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On Left F-derivations of d-algebras

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ABSTRACT

Motivated by some results on derivations in rings and derivations of BCI algebras recently we introduce the notion of derivations on d-algebras and f-derivations on d-algebras. In this paper we introduce the notion of left F-derivations of d-algebras and investigate some simple and interesting results.

Keywords: *d-algebra, edge d-algebras, derivations, f-derivations, endomorphism, left F-derivations.*

Subject Classification*: 03G25, 06F35.*

1. INTRODUCTION

Y. Imai and K. Iseki introduced two classes of abstract algebras BCK-algebras and BCI-algebras ([1] [2] [3]). It is known that the class of BCK-algebras is a proper sub class of the class of BCI-algebras. In ([4] [5]) Q.P.Hu and X. Li introduced a wide class of abstract algebras. BCH-algebras and have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim [6] introduced the notion of d-algebra, which is another generalization of BCK-algebras.

In 2004 Y. B. Jun and X. L. Xin [7] introduced the notion of derivations of BCI-algebras which was motivated from a lot of work done on derivations of rings and near rings. Motivated by the work of Lie and X in recently, we have [8] introduced the notion of derivations and f-derivations [9] on a d-algebras. In this paper we introduce the notion of left F-derivations on d-algebras and study some simple but elegant results.

2. PRELIMINARIES

Definition 2.1 [6] A d-algebra is a non empty set X with a constant 0 and a binary operation * satisfying the following axioms:

1. $x * x = 0$
2. $0 * x = 0$
3. $x * y = 0$ and $y * x = 0$ implies $x = y$.

Definition 2.2 [6] Let S be a non-empty subset of a d-algebra X, then S is called sub algebra of X if $x * y \in S$ for all $x, y \in S$.

Definition 2.3 Let X be a d-algebra and I be a subset of X, then I is called d- ideal of X if it satisfies the following conditions:

1. $0 \in I$
2. $x * y \in I$ and $y \in I$ implies $x \in I$.
3. $x \in I$ and $y \in X$ implies $x * y \in I$.

Definition 2.4 [6] Let (X,*, 0) be a d-algebra and $x \in X$.

Define $x * X = \{x * a \mid a \in X\}$. X is said to be an edge d-algebra if for any $x \in X$, $x * X = \{x, 0\}$.

Properties: In any d-algebra X the following properties hold for all $x, y, z \in X$.

1. $(x * y) * z = (x * z) * y$.
2. $0 * (x * y) = (0 * x) * (0 * y)$.
3. $(x *(x * y)) * y = 0$

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4. $x * (x * y) = y$.
5. $x * (y * z) \geq (x * y) * z$.
6. $((x * y) * (x * z)) \leq (z * y)$.
7. $((x * z) * (y * z)) * (x * y) = 0$.
8. $x * 0 = 0 \Rightarrow x = 0$.
9. $x * a = x * b \Rightarrow a = b$.
10. $a * x = b * x \Rightarrow a = b$.
11. If $x \leq y \Rightarrow x * z \leq y * z$ and $z * y \leq z * x$.

Definition 2.6 For any $x_0 \in X$, the set $A(x_0) = \{x \in X \mid x_0 \leq x\}$ is known as the branch of X determined by x_0. Each branch $A(x_0) \neq \emptyset$ at $x_0 * x_0 = 0 \Rightarrow x_0 \in A(x_0)$.

Clearly we observe that $A(x_0)$ contains all those elements of X that succeed x_0.

Properties. Let X be a d-algebra. The following properties hold.

1. If $x \leq y$, then x and y are contained in the same branch of X.
2. If $x \in A(x_0)$, $y \in A(y_0)$ then $x * y \in A(x_0 * y_0)$.
3. Let $x_0, y_0 \in X$ and $y \in A(y_0)$ then $x_0 * y = x_0 * y_0$.
4. Let $x_0, y_0 \in X$ and $x \in A(x_0)$ then $x * y_0 = x_0 * y_0$.
5. If $A(x_0) \subseteq X$, then $x, y \in A(x_0) \Rightarrow x * y, y * x \in X$.

