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DERIVATIONS ON TM-ALGEBRAS

T. Ganeshkumar* & M. Chandramouleeswaran**

****Department of Mathematics, M.S.S Wakf Board College Madurai-625020, India***
*****Department of Mathematics, S.B.K. College Aruppukottai-626101, India***

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ABSTRACT

In 2010 Tamilarasi and Manimegalai introduced a new class of algebra called TM-algebra. Motivated by the works on derivations on rings and near rings, in this paper we introduced the notion of derivation on TM-algebra.

1. INTRODUCTION

It is well known that BCK and BCI-algebras are two classes of algebras of logic. They were introduced by Imai and Iseki [1] and have been extensively investigated by many researchers. BCK/BCI algebras form an algebraic semantic for CA Meriedith's logic. They are also the generalizations propositional calculi. It is known that the class of BCK-algebras is a proper sub class of the BCI-algebras. In [5], [6] Q.P.Hu and X. Li introduced a wider class of abstract algebras: BCH-algebras and have shown that the class of BCI- algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H.S. Kim [7] introduced the notion of d-algebras which is another generalization of BCK-algebras. Another algebraic formulation of the proportitional calculi is TM-algebra introduced by Tamilarasi and Manimegalai [3].

The notion of derivation on rings is quite old. However it got its significance only after Ponser's work [2] in 1957. After this many researcher started working in this direction.

In [4] the authors introduced the notion of derivations on d-algebras another generalization of BCK-algebras. This notion of derivation is the same as that in ring theory and the usual algebraic theory. Motivated by this, in this paper, we introduce the notion of derivation on a TM-algebra and study some simple but elegant results.

2. PRELIMINARIES

Definition 2.1. A d-algebra (X,*, 0) is a non empty set X with a constant 0 and a binary operation * satisfying the following axioms:

1. $x * x = 0$
2. $0 * x = 0$
3. $x * y = 0$ and $y * x = 0 \Rightarrow x = y$ for all $x, y \in X$.

Definition 2.2. A TM-algebra (X,*, 0) is a non empty set X with a constant 0 and binary operation * satisfying the following axioms:
$x * 0 = x$
$(x * y) * (x * z) = z * y$ for all $x, y, z \in X$.

Definition 2.3. Let X be a d-algebra. A map θ: X → X is a left-right derivation ((l ,r)-derivation) of X it satisfies the identity $\theta(x * y) = ((\theta(x) * y) \wedge (x * \theta(y))$ for all $x, y \in X$. If θ satisfies the identity $\theta(x * y) = (x * \theta(y)) \wedge (\theta(x) * y)$ for all $x, y \in X$, then θ is a right-left derivation ((r, l)-derivation) of X. Moreover if θ is both a (l, r)-derivation and (r, l)-derivation then θ is a derivation of X.

3. DERIVATIONS ON TM-ALGEBRAS

In this section, we introduce the notion of derivation on a TM-algebra and prove some simple results.

Definition 3.1. Let (X,*, 0) be a TM-algebra. A self map d: X → X is said to be a (l, r)-derivation on X if $d(x * y) = (d(x) * y) \wedge (x * d(y))$.

Corresponding author:* M. Chandramouleeswaran*
*****Department of Mathematics, S.B.K. College Aruppukottai-626101, India***

We now present an example of a TM-algebra, in which the notion of derivation can be defined.

Example 3.2. Let (X,*, 0) be a TM-algebra with the following Cayley table.

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

A self map $d : X \to X$ be defined by $d(0) = 1$, $d(1) = 2$, $d(2) = 0$. Then d is a (l, r)-derivation.

