



g^{} -Quasi – FH-closed spaces and g^{**} -Quasi – CH-closed spaces**

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ABSTRACT

*In this paper we have introduced τ - g^{**} boundary, g^{**} -Quasi-FH-closed, g^{**} -FH-closed, g^{**} -Quasi-CH-closed, g^{**} -CH-closed, g^{**} -meager and few results on these new definitions have been established.*

Key Words: τ - g^{**} boundary, g^{**} -Quasi-FH-closed, g^{**} -FH-closed, g^{**} -Quasi-CH-closed, g^{**} -CH-closed, g^{**} -meager.

1. INTRODUCTION

Levine [2] and M.K.R.S. Veerakumar [7] introduced the class of g -closed sets and g^* -closed sets in the year 1970 and 1991 respectively. T.R. Hamlett and D. Jankovic[1] have introduced the class of Quasi-H-closed and H-closed sets in the year 1990. R.L. Newcomb [5] have introduced Quasi-H-closed modulo I in the year 1967. We have introduced g^{**} -closed sets[4], separation axioms via g^{**} -closed sets[5] and strongly g^{**} -regular and strongly g^{**} -normal spaces[6] and investigated many of their properties and in this paper we have introduced g^{**} -Quasi-FH-closed and g^{**} -Quasi-CH-closed and discussed their characteristics.

2. PRELIMINARIES

Definition 2.1: A subset A of a topological space (X, τ) is called

- 1) *generalized closed* (briefly *g -closed*)[2] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- 2) *generalized star closed* (briefly *g^* -closed*)[12] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open in (X, τ) .
- 3) *generalized star star closed* (briefly *g^{**} -closed*)[6] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g^* -open in (X, τ) .

Definition 2.2: [9] A topological space (X, τ) is said to be g^{**} -Lindelof if every g^{**} -open cover has a countable sub cover.

Definition 2.3: [7] The topological space (X, τ) is said to be g^{**} -additive if arbitrary union of g^{**} -closed sets is g^{**} -closed. Equivalently arbitrary intersection of g^{**} -open sets is g^{**} -open.

Definition 2.4: [7] A topological space (X, τ) is said to be g^{**} -multiplicative if arbitrary intersection of g^{**} -closed sets is g^{**} -closed. Equivalently arbitrary union of g^{**} -open sets is g^{**} -open.

Definition 2.5: [9] The topological space (X, τ) is said to be g^{**} -finitely additive if finite union of g^{**} -closed sets is g^{**} -closed.

Definition 2.6:[3] An ideal I on a non empty set X is a collection of subsets of X which satisfies the following properties.(i) $A \in I, B \in I \Rightarrow A \cup B \in I$ (ii) $A \in I, B \subset A \Rightarrow B \in I$. A topological space (X, τ) with an ideal I on X is called an ideal topological space and is denoted by (X, τ, I) .

Definition 2.7: [8] An ideal topological space (X, τ, I) is said to be *g***-compact modulo I if for every *g***-open covering $\{U_\alpha\}_{\alpha \in \Delta}$ of X , there exists a finite subset Δ_0 of Δ such that $X - \bigcup_{\alpha \in \Delta_0} U_\alpha \in I$.

Definition 2.8: [10] An ideal topological space (X, τ) is said to be *g***-Lindelof modulo I if for every *g***-open cover $\{U_\alpha\}_{\alpha \in \Omega}$, there exists a countable subset Ω_0 such that $X - \bigcup_{\alpha \in \Omega_0} U_\alpha \in I$.

Definition 2.9: [7] A topological space (X, τ) is said to be a *g***- T_2 space if for every pair of distinct points x, y in X there exists disjoint *g***-open sets U and V in X such that $x \in U$ and $y \in V$.

Definition 2.10: [8] A topological space (X, τ) is said to be *g***-compact if every *g***-open covering of X contains a finite sub collection that also covers X . A subset A of X is said to be *g***-compact if every *g***-open covering of A contains a finite sub collection that also covers A .

Definition 2.11: [7] Let A be a subset of X . A point $x \in X$ is said to be a *g***- limit point of A if every *g***-neighbourhood of x contains a point of A other than x .

Definition 2.12: [7] Let A be a subset of a topological space (X, τ) . $g^{**}cl(A)$ is defined to be the intersection of all *g***-closed sets containing A .

