

A GENERAL THEOREM ON LAPLACE TRANSFORM AND ITS APPLICATIONS

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ABSTRACT

In this paper we obtain a general theorem on Laplace transform in terms of Fox-Wright hypergeometric function. Laplace transforms of some composite functions are also obtained as special cases.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES:

In the usual notation, the Pochhammer's symbol or shifted factorial or generalized factorial function $(b)_k$ is defined by

$$(b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = \begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if } k \in \{1, 2, 3, \dots\} \\ 1 & ; \text{ if } k = 0 \\ k! & ; \text{ if } b = 1, k \in \{1, 2, 3, \dots\} \end{cases}$$

where b is neither zero nor a negative integer and the notation Γ stands for Gamma function.

$$(\lambda)_{mn} = m^{mn} \left(\frac{\lambda}{m}\right)_n \left(\frac{\lambda+1}{m}\right)_n \left(\frac{\lambda+2}{m}\right)_n \cdots \left(\frac{\lambda+m-1}{m}\right)_n$$

where m is a positive integer, n is a non negative integer and λ may be real or complex number.

The convenient notation $\Delta(N;b)$ is used to denote the array of N number of parameters given by $\frac{b}{N}, \frac{b+1}{N}, \frac{b+2}{N}, \dots, \frac{b+N-1}{N}$, where b is a real or complex number and N is a positive integer.

The generalized Gaussian hypergeometric function ${}_A F_B$ of one variable is defined by

$${}_A F_B \left[\begin{matrix} a_1, a_2, \dots, a_A; \\ b_1, b_2, \dots, b_B; \end{matrix} z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} \tag{1.1}$$

where denominator parameters b_1, b_2, \dots, b_B are neither zero nor negative integers and A, B are non-negative integers.

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If $A \leq B$, then series ${}_A F_B$ is always convergent for all finite values of z (real or complex).

If $A = B + 1$, then series ${}_A F_B$ is convergent for $|z| < 1$.

In 1933, Wright (or Fox-Wright) defined a more interesting generalized hypergeometric function of one variable [12, pp.50-51(1.5.21), p.179(34iii), p.395(23); 4, p.11(1.7.8)] in the following forms:

$${}_p \Psi_q \left[\begin{matrix} (\alpha_1, A_1), (\alpha_2, A_2), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), (\beta_2, B_2), \dots, (\beta_q, B_q); \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + nA_1) \Gamma(\alpha_2 + nA_2) \cdots \Gamma(\alpha_p + nA_p) z^n}{\Gamma(\beta_1 + nB_1) \Gamma(\beta_2 + nB_2) \cdots \Gamma(\beta_q + nB_q) n!} \quad (1.2)$$

$$= \frac{\Gamma(\alpha_1) \Gamma(\alpha_2) \cdots \Gamma(\alpha_p)}{\Gamma(\beta_1) \Gamma(\beta_2) \cdots \Gamma(\beta_q)} {}_p \Psi_q^* \left[\begin{matrix} (\alpha_1, A_1), (\alpha_2, A_2), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), (\beta_2, B_2), \dots, (\beta_q, B_q); \end{matrix} ; z \right]$$

$$= H_{p,q+1}^{1,p} \left[-z \left| \begin{matrix} (1-\alpha_1, A_1), (1-\alpha_2, A_2), \dots, (1-\alpha_p, A_p) \\ (0, 1), (1-\beta_1, B_1), (1-\beta_2, B_2), \dots, (1-\beta_q, B_q) \end{matrix} \right. \right] \quad (1.3)$$

$${}_p \Psi_q^* \left[\begin{matrix} (\alpha_1, A_1), (\alpha_2, A_2), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), (\beta_2, B_2), \dots, (\beta_q, B_q); \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_{nA_1} (\alpha_2)_{nA_2} \cdots (\alpha_p)_{nA_p} z^n}{(\beta_1)_{nB_1} (\beta_2)_{nB_2} \cdots (\beta_q)_{nB_q} n!} \quad (1.4)$$

where the coefficients $A_1, A_2, \dots, A_p, B_1, B_2, \dots, B_q$ are positive real numbers and $\alpha_1, \alpha_2, \dots, \alpha_p, \beta_1, \beta_2, \dots, \beta_q$ are complex parameters.

