International Journal of Mathematical Archive-3(11), 2012, 3997-4003
IJMA Available online through www.ijma.info ISSN 2229 – 5046

BEHAVIOR OF THE SOLUTIONS OF THE FUZZY DIFFERENTIAL EQUATION

*H. M. mohammadinejad & M. H. Moslehi**

Department of Mathematics, Faculty of Science, University of Birjand, Birjand, Iran
Department of Mathematics, University of Birjand, Birjand, Iran

(Received on: 21-02-12; Revised & Accepted on: 11-07-12)

ABSTRACT

In this paper we consider the fuzzy differential equation of the form: $\dot{x}(t) = px(t) + qx([\omega t]/\omega)$, $t \in [0, \infty)$ *and study the existence, the uniqueness and the unboundedness of the solutions of it. for this, we assume that* p, q *are constant real and* ω *is a constant natural numbers but initial value* x_0 *is a fuzzy number.*

AMS Classification: *34A12.*

Keywords*: Fuzzy differential equations with piecewise constant argument,* $\alpha - Level$ *, Fuzzy number, Seikkala derivative.*

1. INTRODUCTION

Differential equations with piecewise constant argument (For detailed study see [1]) are worthwhile studying since describe hybrid dynamical systems (a combination of continues and discrete) and therefore, combine properties of both differential and difference equations. These equations are considerable applied interest since differential equations with piecewise constant argument include, as particular cases, impulsive and loaded equations of control theory and are similar to those found in biomedical models .The initial value of a differential equation, is uncertain and a fuzzy approach is required often.

2. SOME NEEDFUL NOTIONS AND THEOREMS

We now recall some known notions and theorems needed through the paper

- A fuzzy number is a mapping $\mu: \mathbb{R}^+ \to [0,1]$ with the following properties:

(i) μ is upper semi-continuous
(ii) μ is fuzzy convex, i.e., for every $t \in [0,1]$ and $x_1, x_1 \in \mathbb{R}^+$ we have:

$$\mu(tx_1 + (1-t)x_2) \geq \min\{\mu(x_1), \mu(x_2)\}$$

(iii) μ is normal, i.e., there exists an $\mu \in \mathbb{R}^+$, such that $\mu(x) = 1$.
(iv) the support of μ, Supp $\mu = \overline{\{x: \mu(x) > 0\}}$ is compact.

- the $\alpha -$ cut of a fuzzy number μ denoted by $[\mu]_\alpha$, is defined as $[\mu]_\alpha = \{x \in \mathbb{R}^+; \mu(x) \geq \alpha\}$ By using theorems 3.1.5 and 3.1.8 of [2], can be seen that the $\alpha -$ Level of the fuzzy number μ is a closed and bounded interval and by theorem (2.1) of [3], we have :

Theorem 2.1. let $c(\alpha), d(\alpha), \alpha \in (0,1]$ are function such that :
(i) c is a nondecreasing and left continuous,
(ii) d is a nonincreasing and left continuous,
(iii) $c(\alpha) \leq d(\alpha)$ for all $\alpha \in (0,1]$,
(iv) $\overline{\cup_{\alpha \in (0,1]} [c(\alpha), d(\alpha)]}$ is compact,

The $c(\alpha), d(\alpha)$ determine a fuzzy number μ, such that $[\mu]_\alpha = [c(\alpha), d(\alpha)]$, $\alpha \in (0,1]$.

This theorem shows that under the conditions, we can determine a fuzzy number by closed intervals.