Remark 3.3. One can observe that if $d: X \to X$ is a (l, r)-derivation on X, then

$$d(x * y) = d(x) * y.$$

Definition 3.4. Let (X,*,0) be a TM-algebra. A self map $d: X \to X$ is said to be a (r, l)-derivation on X if

$$d(x * y) = (x * d(y)) \wedge (d(x) * y).$$

Remark 3.5. As in remark 3.3, we observe that if the self map $d: X \to X$ is a (r, l)-derivation on X, then

$$d(x * y) = x * d(y).$$

Definition 3.6. Let $d : X \to X$ be a self map on TM-algebra (X,*,0). The map d is said to be a derivation on X if d is both a (l, r)-derivation and a (r, l)-derivation on X.

Remark 3.7. From remarks 3.3 and 3.5 we observe that if d is a derivation on X then,

$$d(x * y) = d(x) * y = x * d(y).$$

We now present another example of a TM-algebra in which one can define a derivation.

Example 3.8. Let (X,*,0) be a TM-algebra with the following Cayley table.

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

The self map $d: X \to X$ be defined by $d(0) = 3$, $d(1) = 2$, $d(2) = 1$, $d(3) = 0$ is a derivation.

Definition 3.9. Let X be a TM-algebra. A self map $d: X \to X$ is said to be regular if $d(0) = 0$.

Definition 3.10. If X is a TM-algebra then we define a partial ordering $\leq$ such that $x \leq y$ whenever $x * y = 0$.

Proposition 3.11. Let (X,*, 0) be a TM-algebra. If $d: X \to X$ is a regular (r, l)-derivation on X then $x \leq d(x)$ for all $x \in X$.

Proof:

$d(0) = 0$
$d(x * x) = 0$ $\quad$ $(x * x = 0)$
$x * d(x) = 0$ $\quad$ (By remark 3.5)
Therefore $x \leq d(x)$ $\quad$ (By definition 3.10)

Proposition 3.12. Let (X,*,0) be a TM-algebra. Let $d: X \to X$ is a derivation.

1 If $x * d(x) = 0$ for all $x \in X$, then d is regular.
2 If $d(x) * x = 0$ for all $x \in X$, then d is regular.

Proof:
1. Given $x * d(x) = 0$ and d is a derivation.

Now,

$$\begin{aligned} d(0) &= d(x * x) \\ &= x * d(x) \quad \text{(By definition 3.4)} \\ &= 0, \text{ thus proving that d is regular.} \end{aligned}$$

2. Given $d(x) * x = 0$ and d is a (l, r)-derivation.

Now,

$$\begin{aligned} d(0) &= d(x * x) \\ &= d(x) * x \quad \text{(By remark 3.3)} \\ &= 0, \text{ thus proving that d is regular.} \end{aligned}$$

Proposition 3.13. Let d be a self map of a TM-algebra X.
1. If d is regular (l, r)-derivation on X, then $d(x) = d(x) \wedge x$.
2. If d is regular (r, l)-derivation on X, then $d(x) = x \wedge d(x)$.

Proof:
1. Given d is regular. Therefore $d(0) = 0$.

Now,

$$\begin{aligned} x &= x * 0 \\ d(x) &= d(x * 0) \\ &= (d(x) * 0) \wedge (x * d(0)) \quad \text{(By definition 3.1)} \\ &= d(x) \wedge (x * 0) \\ &= d(x) \wedge x \end{aligned}$$

2. Given d is regular (r, l)-derivation on X.

$$\begin{aligned} d(x) &= d(x * 0) \\ &= (x * d(0)) \wedge (d(x) * 0) \text{ (By definition 3.4)} \\ &= (x * 0) \wedge d(x) \\ &= x \wedge d(x) \end{aligned}$$

Definition 3.14. Let d_1, d_2 be self maps on a TM-algebra X. We define $d_1 \circ d_2$ as follows.

$$(d_1 \circ d_2)(x) = d_1(d_2(x)) \text{ for all } x \in X.$$

Lemma 3.15. Let (X,*, 0) be a TM-algebra. Let d_1, d_2 be two (l, r)-derivations on X. Then $(d_1 \circ d_2)$ is also a (l, r)-derivation on X.