If (X, τ) is *g***-multiplicative then $g^{**}cl(A)$ is *g***-closed.

Definition 2.13: [9] A subset A of a space (X, τ) is said to be *g***-dense in X , if $g^{**}cl(A) = X$.

Definition 2.14: [7] Let (X, τ) be a topological space and A be a subset of X . A point $x \in A$ is said to be *g***-interior point of A if there exists *g***-open set U such that $x \in U \subseteq A$.

Definition 2.15: [7] Let A be a subset of a topological space (X, τ) . $g^{**}int(A)$ is defined to be the union of all *g***-open sets contained in A .

If (X, τ) is *g***-multiplicative then $g^{**}int(A)$ is *g***-open.

Definition 2.16: [11] A topological space (X, τ) is said to be a *g***-space if (X, τ) is *g***-finitely additive and *g***-multiplicative.

Definition 2.17: [1] A topological space (X, τ) is said to be Quasi-H-closed or QHC if every open cover of a space contains a finite sub collection whose closures cover the space X .

Definition 2.18: [5] Let (X, τ) be a topological space and I be an ideal on X . Then X is said to be Quasi-H-closed modulo I if for every open cover $U = \{U_\alpha / \alpha \in \Omega\}$ of X there exists a finite sub family $\{U_{\alpha_i} / i = 1, 2, \dots, n\}$ of U such that $X - \bigcup_{i=1}^n cl(U_{\alpha_i}) \in I$.

Definition 2.19: [1] A T_2 space which is QHC is said to be H-closed.

Definition 2.20: [5] A space (X, τ, I) is called τ -boundary if $\tau \cap I = \{\emptyset\}$.

Example 2.21: [5] Let X be an infinite set, $\tau = \{A \subseteq X / A^c \text{ is finite}\}$. $I_F = \{\text{finite subsets of } X\}$. Then $\tau \cap I_F = \emptyset$.

3. *g***-Quasi – FH-closed spaces and *g***-Quasi – CH-closed spaces

Definition 3.1 Given an ideal topological space (X, τ, I) , I is called *g***-codense in X if $G^{**}O(X) \cap I = \{\emptyset\}$

Example 3.2: Let X be an infinite set with cofinite topology and $I_F = \{ \text{finite subsets of } X \}$. Then $G^{**}O(X) \cap I_F = \{ \emptyset \}$

Definition 3.3: A space (X, τ) is said to be g^{**} -Quasi-FH-closed (briefly g^{**} -QFHC) if every g^{**} -open cover of X contains a finite sub collection whose g^{**} -closures cover X .

Definition 3.4: A g^{**} - T_2 space which is g^{**} -QFHC is said to be g^{**} -FH-closed.

Definition 3.5: A space (X, τ) is said to be g^{**} -Quasi-CH-closed (briefly g^{**} -QCHC) if every g^{**} -open cover of X contains a countable sub collection whose g^{**} -closures cover X .

Definition 3.6: A g^{**} - T_2 space which is g^{**} -QCHC is said to be g^{**} -CH-closed.

Theorem 3.7: Every g^{**} -compact space is g^{**} -QFHC.

Proof: Let $\{U_\alpha / \alpha \in \Omega\}$ be a g^{**} -open cover for X . Then there exists $\alpha_1, \alpha_2, \dots$ such that $X = \bigcup_{i=1}^n U_{\alpha_i}$ and this implies $X = \overline{\bigcup_{i=1}^n U_{\alpha_i}}$. Therefore X is g^{**} -QFHC.

Note: Any finite space is g^{**} -compact and hence g^{**} -QFHC.

Theorem 3.8: Every g^{**} -Lindelof space is g^{**} -QCHC.

Proof: similar to the above.

Example 3.9: An infinite cofinite topological space (X, τ) is g^{**} -compact and hence g^{**} -QFHC.

Example 3.10: In infinite indiscrete topological space (X, τ) , all subsets are g^{**} -open and g^{**} -closed. $\{\{x\} / x \in X\}$ is a g^{**} -open cover which has no finite sub cover such that g^{**} -closures can cover X and hence (X, τ) is g^{**} - T_2 but not g^{**} -QFHC and hence not g^{**} -FH.