Corresponding author: *M. H. Moslehi**
Department of Mathematics, University of Birjand, Birjand, Iran

- A typical differential equation with piecewise constant argument is an equation of the form:

$$\acute{x}(t) = f(t, x(t)x(k(t))) \tag{1}$$

where the argument $k(t)$ has intervals of constancy. More precisely, if $k(t) = [\omega t]/\omega$, where [.] is the greatest integer number and ω is some natural number, then $k(t)$ is discontinuous and $x(t)$ is a solution of the differential equation with piecewise constant argument if satisfies the following conditions:
(i) $x(t)$ is continuous on $(0,1]$.
(ii) The derivative $\acute{x}(t)$ exists at each $t \in (0, \infty]$. with the possible exception of the points $t = \frac{n}{\omega}$, $n = 0,1 \ldots$
(iii) $x(t)$ satisfies Eq. (1) on each interval $\left[\frac{n}{\omega}, \frac{(n+1)}{\omega}\right)$, $n = 0,1 \ldots$

- We consider the initial value problem:

$$x'(t) = Ax(t) + f(t), \quad x(t_0) = x_0$$

where A is a square matrix and $f(t)$ is a continuous vector function. Then by the variation of constant formula, the solution of the above initial value problem is the following:

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-As} f(s)ds$$

Moreover, if Ahas distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ then by Jordan's Canonical Form:

$$e^{At} = T\,\mathrm{diag}\left(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_m t}\right)T^{-1}$$

where the columns of the matrix T are the corresponding eigenvectors of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ of A.

- Let $x: [0, \infty) \to E$ be a fuzzy map where E is the set of fuzzy numbers, such that:

$$[x(t)]_\alpha = [L_\alpha, R_\alpha], \quad \alpha \in (0,1] \tag{2}$$

The Seikkala derivative $\acute{x}(t)$ defined as follows:

$$[\acute{x}(t)]_\alpha = \left[\acute{L}_\alpha, \acute{R}_\alpha\right], \quad \alpha \in (0,1]$$

This shows that $\acute{x}(t)$ is a fuzzy number for $t \in [0, \infty)$.

- We consider the fuzzy differential equation with piecewise constant argument of the form:

$$\acute{x}(t) = px(t) + qx([\omega t]/\omega), t \in [0, \infty) \tag{3}$$

where p, q are constant real numbers, ω is a constant natural number, the initial value $x(t) = x(0) = c_0$ is a fuzzy number such that:

$$[x(0)]_\alpha = [c_0]_\alpha = [L_\alpha, R_\alpha], \quad \alpha \in (0,1] \tag{4}$$

and $[t]$ is the greater integer function.

Let $x: [0, \infty) \to E$, be a fuzzy map where E is the set of fuzzy numbers, such that (2) holds.We say that $x(t)$ is a solution of (3), if the following conditions are satisfied:

(I) for each $\alpha \in (0,1]$, $L_\alpha(t), R_\alpha(t)$ are continuous functions on [0, 1),
(II) the derivatives $\acute{L}_\alpha(t), \acute{R}_\alpha(t)$ exist at any $t \in [0, \infty)$, may except at $t = \frac{n}{\omega}$, $n = 1,2, \ldots$ where one-sided derivatives exist,
(III) for each $\alpha \in (0,1]$ defines a fuzzy number, such that:

$$[\acute{x}(t)]_\alpha = [\acute{L}_\alpha(t), \acute{R}_\alpha(t)], \ \alpha \in (0,1], \ t \in [0, \infty)$$

(IV) $x(t)$ satisfies (3) at every interval $\left[\frac{n}{\omega}, \frac{(n+1)}{\omega}\right)$, $n = 0,1 \ldots$

- Let $x(t)$ be a solution of (3), which satisfies (4). If L_α, R_α for all $\alpha \in (0,1]$ then we have $L_\alpha(t) = R_\alpha(t)$ for all $t \in [0,\infty)$ and so $x(t)$ is a solution of the corresponding ordinary equation (3), which has been studied in [2]. We call the solution $x(t)$ of (3), which satisfies (4) with $L_\alpha = R_\alpha$ for all $\alpha \in (0,1]$, trivial solution of the fuzzy equation (3).