Proof: Given d_1 is a (l, r)-derivation on X. Hence $d_1(x * y) = d_1(x) * y$, for all $x, y \in X$.

Similarly $d_2(x * y) = d_2(x) * y$.

Now,

$$\begin{aligned} (d_1 \circ d_2)(x * y) &= d_1(d_2(x * y)) \quad \text{(By definition 3.14)} \\ &= d_1(d_2(x) * y) \quad \text{(By remark 3.3)} \\ &= (d_1(d_2(x))) * y \\ &= (d_1 \circ d_2)(x) * y \end{aligned}$$

Therefore $(d_1 \circ d_2)$ is a (l, r)-derivation on X.

Lemma 3.16. Let (X,*,0) be a TM-algebra. Let d_1, d_2 be two (r, l)-derivation on X, then $(d_1 \circ d_2)$ is also a (r, l)-derivation on X.

Proof: Given d_1 is a (r, l)-derivation on X.

$d_1(x * y) = x * d_1(y)$, for all $x, y \in X$. (By remark 3.5)

Similarly $d_2(x * y) = x * d_2(y)$.

$$\begin{aligned}\text{Now } (d_1 \circ d_2)(x * y) &= d_1(d_2(x * y)) \quad \text{(By definition 3.14)}\\ &= d_1(x * d_2(y)) \quad \text{(By remark 3.5)}\\ &= x * (d_1(d_2(y)))\\ &= x * ((d_1 \circ d_2)(y))\end{aligned}$$

Hence $(d_1 \circ d_2)$ is a (r, l)-derivation on X.

By combining the above two lemmas 3.15 and 3.16, we get the following theorem.

Theorem 3.17. Let (X,*,0) be a TM-algebra and d_1, d_2 be derivations on X then $(d_1 \circ d_2)$ is also a derivation on X.

Theorem 3.18. Let (X,*, 0) be a TM-algebra. Let d_1, d_2 be two derivations on X, then $(d_1 \circ d_2) = (d_2 \circ d_1)$.

Proof: Since d_1, d_2 be two derivations on X, d_1, d_2 are both (l, r) and (r, l)-derivations on X.

Now,

$$\begin{aligned}(d_1 \circ d_2)(x * y) &= d_1(d_2(x * y))\\ &= d_1(d_2(x) * y) \quad \text{(By remark 3.3)}\\ &= d_2(x) * d_1(y) \quad \text{(By remark 3.5)} \qquad (1)\end{aligned}$$

Also

$$\begin{aligned}(d_2 \circ d_1)(x * y) &= d_2(d_1(x * y))\\ &= d_2(x * d_1(y)) \quad (d_1 \text{ is (r, l)-derivation})\\ &= d_2(x) * d_1(y) \quad (d_2 \text{ is (l, r)-derivation}) \qquad (2)\end{aligned}$$

From (1) and (2), $(d_1 \circ d_2)(x * y) = (d_2 \circ d_1)(x * y)$, thus proving that $(d_1 \circ d_2) = (d_2 \circ d_1)$.

Definition 3.19. Let (X,*, 0) be a TM-algebra. Let d_1, d_2 be two self maps on X.

We define $(d_1 * d_2) : X \to X$ as $(d_1 * d_2)(x) = d_1(x) * d_2(x)$ for all $x \in X$.

Theorem 3.20. Let (X,*, 0) be a TM-algebra and d_1, d_2 be two derivations of X, then $d_1 * d_2 = d_2 * d_1$.