Example 3.11: A finite indiscrete topological space (X, τ) is g^{**} - T_2 and g^{**} -QFHC and hence it is g^{**} -FH-closed.

Example 3.12: An infinite, cofinite topological space is g^{**} -QFHC but not g^{**} - T_2 and hence it is not g^{**} -FH-closed.

Remark 3.13: Any g^{**} -QFHC space is g^{**} -QCHC space and g^{**} -FH-closed space is g^{**} -CH-closed. Converse need not be true as seen in the following example.

Example 3.14: A countably infinite indiscrete topological space is g^{**} -QCHC and g^{**} -CH-closed but not g^{**} -QFHC and g^{**} -FH-closed.

Definition 3.15: A subset A of (X, τ) is said to be nowhere g^{**} -dense if $g^{**}\text{int}[g^{**}\text{cl}(A)] = \emptyset$.

Theorem 3.16: If A is g^{**} -closed then A is nowhere g^{**} -dense if and only if A^c is g^{**} -dense.

Proof: Let $x \in X$ and $g^{**}\text{cl}(A) = A$. $\therefore g^{**}\text{int}[g^{**}\text{cl}(A)] = g^{**}\text{int}[A] = \emptyset$. Therefore every g^{**} -open set containing x should intersect A^c . $\therefore A^c$ is g^{**} -dense in X . Let A^c be g^{**} -dense in X . $\therefore g^{**}\text{cl}(A^c) = X$. For $x \in X$, every g^{**} -open set containing x should intersect A^c . $\therefore g^{**}\text{int}(A) = \emptyset$. Therefore A is nowhere g^{**} -dense.

Theorem 3.17: Let (X, τ) be a g^{**} -space. Then $g^{**}(I_n) = \{ \text{nowhere } g^{**}\text{-dense subsets of } X \}$ is an ideal in X .

Proof: Let $A \in g^{**}(I_n)$ and $B \subseteq A$ then $g^{**}\text{int}[g^{**}\text{cl}(B)] \subseteq g^{**}\text{int}[g^{**}\text{cl}(A)] = \emptyset$. $\therefore B \in g^{**}(I_n)$. Let $A, B \in g^{**}(I_n)$.

Under the given hypothesis, $g^{**}cl(A \cup B) = g^{**}cl(A) \cup g^{**}cl(B)$.

$$\begin{aligned} \therefore g^{**}int[g^{**}cl(A \cup B) &= g^{**}int[g^{**}cl(A) \cup g^{**}cl(B)] \\ &= g^{**}int[g^{**}cl(A)] \cup [g^{**}int[g^{**}cl(B)]] = \varphi. \end{aligned}$$

$\therefore A \cup B \in g^{**}(I_n)$. Therefore $g^{**}(I_n)$ is an ideal.

Theorem 3.18: If an ideal topological space (X, τ, I) is *g***-multiplicative, *g***-compact modulo I and $G^{**}O(X) \cap I = \{\varphi\}$ then (X, τ) is *g***-QFHC.

Proof: Let $\{U_\alpha / \alpha \in \Omega\}$ be a *g***-open cover for X . Therefore there exists $U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$ such that

$$X - \bigcup_{i=1}^n U_{\alpha_i} \in I.$$

Case (1): If $X - \bigcup_{i=1}^n U_{\alpha_i} = \varphi$, then $X = \bigcup_{i=1}^n U_{\alpha_i} \therefore X = \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i})$ Therefore (X, τ) is *g***-QFHC.

Case (2): If $X - \bigcup_{i=1}^n U_{\alpha_i} \neq \varphi$, then $X - \bigcup_{i=1}^n U_{\alpha_i} = \bigcup_{\alpha \neq \alpha_i} U_\alpha \in I$. But $\bigcup_{\alpha \neq \alpha_i} U_\alpha \in G^{**}O(X)$ (since X is *g***-

multiplicative). $\therefore \bigcup_{\alpha \neq \alpha_i} U_\alpha \in G^{**}O(X) \cap I = \{\varphi\}$. $\therefore X = \bigcup_{i=1}^n U_{\alpha_i}$ and this implies $X = \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i})$.