- We say that the solution $x(t)$ is unbounded if there exists an $\bar{\alpha} \in (0,1]$ such that either $L_{\bar{\alpha}}(t)$ or $R_{\bar{\alpha}}(t)$, for all $t \in [0,\infty)$ is an unbounded function.

3. EXISTENCE AND UNIQUENESS OF SOLUTION

To prove the first proposition we need the following lemma

Lemma 3.1. Consider the functions:

$$\begin{aligned} f = f(p,q,w) &= \left(1+\tfrac{q}{p}\right)e^{\left(\frac{p}{\omega}\right)} - \tfrac{q}{p} \\ g = g(p,q,w) &= \left(1+\left|\tfrac{q}{p}\right|\right)e^{\left(\frac{p}{\omega}\right)} - \left|\tfrac{q}{p}\right| \end{aligned} \tag{5}$$

where p, q are real numbers such that at least one is a negative number and ω is a natural number. Then $|f| \le g$

proof. Using (5) we take:

$$f+g = \begin{cases} \left(1+\frac{q}{p}\right)h(p) - \frac{2q}{p}, & \text{if } p<0, q<0 \\ h(p) + \frac{q}{p}v(p), & \text{if } p<0, q>0 \\ 2e^{p}, & \text{if } p>0, q<0 \end{cases}$$

$$f+g = \begin{cases} \left(1+\frac{q}{p}\right)v(p), & \text{if } p<0, q<0 \\ v(p) + (h(p)-2)\frac{q}{p}, & \text{if } p<0, q>0 \\ \frac{2q}{p}(e^{p}-1), & \text{if } p>0, q<0 \end{cases} \tag{6}$$

where $h(p) = e^{\left(\frac{p}{\omega}\right)} + e^{\left(-\frac{p}{\omega}\right)}$, $v(p) = e^{\left(\frac{p}{\omega}\right)} - e^{\left(-\frac{p}{\omega}\right)}$ Thus, can be seen that:

(i) $h(p) \ge 2$; for all $p \in \mathbb{R}$

(ii) $v(p) > 0$; *for all* $p \in \mathbb{R}^+$

(iii) $v(p) < 2$; *for all* $p \in \mathbb{R}^-$ (7)

Relations (6) and (7) imply that $|f| \le g$ is true.

Proposition 3.2. Let p, q are real constants , ω is a natural constants and c_0 is a fuzzy number. the fuzzy differential equation with piecewise constant argument of the form (3) has a unique solution $x(t)$, which satisfies the initial condition $x(t_0) = x(0) = c_0$.

Proof. For proof, we must consider all possible cases, as follow ;
(1) $p<0, q<0$
(2) $p<0, q>0$
(3) $p>0, q<0$
(4) $p>0, q>0$
(5) $p=0, q<0$
(6) $p=0, q>0$

here, We prove the first case.The other cases will be proven similarly. For this, suppose that

$$p<0, q<0 \tag{8}$$

Let $x: [0,\infty) \to E$ be a function which satisfies (2), (3) and (4). Note that for

$$\frac{n}{\omega} = t_n \le t < t_{n+1} = \frac{n+1}{\omega}, \; n = 0,1,2,\ldots$$

We have: $\frac{[\omega t]}{\omega} = t_n = \frac{n}{\omega}$ Let $\alpha \in [0, \infty)$. Since (8) holds then from (3) we have for $t_n \le t < t_{n+1}$, $n = 0,1,2,\dots$

$$[\acute{L}_\alpha(t), \acute{R}_\alpha(t)] = p[L_\alpha(t), R_\alpha(t)] + q[L_\alpha(\frac{[\omega t]}{\omega}), R_\alpha(\frac{[\omega t]}{\omega})]$$
$$= [pR_\alpha(t), pL_\alpha(t)] + [qR_\alpha(t_n), qL_\alpha(t_n)]$$

and so we take the following parametric system of differential equations for $t_n \le t < t_{n+1}$, $n = 0,1,2,\dots$

$$\begin{aligned} \acute{L}_\alpha(t) &= pR_\alpha(t) + qR_\alpha(t_n) \\ \acute{R}_\alpha(t) &= pL_\alpha(t) + qL_\alpha(t_n) \end{aligned} \tag{9}$$

