}
	(0.4,0.4,0.4)	**2337.03**	8.1×10^{-10}
	(0.6,0.6,0.6)	**1854.79**	5.8×10^{-10}
	(0.8,0.8,0.8)	**4049.49**	2.9×10^{-9}

Table (2)

Comparison of the absolute errors (A.E.) between Lax-Wendroff and the considered method for h=0.2 and k=0.05, for example 2, where **u(.4,.4, .4, .4, 2.5)=544.5** and **u(.4,.4, .4, .4, 5)=984609**

5. CONCLUSION

1. The numerical results presented tables (1), and (2) shows that the absolute errors obtained by the considered methods is almost of order 10^{-11} of that absolute errors obtained by Lax-Wendroff method.

In the case of too large solution for example 2, it is clear from the given data in table (2) that the absolute errors associated with Lax-Wendroff method is too large compared with that of the considered method.

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