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A JOURNEY FROM DE-BOOR'S ALGORITHM TO DE CASTELJAU'S ALGORITHM FOR THE CONSTRUCTION OF B-SPLINE SURFACES & NURBS

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ABSTRACT

In this article we have discussed Bézier and B-spline surfaces with the help of De-Boor and De-Casteljau's algorithms.

Keywords: *Bézier, B-spline curves & surfaces, De-Boor & De-Casteljau's algorithms.*

INTRODUCTION

A historical account of the major developments in the area of curves & surfaces as they entered the area of CAGD (Computer Aided Geometric Design) has been nicely given in an article on "A History of Curves and Surfaces in CAGD" by Gerald Farin [4]. As mentioned in Op.Cit, the term CAGD emerged from a conference on this topic organized by R. Barnhill and R.Riesenfeld in 1974 at the University of Utah. A widely influential proceeding appeared in 1974[4]. The journal CAGD was founded in 1984. Next to lines (simple curves) and planes (surfaces) there are conics and quadric surfaces although they have been around for a long time back. Now, we would like to revisit them in the form of Bézier and B spline curves and surfaces. The most popular objects in many computer aided design and modeling systems are Bézier and B-spline curves and surfaces. We shall discuss Bézier & B-spline surfaces and their constructions with the help of De-Boor and De-Casteljau algorithm.

SURFACES

When comparing mathematics in two and three dimensions, there are many similarities. Very often, the techniques used in the simpler two-dimensional case easily extend to cover three dimensions. Some of the curve representations can easily extend to three dimensions and can therefore represent surfaces.

While creating a curve, we used a single parametric dimension, defined points within this dimension, and then used this to create our curve. For a surface, we need two orthogonal parametric dimensions of points. These form a rectangular mesh. At any point in parametric space, we use two blending functions, one in each parametric direction. For every knot defined, we calculate the Cartesian product of the two blending functions and this is the weight given to that knot. The sum of all the weights will still be one as it was for a curve.

The most commonly used methods of representing curved surfaces in computing are by Bézier Surfaces & B-Spline Surfaces. In many applications of computer graphics require also the possibility to model and display free-form surfaces. The principle of free-form curves will be studied first due to easier understanding of the mathematics used. The method employed can then be extended to surfaces.

The general curves can be defined either by a formula or by specifying some control points, which defined the shape of the curve. It is usually difficult to find an appropriate analytical formula for a specific curve, therefore control points are used most of the times. The curve which is defined by control points are called splines.

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Some of the common variants are cubic spline interpolation, hermite interpolation. While the interpolating curves are those where the curve passes through the control points, approximating curves are those where control points lie close to the curve here Bézier developed an approximating curve in 1960 at Renault to model and describe car bodies. In these curves Bernstein polynomials BEZ (k, n) are used as weighting function for the control points [2]. Every point on the curve is the weighted average of all control points:

$$P(u) = \sum_{k=0}^{n} p_k BEZ_{k,n}(u) \qquad 0 \le u \le 1$$

With $BEZ_{k,n}(u) = \binom{n}{k} u^k (1-u)^{n-k}$

BÉZIER & B-SPLINE SURFACES

If the Cartesian product is constructed over two arrays of curves, the resulting set of points is a free-form surface. This is a natural way of constructing free form surfaces. Depending on which curves are involved, different types of surfaces are obtained, like Bézier surfaces from Bézier curves, B-Spline surfaces from B-Spline curves etc.

$$P(u,v) = \sum_{j=0}^{m} \sum_{k=0}^{n} P_{j,k}\, B_{j,m}(v) B_{k,n}(u)$$

Each pair of parameters (u, v) corresponds to a point on the constructed surface. The curves along the edges of the surface are of the same type, e.g. in the Bézier surface they are Bézier -curves. The other properties of Bézier curves also hold for the corresponding surfaces.

Similar to the Bézier surfaces, B-spline surfaces can be obtained when using B-spline weighting functions in the formula, or NURBS-Surfaces with NURBS-Curves.