Theorem 3.19: Let (X, τ) be a *g***-multiplicative space. If (X, τ, I) is *g***-Lindeloff modulo I and $G^{**}O(X) \cap I = \{\varphi\}$ then (X, τ) is *g***-QCHC.

Proof: Similar to the above proof.

Theorem 3.20: Let (X, τ) be a *g***-space.

(1) Then (X, τ) is *g***-compact modulo $g^{**}(I_n)$ if and only if (X, τ) is *g***-QFHC.

(2) If (X, τ) is *g*** T_2 then (X, τ) is *g***-compact modulo $g^{**}(I_n)$ if and only if (X, τ) is *g***-FH- closed.

Proof:

(1) **Necessity:** (1) Let (X, τ) be *g***-compact modulo $g^{**}(I_n)$ and $\{U_\alpha\}_{\alpha \in \Omega}$ be a *g***-open cover for X . Then there

exists $U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$ such that $X - \bigcup_{i=1}^n U_{\alpha_i} \in g^{**}(I_n)$. $\therefore X - \bigcup_{i=1}^n U_{\alpha_i}$ is nowhere *g***-dense in X and it is

*g***-closed. Therefore $\left[X - \bigcup_{i=1}^n U_{\alpha_i} \right]^c$ is *g***-dense in X . Hence $g^{**}cl\left(\bigcup_{i=1}^n U_{\alpha_i}\right) = X$. In a *g***-space,

$$g^{**}cl\left(\bigcup_{i=1}^n U_{\alpha_i}\right) = \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i}). \therefore X = \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i}).$$
 Therefore (X, τ) is *g***-QFHC.

Sufficiency: Let (X, τ) be *g***-QFHC.

Since (X, τ) is *g***-space, $g^{**}(I_n)$ is an ideal in X . Let $\{U_\alpha / \alpha \in \Omega\}$ be a *g***-open cover. Then there exists

$\alpha_1, \alpha_2, \dots, \alpha_n$ such that $X = \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i}) = g^{**}cl\left[\bigcup_{i=1}^n U_{\alpha_i}\right] \therefore \bigcup_{i=1}^n U_{\alpha_i}$ is *g***-dense in X . $\therefore \left[\bigcup_{i=1}^n U_{\alpha_i}\right]^c$ is

nowhere *g***-dense in X by theorem 3.16 and so $X - \bigcup_{i=1}^n U_{\alpha_i} \in g^{**}(I_n)$.

(2) By (1), (X, τ) is *g***-compact modulo $g^{**}(I_n)$ if and only if (X, τ) is *g***-QFHC. By definition (3.4), (X, τ) is *g***-compact modulo $g^{**}(I_n)$ if and only if it is *g*** FH-closed.

Theorem 3.21: Let (X, τ) be *g***-multiplicative, *g***-countably additive space. Then

- 1) (X, τ) is *g***-QCHC space if and only if (X, τ) is *g***-Lindeloff modulo *g***(I_n).
- 2) If (X, τ) is *g***- T_2 then (X, τ) is *g***-Lindeloff modulo *g***(I_n) if and only if (X, τ) is *g***-CH-closed.

Proof: Since (X, τ) is *g***-multiplicative and *g***-countably additive, $\bigcup_{i=1}^{\infty} g^{**}cl(U_{\alpha_i}) = g^{**}cl\left[\bigcup_{i=1}^{\infty} U_{\alpha_i}\right]$

The rest of the proof is similar to the proof of theorem (3.20)

Definition 3.24: A space (X, τ) is said to be *g***-meager or *g***- first category if it is a countable union of nowhere *g***-dense sets.

Theorem 3.25: Let (X, τ) be a *g***-space and let $g^{**}(I_m) = \{g^{**}\text{-meager subsets of } X\}$. Then $g^{**}(I_m)$ is an ideal in X .