Definition 2.6 Let X be a d-algebra. A map $\theta : X \rightarrow X$ is a left-right derivations (briefly (l,r)-derivation) of X if it satisfies the identity $\theta(x * y) = (\theta(x) * y \wedge x * \theta(y))$ for all $x, y \in X$.

If θ satisfies the identity $\theta(x*y) = (x * \theta(y) \wedge \theta(x) * y)$ for all $x, y \in X$, then θ is a right-left derivation (briefly (r,l)-derivation) of X. Moreover if θ is both a (l,r)-derivation and (r,l)-derivation, then θ is a derivation of X.

Definition 2.7 A mapping f of a d-algebra X into itself is called an endomorphism if $f(x*y) = f(x) * f(y)$. Note that $f(0) = 0$.

3. LEFT F-DERIVATIONS

In this section we introduce the notion of left F-derivation of a d-algebra and give some examples to explain the theory of left F-derivation in d-algebras and prove some simple but elegant properties.

Definition 3.1 Let X be a d-algebra. By a left F-derivation of X, we mean a self map θ_F of X satisfying the identity $\theta_F(x*y) = (\theta_F(x) * F(y)) \wedge \theta_F(y) * F(x))$ for all $x, y \in X$, where F is an endomorphism of X.

Definition 3.2 A left F-derivation θ_F of a d-algebra X is said to be regular if $\theta_F(0) = 0$. Otherwise it is called an irregular left F-derivation.

Example 3.3 Let X = {0, 1, 2, 3} be a d-algebra with the following Cayley table

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

Define a self map $\theta_F : X \rightarrow X$ as follows $\theta_F(x) = 3$ if $x = 0, 1, 2, 3$.

Define an endomorphism $F: X \rightarrow X$ as follows $F(x) = 0$ if $x = 0, 1, 2$ and $F(x) = 2$ if $x = 3$.

Then it is easily checked that θ_F is a left F-derivation of X.

Theorem 3.4 Let θ_F be a regular left F-derivation of a d-algebra X. If F(x) and $\theta_F(x)$ belong to the same branch of X, then $\theta_F(x) = F(x)$.

Proof: Since $F(x)$ and $\theta_F(x)$ belong to the same branch of X,

$F(x) \le \theta_F(x)$ (1)

Since θ_F is a regular left derivation of X, $\theta_F(0) = 0$.

Now $\theta_F(0) = \theta_F(x*x)$
$= (\theta_F(x) * F(x)) \wedge (\theta_F(x) * F(x))$ (using left F-derivation)
$= (\theta_F(x) * F(x)) * ((\theta_F(x) * F(x)) * (\theta_F(x) * F(x)))$ (using $x \wedge y = y * (y * x)$)
$= \theta_F(x) * F(x)$ (using $x * (x * y) = y$)

Since $\theta_F(0) = 0$, $\theta_F(x) * F(x) = 0$.

Hence $\theta_F(x) \le F(x)$ (2)

From (1) and (2), it follows $\theta_F(x) = F(x)$.

Theorem 3.5 Let θ_F be a self map and $A(x_0)$ be any branch of a d-algebra X. If for any $x \in A(x_0)$, $\theta_F(x) = F(x_0)$, then θ_F is a left F-derivation.

Proof: Let θ_F be a self map and $A(x_0)$ be any branch of a d-algebra X determined by x_0.

According to given condition for any $x \in A(x_0)$, $\theta_F(x) = F(x_0)$ (1)

Now for $x, y \in X$ following two cases arise.

Case 1: Both x and y belongs to the same branch of X.

Case 2: x and y belongs to different branches of X.

Case 1: Let $x, y \in A(x_0)$, So $x_0 \le x$ and $x_0 \le y$.