Convergence Conditions:

(i) The Fox H -function makes sense when

$$\delta \equiv (1 + B_1 + B_2 + \cdots + B_q) - (A_1 + A_2 + \cdots + A_p) > 0 \quad \text{and} \quad 0 < |z| < \infty; \quad z \neq 0$$

(ii) The equality holds true only for suitably constrained values of $|z|$ or appropriately

$$\text{bounded values of } |z| \text{ i.e. } \delta = 0 \text{ and } 0 < |z| < R \equiv A_1^{-A_1} A_2^{-A_2} \cdots A_p^{-A_p} B_1^{B_1} B_2^{B_2} \cdots B_q^{B_q}.$$

Suppose given function $F(t)$ is well defined for all values of $t > 0$, then Laplace transform of $F(t)$ is given by

$$L\{F(t)\} = \int_0^{\infty} e^{-st} F(t) dt = f(s) \quad (1.5)$$

$$L^{-1}\{f(s)\} = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} e^{st} f(s) ds = F(t), \quad h > 0 \quad (1.6)$$

provided that above integrals exist and s is a real or complex number.

$$\int_0^{\infty} e^{-sx} x^{\alpha-1} dx = \frac{\Gamma(\alpha)}{s^\alpha} \quad (1.7)$$

$$((\text{Re}(s) > 0, \text{Re}(\alpha) > 0) \text{ or } (\text{Re}(s) = 0, 0 < \text{Re}(\alpha) < 1))$$

$$\int_0^{\infty} e^{-\frac{\alpha}{x}} x^{s-1} dx = \alpha^s \Gamma(-s) \quad (1.8)$$

$$(\text{Re}(s) < 0, \text{Re}(\alpha) > 0)$$

$$\int_0^{\infty} (e^{-x} - 1) x^{s-1} dx = \Gamma(s) \quad (1.9)$$

$$(-1 < \text{Re}(s) < 0)$$

Hankel's Contour Integral:

$$\int_{h-i\infty}^{h+i\infty} e^t t^{-A} dt = \frac{2\pi i}{\Gamma(A)} \quad (1.10)$$

(Re(A) > 0, h > 0)

Fractional Derivative:

The fractional derivative of t^{P-1} with respect to t, λ times is given by the following formula

$$\frac{d^\lambda}{dt^\lambda} (t^{P-1}) = \frac{\Gamma(P)}{\Gamma(P-\lambda)} t^{P-\lambda-1} \quad (1.11)$$

where λ is arbitrary complex number. Any values of λ and P leading to the result which do not make sense, are tacitly excluded.

2. GENERAL THEOREM:

Since Pochhammer's symbol is associated with Gamma function and Gamma function is undefined for zero and negative integers, therefore arguments, numerator and denominator parameters are adjusted in such a way that each term of following results is completely well defined and meaningful then without any loss of convergence, we have

$$\begin{aligned} & \mathbf{L} \left\{ t^\nu \frac{d^\lambda}{dt^\lambda} \left(t^\mu {}_A F_B \left[\begin{matrix} (a_A) \\ (b_B) \end{matrix} ; xt^k \right] \right) \right\} = \frac{\Gamma(\mu+1)\Gamma(\nu+\mu-\lambda+1)}{s^{\nu+\mu-\lambda+1} \Gamma(\mu-\lambda+1)} \times \\ & \times {}_{2+A} \Psi_{1+B}^* \left[\begin{matrix} (a_1, 1), (a_2, 1), \dots, (a_A, 1), (\mu+1, k), (\nu+\mu-\lambda+1, k) \\ (b_1, 1), (b_2, 1), \dots, (b_B, 1), (\mu-\lambda+1, k) \end{matrix} ; \frac{x}{s^k} \right] \quad (2.1) \\ & (\text{Re}(\nu+\mu-\lambda) > -1; k \text{ is a positive real number}) \end{aligned}$$

Proof:

$$\begin{aligned} & \mathbf{L} \left\{ t^\nu \frac{d^\lambda}{dt^\lambda} \left(t^\mu {}_A F_B \left[\begin{matrix} (a_A) \\ (b_B) \end{matrix} ; xt^k \right] \right) \right\} = \mathbf{L} \left\{ t^\nu \sum_{r=0}^{\infty} \frac{[(a_A)]_r x^r}{[(b_B)]_r r!} \frac{d^\lambda}{dt^\lambda} t^{\mu+kr} \right\} \\ & = \mathbf{L} \left\{ t^\nu \sum_{r=0}^{\infty} \frac{[(a_A)]_r x^r \Gamma(\mu+kr+1) t^{\mu+kr-\lambda}}{[(b_B)]_r r! \Gamma(\mu+kr-\lambda+1)} \right\} \\ & = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\lambda+1)} \sum_{r=0}^{\infty} \frac{[(a_A)]_r x^r (\mu+1)_{kr}}{[(b_B)]_r r! (\mu-\lambda+1)_{kr}} \mathbf{L} \{ t^{\nu+\mu+kr-\lambda} \} \\ & = \frac{\Gamma(\mu+1)\Gamma(\nu+\mu-\lambda+1)}{\Gamma(\mu-\lambda+1) s^{\nu+\mu-\lambda+1}} \sum_{r=0}^{\infty} \frac{[(a_A)]_r (\mu+1)_{kr} (\nu+\mu-\lambda+1)_{kr} x^r}{[(b_B)]_r (\mu-\lambda+1)_{kr} s^{kr} r!} \end{aligned}$$

Now writing above series in hypergeometric notation (1.4) of Fox-Wright, we get the right hand side of (2.1).