Proof: Let $A \in g^{**}(I_m)$ and $B \subseteq A$. Then $A = \bigcup_{i=1}^{\infty} G_i$ where each G_i is nowhere *g***-dense subsets. Now

$$B = \bigcup_{i=1}^{\infty} (B \cap G_i). \text{ and } g^{**}cl(B \cap G_i) \subseteq g^{**}cl(B) \cap g^{**}cl(G_i).$$

Therefore

$$g^{**}int[g^{**}cl(B \cap G_i)] \subseteq g^{**}int[g^{**}cl(B) \cap g^{**}cl(G_i)] = g^{**}int[g^{**}cl(B)] \cap g^{**}int[g^{**}cl(G_i),$$

Since X is a *g***-space. Therefore $B \cap G_i$ is nowhere *g***-dense for all i and so $B \in g^{**}(I_m)$. Obviously $A, B \in g^{**}(I_m) \Rightarrow A \cup B \in g^{**}(I_m)$. Therefore $g^{**}(I_m)$ is an ideal in X .

Definition 3.26: A topological space (X, τ) is said to be of *g***-second category if it is not of *g***-first category.

Definition 3.27: A *g***-space (X, τ) is said to be a *g***-Baire space if $G^{**}O(X) \cap g^{**}(I_m) = \{\varnothing\}$.

Theorem 3.28: Let (X, τ) be a *g***-baire space. Then

- (1) (X, τ) is *g***-compact modulo $g^{**}(I_m)$ if and only if (X, τ) is *g***-QFHC.
- (2) In addition if (X, τ) is *g***- T_2 then (X, τ) is *g***-compact modulo $g^{**}(I_m)$ if and only if (X, τ) is *g***-FH-closed.

Proof: (1) Let (X, τ) be *g***-Baire space and *g***-compact modulo $g^{**}(I_m)$. Let $\{U_{\alpha}\}$ be a *g***-open cover for

X . Then there exists $U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$ such that $X - \bigcup_{i=1}^n U_{\alpha_i} \in g^{**}(I_m)$.

Case (i): $X - \bigcup_{i=1}^n U_{\alpha_i} = \varnothing$ then $X = \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i})$ and so it is *g***-QFHC.

Case (ii): $X - \bigcup_{i=1}^n U_{\alpha_i} \neq \varnothing$ then $X - \bigcup_{i=1}^n U_{\alpha_i} = \bigcup_{\alpha \neq \alpha_i} U_{\alpha_i} \in G^{**}O(X)$ (since (X, τ) is a *g***-space).

$\therefore X - \bigcup_{i=1}^n U_{\alpha_i} \in G^{**}O(X) \cap g^{**}(I_m) = \varnothing$. $\therefore X = \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i})$. (by case(i)). Therefore X is *g***-QFHC.

Conversely, let (X, τ) be *g***-QFHC. By theorem (3.20), (X, τ) is *g***-compact modulo $g^{**}(I_n)$.

This implies that (X, τ) is *g***-compact modulo $g^{**}(I_m)$, since $g^{**}(I_n) \subseteq g^{**}(I_m)$. (2) Follows from (1) and definition of *g***-FH-closed space.

Theorem 3.29: Let (X, τ) be a *g***-baire space which is *g***-countably additive. Then

- (1) (X, τ) is *g***-Lindeloff modulo $g^{**}(I_m)$ if and only if (X, τ) is *g***-QCHC.

(2) In addition if (X, τ) is *g***- T_2 then (X, τ) is *g***-Lindeloff modulo $g^{**}(I_m)$ if and only if (X, τ) is *g***-CH-closed.

Proof: Similar to the above proof.

4. *g*-Quasi-H-closed modulo an ideal**

Definition 4.1: An ideal topological space (X, τ, I) is said to be *g***-QFHC modulo an ideal if for every *g***-open cover $\{U_\alpha / \alpha \in \Omega\}$ of X there exists a finite sub family $\{U_{\alpha_i} / i = 1, 2, \dots, n\}$ such that $X - \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i}) \in I$. Such a sub family is said to be proximate *g***-sub cover modulo I .

Definition 4.2: A subset A of (X, τ) is said to be *g***-pre-open if $A \subseteq g^{**}(g^{**}int(A))$. The collection of all *g***-preopen sets is denoted by $G^{**}PO(X)$.

Definition 4.3: An ideal I in (X, τ) is said to be completely *g***-codense if $I \cap G^{**}PO(X) = \varnothing$.

Note: (1) $G^{**}O(X) \subseteq G^{**}PO(X)$.

(2) $I \cap G^{**}PO(X) = \varnothing \Rightarrow I \cap G^{**}O(X) = \varnothing$.