Proof:

$$\begin{aligned}(d_1 \circ d_2)(x * y) &= d_1(d_2(x * y))\\ &= d_1(d_2(x) * y) \quad \text{(By remark 3.3)}\\ &= d_2(x) * d_1(y) \quad \text{(By remark 3.5)} \qquad (1)\end{aligned}$$

Again,

$$\begin{aligned}(d_1 \circ d_2)(x * y) &= d_1(d_2(x * y))\\ &= d_1(x * d_2(y)) \quad \text{(By remark 3.5)}\\ &= d_1(x) * d_2(y) \quad \text{(By remark 3.3)} \qquad (2)\end{aligned}$$

Combining (1) and (2), we get $d_2(x) * d_1(y) = d_1(x) * d_2(y)$ (3)

Substituting y = x in (3) we get,

$$d_2(x) * d_1(x) = d_1(x) * d_2(x).$$

$$(d_2 * d_1)(x) = (d_1 * d_2)(x).$$

Since this is true for all elements x in X, we conclude that $d_2 * d_1 = d_1 * d_2$

Lemma 3.21. In a TM-algebra both right and left cancellation law hold good.

Proof: Let (X,*, 0) be a TM-algebra. Assume that $x * y = x * z$ for all $x, y, z \in X$.

$$\begin{aligned}\text{Now} \quad y &= x * (x * y)\\ &= x * (x * z)\\ &= z\end{aligned}$$

This proves that the left cancellation law holds in X.

Assume now that $y * x = z * x$.

Consider $x * y = (y * y) * (y * x)$ (By definition)

$= 0 * (z * x)$

$= (z * z) * (z * x)$

Thus $x * y = x * z$

Therefore $y = z$ (By Left Cancellation Law)

Hence the Right Cancellation Law holds in X.

Theorem 3.22. Let d be a (l, r)-derivation of TM-algebra X, then

1. $d(0) = d(x) * x$.
2. d is 1-1.
3. If d is regular then d is the identity map.
4. If there is an element $x \in X$ such that $d(x) = x$, then d is the identity map.
5. If there is an element $x \in X$ such that $d(y) * x = 0$ or $x * d(y) = 0$ for all $y \in X$, then $d(y) = x$, (ie) d is a constant map.

Proof:

1. $x * x = 0$, therefore $d(0) = d(x * x) = d(x) * x$ (Since d is (l, r)-derivation)
2. Let $x, y \in X$ and $d(x) = d(y)$.

Now $d(0) = d(x * x) = d(x) * x$ (1)

Again $d(0) = d(y * y) = d(y) * y = d(x) * y$ (Since $d(x) = d(y)$) (2)

From (1) and (2), $d(x) * x = d(x) * y$.

$\Rightarrow x = y$ (By L.C.L)

3. Given d is regular. Therefore $d(0) = 0$.

$d(0) = d(x) * x$ (By (1)).

$0 = d(x) * x$.

$x * x = d(x) * x$

Applying Right Cancellation Law in a TM-algebra,

we get $x = d(x)$, proving that d is the identity map.

4. Let $x \neq y$, $x, y \in X$.

Given that there is an element $x \in X$ such that $d(x) = x$ (3)

Now,

$y = x * (x * y)$

$d(y) = d(x) * (x * y)$ (since d is (l, r)-derivation)

$= x * (x * y)$ (using (3))

$= y$

Therefore d is the identity map.

5. Given $d(y) * x = 0$

$d(y) * x = x * x$.

$\Rightarrow d(y) = x$ (By R.C.L)

Again if $x * d(y) = 0$

$x * d(y) = x * x$

$\Rightarrow d(y) = x$ (By L.C.L)

Hence $d(y) = x$, for all $y \in X$.

Therefore d is a constant map.

Theorem 3.23. Let d be a (r, l)-derivation of TM-algebra X, then

1. $d(0) = x * d(x)$.
2. $d(x) = d(x) \wedge x$ for all $x \in X$.
3. d is 1-1.
4. If d is regular then d is the identity map.
5. If there is an element $x \in X$ such that $d(x) = x$, then d is the identity map.
6. If there is an element $x \in X$ such that $d(y) * x = 0$ or $x * d(y) = 0$ for all $y \in X$ then $d(y) = x$ (ie) d is a constant map

Proof: (1), (3), (4), (5) and (6) are analogous to results (1) to (5) of the above theorem 3.22.

Hence we prove only the property (2).