Then $x * y \in A(x_0 * x_0) = A(0)$. (By result)

So using (1), $\theta_F(x * y) = F(0) = 0$ (2)

Also $x_0 \le y \Rightarrow x_0 * y = 0$ and $x_0 \le x \Rightarrow x_0 * x = 0$ (3)

Further, as F is an endomorphism,

$0 = F(0) = F(x_0 * y) = F(x_0) * F(y)$ and $0 = F(0) = F(x_0 * x) = F(x_0) * F(x)$

Now
$(\theta_F(x) * F(y)) \wedge (\theta_F(y) * F(x)) = (F(x_0) * F(y)) \wedge (F(x_0) * F(x))$ (using (1)).
$= 0 \wedge 0 = 0 * (0 * 0) = 0 * 0 = 0.$

That is, $(\theta_F(x) * F(y)) \wedge (\theta_F(y) * F(x)) = 0 = \theta_F(x*y)$ (using (2))

which implies θ_F is a left F-derivation.

Case 2: Let $x \in A(x_0)$ and $y \in A(y_0)$.$

Then $x*y \in A(x_0 * y_0)$ (By result)

So using (1) $\theta_F(x * y) = F(x_0 * y_0)$ (4)

Now

$(\theta_F(x) * F(y)) \wedge (\theta_F(y) * F(x)) = (F(x_0) *F(y)) \wedge (F(y_0) * F(x))$
$= (F(y_0) * F(x)) * ((F(y_0) *F(x)) * (F(x_0) * F(y)))$
$= F(x_0) *F(y)$
$= F(x_0 *y)$ (Since F is an endomorphism)

$= F(x_0 * y_0)$ (By result)
$= \theta_F(x * y)$ (using (4))

which implies θ_F is a left F-derivation.

This completes the proof.

Lemma 3.6 Let θ_F be a self map of a d-algebra X. If θ_F is a left F-derivation of X. Then the following hold.

1. $\theta_F(x) * F(x) = \theta_F(y) * F(y)$.
2. $\theta_F(x) = \theta_F(x) \wedge \theta_F(0)$.

Proof:

1. Let x, y ∈ X. Then

$$\begin{aligned}\theta_F(0) &= \theta_F(x*x)\\ &= (\theta_F(x) * F(x)) \wedge (\theta_F(x) * F(x))\\ &= (\theta_F(x) * F(x)) * ((\theta_F(x) * F(x)) * (\theta_F(x) * F(x)))\\ &= \theta_F(x) * F(x) \qquad (1)\end{aligned}$$

Similarly $\theta_F(0) = \theta_F(y * y) = \theta_F(y) * F(y)$ (2)

From (1) and (2), it follows that $\theta_F(x) * F(x) = \theta_F(y) * F(y)$.

2. Let x ∈ X. Then

$$\begin{aligned}\theta_F(x) &= \theta_F(x*0)\\ &= (\theta_F(x) * F(0)) \wedge (\theta_F(0) * F(x))\\ &= (\theta_F(x) * 0) \wedge (\theta_F(0) * F(x))\\ &= \theta_F(x) \wedge (\theta_F(0) * F(x))\\ &= (\theta_F(0) * F(x)) * ((\theta_F(0) * F(x)) * \theta_F(x))\\ &= (\theta_F(0) * F(x)) * ((\theta_F(0) * \theta_F(x)) * F(x)) && ((x*y) * z = (x*z) * y)\\ &\leq \theta_F(0) * (\theta_F(0) * \theta_F(x)) && ((x*y) * (z*y) \leq x*z)\\ &= \theta_F(x) && (x*(x*y) = y)\end{aligned}$$

Thus $\theta_F(x) \leq \theta_F(0) * (\theta_F(0) * \theta_F(x)) \leq \theta_F(x)$.

Therefore $\theta_F(x) = \theta_F(0) * (\theta_F(0) * \theta_F(x))$, which implies that $\theta_F(x) = \theta_F(x) \wedge \theta_F(0)$.

Theorem 3.7 A self map θ_F of a d-algebra X, defined as $\theta_F(x) = F(x)$ for all x ∈ X is a left F-derivation of X, where F is an endomorphism of X.

Proof: Let θ_F be a self map of a d-algebra X, where F is an endomorphism of X defined as follows

$\theta_F(x) = F(x)$ for all x ∈ X (1)

As for x, y ∈ X, $x * y \in X$.