Therefore every completely *g***-codense ideal is *g***-codense. But the converse is not true as seen in the following example.

Example 4.4: Consider R with cofinite topology. A subset is *g***-closed if and only if it is finite. Let I_c be the ideal of all countable subsets. Then $G^{**}O(R) \cap I_c = \varnothing$ and so I_c is *g***-codense.

$g^{**}cl(Q) = R$. $g^{**}int(g^{**}cl(Q)) = g^{**}int(R) = R$. $\therefore Q \in G^{**}PO(R)$. $\therefore Q \in I_c \cap G^{**}PO(R)$.

Therefore I_c is not completely *g***-codense in this space.

Theorem 4.5: For a space (X, τ) , the following statements are equivalent.

- (1) (X, τ) is *g***-QFHC.
- (2) (X, τ) is *g***-QFHC modulo \varnothing .
- (3) (X, τ) is *g***-QFHC modulo I_F

If (X, τ) is a *g***-space then these are equivalent to

- (4) (X, τ) is *g***-QFHC modulo I for every *g***-codense ideal I .

Proof: (1) \Leftrightarrow (2) is obvious. (2) \Rightarrow (3) is obvious.

(3) \Rightarrow (1) : Let $\{U_\alpha\}$ be a *g***-open cover for X . Then there exists $U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$ such that $X - \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i}) \in I_F$. Let $X - \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i}) = \{x_1, x_2, \dots, x_k\}$. Choose U_{β_i} such that $x_i \in U_{\beta_i}$. Let $\Delta_0 = \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_k\}$. Then Δ_0 is finite and $X = \bigcup_{\alpha \in \Delta_0} g^{**}cl(U_\alpha)$.

Therefore X is *g***-QFHC.

(1) \Rightarrow (4) is obvious.

(4) \Rightarrow (1) : Let I be *g***-codense ideal. Let $\{U_\alpha\}_{\alpha \in \Omega}$ be a *g***-open cover in X . Then there exists $\{U_{\alpha_i} / i = 1, 2, \dots, n\}$ such that $X - \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i}) \in I$. But $X - \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i}) \in G^{**}O(X)$, since X is a *g***-space. But $G^{**}O(X) \cap I = \varnothing$. $\therefore X = \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i})$. $\therefore (X, \tau)$ is *g***-QFHC.

Theorem 4.6: For a topological space (X, τ) the following statements are equivalent.

- (1) (X, τ) is *g***-QCHC.
- (2) (X, τ) is *g***-QCHC modulo $\{\varnothing\}$.
- (3) (X, τ) is *g***-QCHC modulo $\{I_c\}$.

If (X, τ) is a *g***-multiplicative and *g***-countably additive then these are equivalent to

(4) (X, τ) is *g***-QCHC modulo I for every *g***-codense ideal I .

Proof: Similar to the above proof, since (X, τ) is *g***-countably additive implies $\bigcup_{i=1}^{\infty} g^{**}cl(U_{\alpha_i})$ is *g***-closed.

Remark 4.7: In theorem (4.6) the condition I is *g***-codense is necessary as seen in the following example.

Example 4.8: Consider R with indiscrete topology τ . Let $A = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a finite subset of R . then $I = \wp(R - A)$ is an ideal in R . Since all subsets are *g***-open, $G^{**}O(X) \cap I \neq \varnothing$.

Therefore I is not *g***-codense. Let $\{U_{\alpha} / \alpha \in \Omega\}$ be a *g***-open cover for X . Choose U_{α_i} such that $a_i \in U_{\alpha_i}$. Then

$$R - \bigcup_{i=1}^n U_{\alpha_i} \in I. \text{ But } R - \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i}) \subseteq R - \bigcup_{i=1}^n U_{\alpha_i} \therefore R - \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i}) \in I.$$

Therefore R is *g***-QFHC modulo I . $\{\{x\} / x \in R\}$ is a *g***-open cover which has no finite sub cover whose *g***-closures cover X . Therefore R is not *g***-QFHC.

Remark 4.9: In theorem (4.7) the condition I is *g***-codense is necessary as seen in the following example.

Example 4.10: In example(4.8), (R, τ) is *g***-QCHC modulo I but not *g***-QCHC.

The following theorem contains a number of characterizations of *g***-QFHC modulo I spaces.