If $y_\alpha(t) = \begin{pmatrix} L_\alpha(t) \\ R_\alpha(t) \end{pmatrix}$ then we have from (9):

$$\acute{y}_\alpha(t) = Ay_\alpha(t) + By_\alpha(t_n) \tag{10}$$

where:

$$A = \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix} \ , \ B = \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} \tag{11}$$

Then by variation of constants formula, we take for $t_n \le t < t_{n+1}$, $n = 0,1,2,\dots$

$$y_\alpha(t) = e^{A(t-t_n)} y_\alpha(t_n) + \int_{t_n}^{t} e^{A(t-s)} B y_\alpha(t_n) ds \tag{12}$$

Using Jordan's Canonical Form we can find a matrix: $T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ such that:

$$T^{-1}AT = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} = D$$

Then $A = TDT^{-1}$ and so

$$e^{At} = T \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix} T^{-1} = \begin{pmatrix} \cosh(pt) & \sinh(pt) \\ \sinh(pt) & \cosh(pt) \end{pmatrix} \tag{13}$$

From (12) and (13) we have for $t_n \le t < t_{n+1}$, $n = 0,1,2,\dots$ and $\alpha \in (0,1]$:

$$y_\alpha(t) = \begin{pmatrix} \cosh(p(t-t_n) & \sinh(p(t-t_n)) \\ \sinh(p(t-t_n)) & \cosh(p(t-t_n)) \end{pmatrix} y_\alpha(t_n) + \begin{pmatrix} \int_{t_n}^{t} \cosh(p(t-s)ds & \int_{t_n}^{t} \cosh(p(t-s)ds \\ \int_{t_n}^{t} \sinh(p(t-s)ds & \int_{t_n}^{t} \sinh(p(t-s)ds \end{pmatrix} \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} y_\alpha(t_n)$$

$$= \begin{pmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{pmatrix} y_\alpha(t_n)$$

$$\Rightarrow y_\alpha(t) = C(t) y_\alpha(t_n) \tag{14}$$

where :

$$c_{11}(t) = c_{22}(t) = (1 + \frac{q}{p})\cosh(p(t-t_n)) - \frac{q}{p}$$
$$c_{12}(t) = c_{21}(t) = (1 + \frac{q}{p})\sinh(p(t-t_n))$$

If $t \to t_{n+1}$ in (14) we get:

$$y_\alpha(t_{n+1}) = C y_\alpha(t_n) \ , \ C = C(t_{n+1}) \tag{15}$$

Remark. note that C is a matrix with constant element Especially the elements of C is independent of n .

By Jordan's Canonical Form we have: $T^{-1}AT = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$, where f, g was defined in (5). Then

$$C^n = T \begin{pmatrix} f^n & 0 \\ 0 & g^n \end{pmatrix} T^{-1} = \begin{pmatrix} \frac{1}{2}(f^n + g^n) & \frac{1}{2}(f^n - g^n) \\ \frac{1}{2}(f^n - g^n) & \frac{1}{2}(f^n + g^n) \end{pmatrix} \tag{16}$$

By using (15) and (16) we have:

$$\begin{aligned} y_\alpha(t_n) &= Cy_\alpha(t_{n-1}) \\ &\vdots \\ &= C^n y_\alpha(t_0) \\ &= C^n y_\alpha(0) \\ &= \begin{pmatrix} \frac{1}{2}(f^n+g^n) & \frac{1}{2}(f^n-g^n) \\ \frac{1}{2}(f^n-g^n) & \frac{1}{2}(f^n+g^n) \end{pmatrix} y_\alpha(0) \end{aligned} \tag{17}$$