Therefore $\theta_F(x*y) = F(x*y) = F(x) * F(y)$ (2)

$$\begin{aligned}\text{Now } (\theta_F(x) * F(y)) \wedge (\theta_F(y) * F(x)) &= (F(x) * F(y)) \wedge (F(y) * F(x)) && \text{(using (1))}\\ &= (F(y) * F(x)) * ((F(y) * F(x)) * (F(x) * F(y)))\\ &= F(x) * F(y)\\ &= F(x * y) && \text{(Since F is an endomorphism)}\\ &= \theta_F(x * y) && \text{(using (2))}\end{aligned}$$

Thus $(\theta_F(x) * F(y)) \wedge (\theta_F(y) * F(x)) = \theta_F(x * y)$.

which implies that θ_F is a left F-derivation.

Theorem 3.8. Let θ_F be a left F-derivation of a d-algebra X where F is an endomorphism of X. Then

1. $x \leq y$ implies $\theta_F(x)$ and $\theta_F(y)$ belongs to the same branch of X.
2. $y \leq x$ implies $\theta_F(y)$ and $\theta_F(x)$ belongs to the same branch of X.

Proof:
1. Let θ_F be a left F-derivation of a d-algebra X, where F is an endomorphism of X.

Since X is a d-algebra, $x \leq y$ implies $x * y = 0$.

when θ_F is a left F-derivation

$$\begin{aligned} \text{Now } \theta_F(x) &= \theta_F(y * (y * x)) \qquad (y*(y*x) = x)\\ &= (\theta_F(y) * F(y * x)) \wedge (\theta_F(y * x) * F(y))\\ &= (\theta_F(y * x) * F(y)) * ((\theta_F(y * x) * F(y)) * (\theta_F(y) * F(y*x)))\\ &= \theta_F(y) * F(y * x). \end{aligned}$$

which implies $\theta_F(x) = \theta_F(y) * (F(y) * F(x))$ (Since F is an endomorphism) (1)

Since $x \leq y$ implies $x * y = 0$.

Therefore $0 = F(0) = F(x * y) = F(x) * F(y)$.

That is $F(x) * F(y) = 0$ implies $F(x) \leq F(y)$.

As $F(x) * F(y) = 0$, So $F(y) * F(x) \neq 0$,

Otherwise because of property of $d-$algebra $F(x) * F(y) = 0 = F(y) * F(x)$.

$\Rightarrow F(x) = F(y)$, a contradiction.

(1) $\Rightarrow \theta_F(x) = \theta_F(y) * (F(y) * F(x))$.

$$\begin{aligned} \theta_F(x) * \theta_F(y) &= (\theta_F(y) * (F(y) * F(x))) * \theta_F(y)\\ &= (\theta_F(y) * \theta_F(y)) * (F(y) * F(x))\\ &= 0 * (F(y) * F(x))\\ &= 0 \end{aligned}$$

Which implies $\theta_F(x) \leq \theta_F(y)$.

By property, it follows that $\theta_F(x)$ and $\theta_F(y)$ belong to the same branch of X.

2. Interchanging the role of x and y in (1), we have $y \leq x$ implies $\theta_F(y) \leq \theta_F(x)$.

This implies $\theta_F(y)$ and $\theta_F(x)$ belong to the same branch of X.

Definition 3.9 Let X be a d-algebra and θ_F, θ'_F be two self maps of X. We define $\theta_F \circ \theta'_F : X \rightarrow X$ as

$$\theta_F \circ \theta'_F(x) = \theta_F(\theta'_F(x)) \text{ for all } x \in X.$$

Notation: Der(X) denotes the set of all F-derivations (both right F-derivation and left F-derivation) on X.

Definition 3.10. Let θ_F, $\theta'_F \in$ Der(X). Define the binary operation $\wedge$ as

$$(\theta_F \wedge \theta'_F)(x) = \theta_F(x) \wedge \theta'_F(x).$$

Lemma 3.11. Let X be d-algebra. θ_F and θ'_F are left F-derivation of X. Then $\theta_F \wedge \theta'_F$ is also a left F-derivation of X.