Now, $d(x) \wedge x = x * (x * d(x)) = d(x)$ for all $x \in X$. (Since $x * (x * y) = y$)

Theorem 3.24. Let X be a TM-algebra and $d_1, d_2, \ldots, d_n$ be derivations on X, then $d_n(d_{n-1}(d_{n-2}(d_{n-3} \ldots (d_2(d_1(x))))) \leq x$.

Proof:
$$\begin{aligned} d_n(d_{n-1}(d_{n-2}(d_{n-3} \ldots (d_2(d_1(x))))) &= d_n(d_{n-1}(d_{n-2}(d_{n-3} \ldots (d_2(d_1(x))) \ldots))) \\ &\leq d_{n-1}(d_{n-2}(\ldots (d_2(d_1(x))) \ldots)) \\ &\;\;. \\ &\;\;. \\ &\;\;. \\ &\leq d_1(x) \\ &\leq x. \end{aligned}$$

Definition 3.25. Let L Der(X) denote the set of all (l, r)-derivations on X. Define the binary operation $\wedge$ on L Der(X) as follows. For $d_1, d_2 \in$ L Der(X), define $(d_1 \wedge d_2)(x) = d_1(x) \wedge d_2(x)$ for all $x \in X$.

Lemma 3.26. If d_1 and d_2 are (l, r)-derivations on X, then $(d_1 \wedge d_2)$ is also a (l, r)-derivation.

Proof: To Prove: $(d_1 \wedge d_2)(x * y) = (d_1 \wedge d_2)(x) * y$ for all $x, y \in X$.

$$\begin{aligned} (d_1 \wedge d_2)(x * y) &= d_1(x * y) \wedge d_2(x * y) \quad \text{[By definition 3.25]} \\ &= (d_1(x) * y) \wedge (d_2(x) * y) \\ &= (d_2(x) * y) * ((d_2(x) * y) * (d_1(x) * y)) \\ &= d_1(x) * y \end{aligned} \tag{1}$$

$$\begin{aligned} (d_1 \wedge d_2)(x) * y &= (d_1(x) \wedge d_2(x)) * y \\ &= (d_2(x) * (d_2(x) * d_1(x))) * y \\ &= d_1(x) * y \end{aligned} \tag{2}$$

From (1) and (2), $(d_1 \wedge d_2)(x * y) = (d_1 \wedge d_2)(x) * y$.

Therefore $(d_1 \wedge d_2)$ is a (l, r)-derivation.

Lemma 3.27. The binary composition $\wedge$ defined on L Der(X) is associative.

Proof: Let X be a TM-algebra.

Let d_1, d_2, d_3 are (l, r)-derivations.

Now,

$$\begin{aligned} ((d_1 \wedge d_2) \wedge d_3)(x * y) &= (d_1 \wedge d_2)(x * y) \wedge d_3(x * y) \\ &= (d_1(x) * y) \wedge (d_3(x) * y) \text{ (using lemma 3.26\} in (1))} \\ &= (d_3(x) * y) * ((d_3(x) * y) * (d_1(x) * y)) \\ &= d_1(x) * y \end{aligned} \tag{1}$$

Again,

$$\begin{aligned} (d_1 \wedge (d_2 \wedge d_3))(x * y) &= d_1(x * y) \wedge ((d_2 \wedge d_3)(x * y)) \\ &= (d_1(x) * y) \wedge (d_2(x) * y) \\ &= (d_2(x) * y) * ((d_2(x) * y) * (d_1(x) * y)) \\ &= d_1(x) * y \end{aligned} \tag{2}$$

Combining (1) and (2) we get, $(d_1 \wedge d_2) \wedge d_3 = d_1 \wedge (d_2 \wedge d_3)$.

Combining the above two lemmas we get following theorem.

Theorem 3.28. LDer(X) is a semi-group under the binary composition $\wedge$ defined by $(d_1 \wedge d_2)(x) = d_1(x) \wedge d_2(x)$ for all $x \in X$ and $d_1, d_2 \in$ LDer(X).