Then from (2), (4), (14), and (17), we have:

$$\begin{aligned} y_\alpha(t) &= \begin{pmatrix} L_\alpha(t) \\ R_\alpha(t) \end{pmatrix} \\ & C(t) \times \begin{pmatrix} \frac{1}{2}(f^n+g^n) & \frac{1}{2}(f^n-g^n) \\ \frac{1}{2}(f^n-g^n) & \frac{1}{2}(f^n+g^n) \end{pmatrix} \times \begin{pmatrix} L_\alpha(t) \\ R_\alpha(t) \end{pmatrix} \end{aligned}$$

Therefore we have for $t_n \le t < t_{n+1}$, $n = 0,1,2,...$ and $\alpha \in (0,1]$:

$$\begin{aligned} L_\alpha = \frac{1}{2}((f^n+g^n)L_\alpha + (f^n-g^n)R_\alpha)(\left(1+\frac{q}{p}\right)\cosh\big(p(t-t_n)\big) - \frac{q}{p}) \\ + \frac{1}{2}((f^n-g^n)L_\alpha + (f^n+g^n)R_\alpha)\left(1+\frac{q}{p}\right)\sinh\big(p(t-t_n)\big) \end{aligned} \tag{18}$$

$$\begin{aligned} R_\alpha = \frac{1}{2}((f^n-g^n)L_\alpha + (f^n+g^n)R_\alpha)(\left(1+\frac{q}{p}\right)\cosh\big(p(t-t_n)\big) - \frac{q}{p}) \\ + \frac{1}{2}((f^n+g^n)L_\alpha + (f^n-g^n)R_\alpha)\left(1+\frac{q}{p}\right)\sinh\big(p(t-t_n)\big) \end{aligned} \tag{19}$$

For $L_\alpha(t)$ and $R_\alpha(t)$ Presented above, $\alpha \in (0,1]$: and $t \in [0,\infty)$, we have:

(i) Using Lemma 3.1 , it is clear that $L_\alpha(t)$ is an increasing function and left continuous.
(ii) Using Lemma 3.1 , it is clear that $R_\alpha(t)$ is an decreasing function and left continuous.
(iii) $L_\alpha(t) \le R_\alpha(t)$.
(iv) $supp(x(t))$ is compact because $C_0 = [L_\alpha, R_\alpha]$ is a fuzzy number and $supp(C_0)$ is compact.

Therefore, using Theorem (2.1), $[L_\alpha(t), R_\alpha(t)]$ (such that $L_\alpha(t)$ defined by (19) and $R_\alpha(t)$ defined by (20)), for every $t \in [0,\infty)$ defined a fuzzy number $x(t)$ such that:

$$[x(t)]_\alpha = [L_\alpha(t), R_\alpha(t)] \quad , \quad \alpha \in (0,1]$$

Conversely, let $x: [0,\infty) \to E$ be the map such that (2) holds, where $L_\alpha(t), R_\alpha(t)$ defined in (19) and (20). We can easily prove that $x(t)$ is a solution of (3), which satisfies (4).

4. BOUNDEDNESS OF SOLUTION

In the following proposition, we prove that every nontrivial solution of (3) is unbounded.

Proposition 4.1. Consider the fuzzy differential equation with piecewise constant argument of the form (3), where p,q are real numbers. Then every nontrivial solution of (3)is unbounded. Moreover, the fuzziness $R_\alpha(t) - L_\alpha(t)$, $\alpha \in (0,1]$ tends to ∞ as $t \to \infty$ for all α such that $R_\alpha(t) \ne L_\alpha(t)$.