Proof: Let X be a d-algebra, θ_F and θ'_F are left F-derivation of X. Then

$$\begin{aligned} (\theta_F \wedge \theta'_F)(x * y) &= \theta_F(x * y) \wedge \theta'_F(x * y)\\ &= [(\theta_F(x) * F(y)) \wedge (\theta_F(y) * F(x))] \wedge [(\theta'_F(x) * F(y)) \wedge (\theta'_F(y) * F(x))]\\ &= (\theta_F(x) * F(y)) \wedge (\theta'_F(x) * F(y))\\ &= \theta_F(x) * F(y)\\ &= (\theta'_F(x) * (\theta'_F(x) * \theta'_F(x))) * F(y)\\ &= (\theta_F(x) \wedge \theta'_F(x)) * F(y)\\ &= (\theta_F \wedge \theta'_F)(x) * F(y)\\ &= ((\theta_F \wedge \theta'_F)(y) * F(x)) * (((\theta_F \wedge \theta'_F)(y) * F(x) * ((\theta_F \wedge \theta'_F)(x) * F(y)))\\ &= ((\theta_F \wedge \theta'_F)(x) * F(y)) \wedge ((\theta_F \wedge \theta'_F)(y) * F(x)) \end{aligned}$$

This shows that $(\theta_F \wedge \theta'_F)$ is a left F-derivation of X.

This completes the proof.

Theorem 3.12 Let X be a d-algebra and θ_F, θ'_F, θ''_F are left F-derivations of X. Then

$$(\theta_F \wedge \theta'_F) \wedge \theta''_F = \theta_F \wedge (\theta'_F \wedge \theta''_F).$$

Proof: Let θ_F, θ'_F and $\theta''_F \in Der(X)$.

Then by definition

$$\begin{aligned}
((\theta_F \wedge \theta'_F) \wedge \theta''_F)(x * y) &= (\theta_F \wedge \theta'_F)(x * y) \wedge \theta''_F(x * y) \\
&= \theta''_F(x * y) * (\theta''_F(x * y) * (\theta_F \wedge \theta'_F)(x * y)) \\
&= (\theta_F \wedge \theta'_F)(x * y) \\
&= \theta_F(x*y) \wedge \theta'_F(x*y) \\
&= [(\theta_F(x) * F(y)) \wedge (\theta_F(y) * F(x))] \wedge \\
&\quad [(\theta'_F(x) * F(y)) \wedge (\theta'_F(y)) \wedge (\theta'_F(y) * F(x))] \\
&= (\theta_F(x) * F(y)) \wedge (\theta'_F(x) * F(y)) \\
&= \theta_F(x) * F(y) \qquad (1)
\end{aligned}$$

Also consider the following

$$\begin{aligned}
(\theta_F \wedge (\theta'_F \wedge \theta''_F))(x * y) &= \theta_F(x * y) \wedge (\theta'_F \wedge \theta''_F)(x * y) \\
&= \theta_F(x * y) \wedge (\theta'_F(x * y) \wedge \theta''_F(x * y)) \\
&= \theta_F(x * y) \wedge \theta'_F(x * y) \\
&= [(\theta_F(x) * F(y)) \wedge (\theta_F(y) * F(x))] \wedge [(\theta'_F(x) * F(y)) \wedge (\theta'_F(y) * F(x)))] \\
&= (\theta_F(x) * F(y)) \wedge (\theta'_F(x) * F(y)) \\
&= \theta_F(x) * F(y) \qquad (2)
\end{aligned}$$

From (1) and (2) it follows that

$((\theta_F \wedge \theta'_F) \wedge \theta''_F)(x * y) = (\theta_F \wedge (\theta'_F \wedge \theta''_F))(x * y)$.

Put $y = 0$, we have $((\theta_F \wedge \theta'_F) \wedge \theta''_F)(x) = (\theta_F \wedge (\theta'_F \wedge \theta''_F))(x)$.

which implies that $(\theta_F \wedge \theta'_F) \wedge \theta''_F = \theta_F \wedge (\theta'_F \wedge \theta''_F)$.

Thus Der(X) form a semi group.

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