3. SOME DEDUCTIONS OF (2.1):

If k is a positive integer, then (2.1) reduces to

$$\begin{aligned} & \mathbf{L} \left\{ t^\nu \frac{d^\lambda}{dt^\lambda} \left(t^\mu {}_A F_B \left[\begin{matrix} (a_A) \\ (b_B) \end{matrix} ; xt^k \right] \right) \right\} \\ & = \frac{\Gamma(\mu+1)\Gamma(\nu+\mu-\lambda+1)}{s^{\nu+\mu-\lambda+1} \Gamma(\mu-\lambda+1)} {}_{A+2k} F_{B+k} \left[\begin{matrix} (a_A), \Delta(k; \mu+1), \Delta(k; \nu+\mu-\lambda+1) \\ (b_B), \Delta(k; \mu-\lambda+1) \end{matrix} ; \frac{xk^k}{s^k} \right] \quad (3.1) \end{aligned}$$

$$(\operatorname{Re}(\nu + \mu - \lambda) > -1; A + k \leq B + 1)$$

In (3.1), setting $\lambda = \mu = 0$ and then adjusting parameters and variables suitably, we get on simplification

$$\mathcal{L}\left\{t^{\nu-1} {}_A F_B \left[\begin{matrix} (a_A) \\ (b_B) \end{matrix}; (xt)^k \right]\right\} = \frac{\Gamma(\nu)}{s^\nu} {}_{A+k} F_B \left[\begin{matrix} (a_A), \Delta(k; \nu) \\ (b_B) \end{matrix}; \frac{(xk)^k}{s^k} \right] \quad (3.2)$$

$$(\operatorname{Re}(\nu) > 0; \operatorname{Re}(s) > 0; A + k \leq B + 1)$$

$$\mathcal{L}\left\{t^{k\sigma-1} {}_A F_{B+k} \left[\begin{matrix} (a_A) \\ (b_B), \Delta(k; k\sigma) \end{matrix}; \frac{\lambda^k t^k}{k^k} \right]\right\} = \frac{\Gamma(k\sigma)}{s^{k\sigma}} {}_{A+1} F_B \left[\begin{matrix} 1, (a_A) \\ (b_B) \end{matrix}; \left(\frac{\lambda}{s}\right)^k \right] \quad (3.3)$$

$$(\operatorname{Re}(\sigma) > 0; \operatorname{Re}(s) > 0; A \leq B)$$

4. APPLICATIONS:

Making suitable adjustments of parameters and variables in (3.2) and (3.3), we obtain following results

$$\mathcal{L}\{t^n \sin(bt)\} = \frac{b\Gamma(n+2)}{s^{n+2}} {}_2 F_1 \left[\begin{matrix} \frac{n+2}{2}, \frac{n+3}{2} \\ \frac{3}{2} \end{matrix}; -\frac{b^2}{s^2} \right]; n > -2, \operatorname{Re}(s) > |\operatorname{Im}(b)| \quad (4.1)$$

$$\mathcal{L}\{t^n \cos(bt)\} = \frac{\Gamma(n+1)}{s^{n+1}} {}_2 F_1 \left[\begin{matrix} \frac{n+1}{2}, \frac{n+2}{2} \\ \frac{1}{2} \end{matrix}; -\frac{b^2}{s^2} \right]; n > -1, \operatorname{Re}(s) > |\operatorname{Im}(b)| \quad (4.2)$$

$$\mathcal{L}\{\sin(\sqrt{t})\} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}} \exp\left(-\frac{1}{4s}\right); \operatorname{Re}(s) > 0 \quad (4.3)$$

$$\mathcal{L}\left\{\frac{\sin(\sqrt{t})}{\sqrt{t}}\right\} = \frac{1}{s} {}_1 F_1 \left[\begin{matrix} 1 \\ \frac{3}{2} \end{matrix}; -\frac{1}{4s} \right] \quad (4.4)$$

$$\mathcal{L}\left\{\frac{\cos(\sqrt{t})}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}} \exp\left(-\frac{1}{4s}\right) \quad (4.5)$$

$$\mathcal{L}\{\sqrt{t} \cos(\sqrt{t})\} = \frac{1}{2s\sqrt{\pi s}} {}_1 F_1 \left[\begin{matrix} \frac{3}{2} \\ \frac{1}{2} \end{matrix}; -\frac{1}{4s} \right] \quad (4.6)$$