Proof. similar to the previous proof for proposition (3.2), we suppose that $p < 0$, $q < 0$. for other cases,we can prove that the solution $x(t)$ is unbounded and (25) holds, similarly. Let $x(t)$ be a nontrivial solution of Eq. (3), such that, (4) holds. Since $x(t)$ is a nontrivial solution of equation of (3), there exists an $\overline{\alpha} \in (0,1]$ such that:

$$L_{\overline{\alpha}} < R_{\overline{\alpha}} \tag{20}$$

From (8), it is obvious that:

$$f > 1 \, , g > 1 \tag{21}$$

Relation (17) implies that:

$$L_{\bar{\alpha}}(t_n) = \left|\frac{1}{2}((f^n + g^n)L_{\bar{\alpha}} + (f^n - g^n)R_{\bar{\alpha}})\right|$$

$$= \left|\frac{1}{2}g^n\left(\left(\frac{f}{g}\right)^n (L_{\bar{\alpha}} + R_{\bar{\alpha}}) + (L_{\bar{\alpha}} - R_{\bar{\alpha}})\right)\right|$$

$$\geq \frac{1}{2}g^n \left|\left|\frac{f}{g}\right|^n |L_{\bar{\alpha}} + R_{\bar{\alpha}}| - |L_{\bar{\alpha}} - R_{\bar{\alpha}}|\right|$$

From above relations, we have:

$$\lim_{n\to\infty} L_{\bar{\alpha}}(t_n) = \infty$$

Therefore, every solution of (3) is unbounded. From (19) and (20) we get for $t_n \leq t < t_{n+1}$, $n = 0,1,2,\ldots$ and $\alpha \in (0,1]$:

$z_\alpha(t) = R_\alpha(t) - L_\alpha(t).$

$$= \left(\left(1 + \frac{q}{p}\right)e^{-p(t-t_n)} - \frac{q}{p}\right)(R_\alpha - L_\alpha)g^n \tag{22}$$

Therefore, $\lim_{t\to\infty} z_\alpha(t) = \infty$

Thus we have shown that the solution $x(t)$ is unbounded and (23) holds.

5. EVALUATION OF RESULTS BY USING EXAMPLE

To illustrate our results, we give an example.

Example 5.1. Consider the fuzzy differential equation with piecewise constant argument $\acute{x}(t) = -x(t) - 2x([\omega t]/\omega$, $t_n \leq t < t_{n+1}$, $n=0,1,2,\ldots$ we find the solution of it such that:

$$x(0) = \begin{cases} x - 1, & \text{if } 1 \leq x \leq 2 \\ 3 - x, & \text{if } 2 \leq x \leq 3 \end{cases}$$