Theorem 4.11: For a topological space (X, τ) and an ideal I on X , the following statements are equivalent.

- (1) (X, τ) is *G***-QFHC modulo I .
- (2) For each family $A' = \{A_{\alpha} / \alpha \in \Omega\}$ of *g***-closed sets having empty intersection there exists a finite sub family $\{A_{\alpha_1}, A_{\alpha_2}, \dots, A_{\alpha_n}\}$ such that $\bigcap_{i=1}^n g^{**}int(A_{\alpha_i}) \in I$.
- (3) For each family $A' = \{A_{\alpha} / \alpha \in \Omega\}$ of *g***-closed sets such that $\{g^{**}int(A_{\alpha}) / \alpha \in \Omega\}$ with FIP modulo I one has, $\bigcap_{A \in A'} A \neq \varnothing$.

Proof: (1) \Rightarrow (2) Let $A' = \{A_{\alpha} / \alpha \in \Omega\}$ be a family of *g***-closed sets such that $\bigcap_{\alpha \in \Omega} A_{\alpha} = \varnothing$. Then $X = \bigcup_{\alpha \in \Omega} (G_{\alpha})$ where $G_{\alpha} = X - A_{\alpha}$ is *g***-open. By (1), there exists $\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}\}$ such that $X - \bigcup_{i=1}^n g^{**}cl(G_{\alpha_i}) \in I$. (ie) $\bigcap_{i=1}^n [g^{**}cl(G_{\alpha_i})]^c \in I$. But $[g^{**}cl(G_{\alpha_i})]^c = X - g^{**}cl(G_{\alpha_i})$ which is equal to $g^{**}int(X - G_{\alpha_i}) = g^{**}int(A_{\alpha_i})$. $\therefore \bigcap_{i=1}^n g^{**}int(A_{\alpha_i}) \in I$.

(2) \Rightarrow (3) : Let A' be a family of *g***-closed sets such that $\{g^{**}int(A_{\alpha}) / \alpha \in \Omega\}$ has FIP modulo I .

Suppose $\bigcap_{\alpha \in \Omega} A_{\alpha} = \varnothing$. Then by (2) there exists a finite sub family $\{A_{\alpha_1}, A_{\alpha_2}, \dots, A_{\alpha_n}\}$ such that $\bigcap_{i=1}^n g^{**}int(A_{\alpha_i}) \in I$ which is a contradiction. $\therefore \bigcap_{\alpha \in \Omega} A_{\alpha} \neq \varnothing$.

(3) \Rightarrow (1) : Let $\{U_{\alpha} / \alpha \in \Omega\}$ be a *g***-open cover for X . To prove that there exists a finite sub family $\{U_{\alpha_i} / i = 1, 2, \dots, n\}$ such that $X - \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i}) \in I$. If not there is no finite sub family with this property.

Now $\{X - U_{\alpha} / \alpha \in \Omega\}$ is a family of *g***-closed sets. For any finite sub family $U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$, $X - \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i}) \notin I$. (ie) $\bigcap_{i=1}^n g^{**}int(X - U_{\alpha_i}) \notin I$. By (3), $\bigcap_{\alpha \in \Omega} (X - U_{\alpha}) \neq \varnothing$. $\therefore \bigcup_{\alpha \in \Omega} (U_{\alpha}) \neq X$ which is a contradiction. Therefore (X, τ) is *g***-QFHC.

Theorem 4.12: For a topological space (X, τ) and an ideal I on X , the following statements are equivalent.

- (1) (X, τ) is G^{**} -QCHC modulo I .
- (2) For each family $A' = \{A_\alpha / \alpha \in \Omega\}$ of g^{**} -closed sets having empty intersection there exists a countable sub family $\{A_{\alpha_1}, A_{\alpha_2}, \dots\}$ such that $\bigcap_{i=1}^{\infty} g^{**}\text{int}(A_{\alpha_i}) \in I$.
- (3) For each family $A' = \{A_\alpha / \alpha \in \Omega\}$ of g^{**} -closed sets such that $\{g^{**}\text{int}(A_\alpha) / \alpha \in \Omega\}$ with CIP modulo I one has $\bigcap_{A \in A'} A \neq \emptyset$.

Proof: Similar to the above proof.

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