$$\mathcal{L}\{J_1(t)\} = 1 - \frac{s}{\sqrt{s^2 + 1}} \quad (4.7)$$

$$\mathcal{L}\{\sqrt{t} J_1(\sqrt{t})\} = \frac{1}{2s^2} \exp\left(-\frac{1}{4s}\right) \quad (4.8)$$

$$\mathcal{L}\{\sqrt{t} J_1(t)\} = \frac{3}{8s^2} \sqrt{\frac{\pi}{s}} {}_2 F_1 \left[\begin{matrix} \frac{5}{4}, \frac{7}{4} \\ 1 \end{matrix}; -\frac{25}{16s^2} \right] \quad (4.9)$$

$$\mathcal{L}\{J_0(\sqrt{t})\} = \frac{1}{s} \exp\left(-\frac{1}{4s}\right) \quad (4.10)$$

$$L\left\{\frac{1}{t^{\frac{3}{2}}}\exp\left(-\frac{1}{4t}\right)\right\}=2\sqrt{\pi}\exp(-\sqrt{s}) \quad (4.11)$$

$$L\left\{\frac{1}{\sqrt{t}}\exp\left(-\frac{1}{4t}\right)\right\}=\sqrt{\frac{\pi}{s}}\exp(-\sqrt{s}) \quad (4.12)$$

$$L\{J_0(t)\}=\frac{1}{\sqrt{s^2+1}} \quad (4.13)$$

$$L\{I_0(t)\}=\frac{1}{\sqrt{s^2-1}} \quad (4.14)$$

$$L\left\{\frac{\exp(-2t)}{\sqrt{t}}\right\}=\sqrt{\frac{\pi}{s+2}} \quad (4.15)$$

$$L\{\operatorname{erf}(\sqrt{t})\}=\frac{1}{s\sqrt{s+1}} \quad (4.16)$$

$$L\left\{\operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right)\right\}=\frac{\exp(-2\sqrt{s})}{s} \quad (4.17)$$

$$L\left\{t^2 {}_0F_2\left[\begin{matrix} - & - & - \\ \frac{4}{3}, \frac{5}{3} \end{matrix}; -\frac{t^3}{27}\right]\right\}=\frac{2}{1+s^3} \quad (4.18)$$

$$L\left\{{}_0F_2\left[\begin{matrix} - & - \\ 1, 1 \end{matrix}; -t\right]\right\}=\frac{1}{s} {}_0F_1\left[\begin{matrix} - & - \\ 1 \end{matrix}; -\frac{1}{s}\right]=\frac{1}{s} J_0\left(\frac{2}{\sqrt{s}}\right) \quad (4.19)$$

$$L\left\{t^{m+n-1} {}_1\Psi_2\left[\begin{matrix} (1, 1) \\ (2, 2), (m+n, 2m) \end{matrix}; -b^2 t^{2m}\right]\right\}=\frac{1}{b s^n} \sin\left(\frac{b}{s^m}\right) \quad (4.20)$$

$$L\left\{t^{n-1} {}_1\Psi_2\left[\begin{matrix} (1, 1) \\ (1, 2), (n, 2m) \end{matrix}; -b^2 t^{2m}\right]\right\}=\frac{1}{s^n} \cos\left(\frac{b}{s^m}\right) \quad (4.21)$$

$$L\left\{t^{m+n-1} {}_2\Psi_2\left[\begin{matrix} (1, 1), (1, 2) \\ (2, 2), (m+n, 2m) \end{matrix}; -b^2 t^{2m}\right]\right\}=\frac{1}{b s^n} \tan^{-1}\left(\frac{b}{s^m}\right) \quad (4.22)$$

$$L\left\{t^{m+n-1} {}_2\Psi_2\left[\begin{matrix} (1, 1), (1, 1) \\ (2, 1), (m+n, m) \end{matrix}; -b t^m\right]\right\}=\frac{1}{b s^n} \ln\left(1+\frac{b}{s^m}\right) \quad (4.23)$$

$$L\left\{\frac{\sin(bt)}{t}\right\}=\tan^{-1}\left(\frac{b}{s}\right), \operatorname{Re}(s) > |\operatorname{Im}(b)| \quad (4.24)$$

$$L\left\{\frac{\sin(\sqrt{t})}{t}\right\}=\pi \operatorname{erf}\left(\frac{1}{2\sqrt{s}}\right) \quad (4.25)$$

$$L\{\sinh(\sqrt{t})\}=\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\exp\left(\frac{1}{4s}\right); \operatorname{Re}(s) > 0 \quad (4.26)$$

$$L\{I_0(\sqrt{t})\}=\frac{1}{s}\exp\left(\frac{1}{4s}\right) \quad (4.27)$$

where m is a positive real number and standard notations have their usual meanings.

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