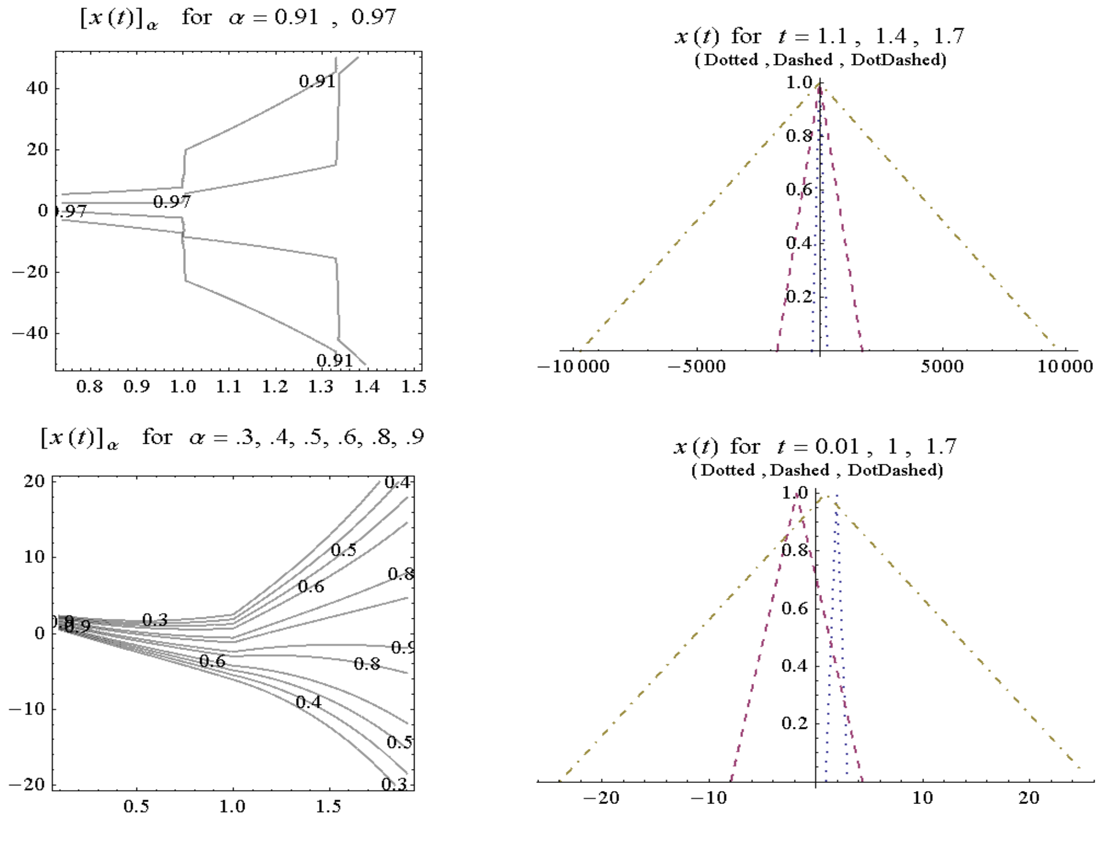

Fig: 1 example 5.1 $\omega = 3$ for first row and $\omega = 1$ for second row)

We can easily show that $[x(0)]_\alpha = [\alpha + 1, 3 - \alpha]$, $\alpha \in (0,1]$. Then by relations (19) and (20) we take for $t_n \le t < t_{n+1}$:

$$L_\alpha(t) = z_1(t)(\alpha - 1) + K_1(t)$$

$$R_\alpha(t) = z_1(t)(1 - \alpha) + K_1(t) \tag{23}$$

where $n = 0,1,2,\ldots$ and $\alpha \in (0,1]$ and

$$z_1(t) = (3e - 2)^n \left(3e^{\left(t - \frac{n}{\omega}\right)} - 2\right),$$

$$K_1(t) = 2(3e^{-1} - 2)^n \left(3e^{-\left(t - \frac{n}{\omega}\right)} - 2\right) \tag{24}$$

By using (24), the solution of above equation is: $[x(t)]_\alpha = [L_\alpha(t), R_\alpha(t)]$ and so

$$x(0) = \begin{cases} 1 + \dfrac{y_n(t) - K_1(t)}{z_1(t)}, & \text{if } -z_1(t) + K_1(t) \le y_n(t) \le K_1(t) \\ 1 - \dfrac{y_n(t) - K_1(t)}{z_1(t)}, & \text{if } -K_1(t) < y_n(t) \le K_1(t) + z_1(t) \end{cases}$$

See figure1.

6. CONCLUSION

In this paper, we studied the existence and the uniqueness of the solutions of (3). Moreover, we prove that every nontrivial solution of (3) is unbounded.

7. REFERENCES

[1] A.R. Aftabizadeh, J. Wiener, Oscillatory and periodic solutions for systems of two first order linear differential equations with piecewise constant argument, Appl. Anal. 26 (1998), 327-333.

[2] H.T. Nguyen, E.A. Walker, A first Course in Fuzzy Logic, CRC Press, New York, London, Tokyo, 1997.

[3] C. Wu, B. Zhang, Embedding problem of noncompact fuzzy number space $\widetilde{E}$, Fuzzy Sets Syst. 105 (1999), 165-169.

Source of support: Nil, Conflict of interest: None Declared