Analogously we can prove that

Theorem 3.29. RDer(X) is a semi-group under the binary operation $\wedge$ defined by $(d_1 \wedge d_2)(x) = d_1(x) \wedge d_2(x)$, for all $x \in X$ and $d_1, d_2 \in$ RDer(X).

Combining the above two theorem, we get the following theorem.

Theorem 3.30 If Der(X) denotes the set of all derivations on X, it is a semi-group under the binary operation $\wedge$ defined by $(d_1 \wedge d_2)(x) = d_1(x) \wedge d_2(x)$, for all $x \in X$ and $d_1, d_2 \in$ Der(X).

4. 0 COMMUTATIVE

It is to be observed that many of the TM-algebras are not commutative in the sense of commutativeness defined for BCI-algebras. However, we observe that 0-commutativeness can be defined in a TM-algebra. This section introduces the notion of 0-commutative in a TM-algebra and give some simple properties.

Definition 4.1. A TM-algebra (X,*, 0) is said to be 0-commutative if $x * (0 * y) = y * (0 * x)$ for all $x, y \in X$.

Example 4.2. Let (X,*, 0) be a TM-algebra with the Cayley table.

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

(X,*, 0) form the 0-commutative TM-algebra.

Lemma 4.3. If (X,*,0) is a 0-commutative TM-algebra then

1. $(0 * x) * (0 * y) = y * x$
2. $(z * y) * (z * x) = x * y$
3. $(x * y) * z = (x * z) * y$
4. $(x * (x * y)) * y = 0$
5. $(x * z) * (y * t) = (t * y) * (z * x)$ for all $x, y, z, t \in X$.
6. $x * (x * y) = y$

Proof: Results 1-4 follows easily. We give the proofs for 5 and 6 only. 5 is true because,

$$\begin{aligned}(x * z) * (y * t) &= (0 * (y * t)) * (0 * (x * z)) \quad \text{[In TM algebra } (y * z) = (0 * y) * (0 * z)]\\ &= ((0 * y) * (0 * t)) * ((0 * x) * (0 * z))\\ &= (t * (0 * (0 * y))) * (z * (0 * (0 * x)) \quad \text{[By definition 3.22]}\\ &= (t * y) * (z * x)\end{aligned}$$

Similarly 6 follows as $x * (x * y) = (x * 0) * (x * y) = y * 0 = y$.

Theorem 4.4/ Let (X,*, 0) be a 0-commutative TM-algebra and d be a derivation on X. Then $d(x) * d(y) = x * y$.

Proof: Since X is 0-commutative, by definition $x * (0 * y) = y * (0 * x)$ for all $x, y \in X$.

$$\begin{aligned}&d(x * (0 * y)) = d(y * (0 * x))\\ &d(x) * (0 * y) = d(y) * (0 * x)\\ &[d(x) * (0 * y)] * y = [d(y) * (0 * x)] * y\\ &(d(x) * y) * (0 * y) = (d(y) * y) * (0 * x) \quad (\text{since } (x * y) * z = (x * z) * y)\\ &\qquad = 0 * (0 * x) \quad (\text{since } d(y) \leq y) \ \$\\ &\qquad = x \quad (\text{since } x * (x * y) = y)\end{aligned}$$

That is $(d(x) * y) * (0 * y) = x \quad (1)$

Interchanging x and y in (1) we have

$(d(y) * x) * (0 * x) = y \quad (2)$

From (1) and (2)

$$\begin{aligned}(x * y) &= ((d(x) * y) * (0 * y)) * ((d(y) * x) * (0 * x)) \\ &= ((y * 0) * (y * d(x)) * ((x * 0) * (x * d(y)) \quad \text{[By lemma 4.3(5)]} \\ &= [y * (y * d(x))] * [x * (x * d(y))] \\ &= d(x) * d(y) \qquad (x * (x * y) = y)\end{aligned}$$

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