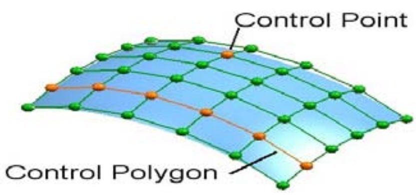

Fig: 1

As with the Bézier curve, a Bézier surface is defined by a set of control points. Similar to interpolation in many respects, a key difference is that the surface does not, in general, pass through the central control points; rather, it is "stretched" toward them as though each was an attractive force. They are visually intuitive, and for many applications mathematically convenient.

EQUATION OF BÉZIER SURFACES

A given Bézier surface of order (n, m) is defined by a set of (n+1) (m+1) control points K_{ij}. It maps the unit square into a smooth continuous surface embedded within a space of the same dimensionality as $\{K_{ij}\}$. For example, if K are all points in a four dimensional space, then the surface will be within a four dimensional space[6]. A two dimensional Bézier surface can be defined as a parametric surface where the position of a point P as a function of the parametric coordinates u, v is given by:

$$P(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} B_i^n(u) B_j^m(v) K_{i,j}$$

evaluated over the unit square,

Where $B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i}$ is a Bernstein Polynomial,

and $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ is the binomial coefficient.

Sample Bézier surface is shown in the following Fig 2.

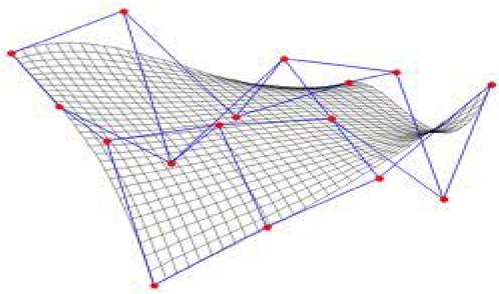

Fig: 2

Generally, the most common use of Bézier surfaces is as nets of bicubic patches (where m=n=3). The geometry of a single bicubic patch is thus completely defined by a set of 16 control points. These are typically linked up to form a B-spline surface in a similar way as Bézier curves are linked up to form a B-Spline curve.

Simpler Bézier surface are formed from biquadratic patches (m=n=2) or Bézier triangles.

There are two types of Bézier Surfaces:

1. Tensor Product Bézier Surfaces:
   Given degrees $m, n > 0$ and points $b_{ij} \epsilon R^3$, $i = 0, \ldots \ldots . . m$, $j = 0, \ldots \ldots . . n$ the parametric surface $q: [0,1]^2 \rightarrow R^3$ defined by

$$q(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_i^m(u) B_j^n(v) b_{i,j} \qquad u, v \in [0,1]$$

is called a tensor product Bézier Surface, where $B_i^m(u)$ and $B_j^n(v)$ are Bernstein polynomial of degree m, n resp. The points $b_{i,j}$ are called control points or Bézier net of the surface q. analogously as for Bézier curves, tensor product Bézier Surfaces having an arbitrary rectangle $[0, r] * [s, t]$ as domain may be introduced.

Many properties of Bézier curves carry over to tensor product Bézier surfaces. Any isoparametric curve $v = v_0 = const$ of q is a Bézier curve.

$P(u) = q(u_0, v_0) = \sum_{i=0}^{m} \left(\sum_{j=0}^{n} B_j^n(v_0) b_{ij} \right) B_i^m(u)$ of degree m.

The Bézier points of P can be obtained by applying m+1 de casteljau algorithms using Bézier points $b_{i,j}$ $j = 0, \ldots \ldots . . n$. A point $q(u_0, v_0)$ on the surface is then obtained by performing one more de casteljau algorithm furthermore; the surface lies in the convex hull of its Bézier points. The boundary curves of the surface are Bézier curves whose Bézier points are the corresponding boundary points of the Bézier net.

2. Triangular Bézier Surfaces:
   Triangular Bézier Surfaces have a triangle as domain, therefore they are also called Bézier triangles. Thus, let there be given an arbitrary, non-degenerate triangle with vertices $r, s, t \in R^2$
   Then any point $P \in R^2$ has unique barycentric coordinates $u, v, w \in R$ w.r.t r, s, t i.e

   $P = ur + vs + wt$ with $u + v + w = 1$ for any $n > 0$, the bivariate Bernstein polynomial B_{ijk}^n are defined by
   $B_{ijk}^n(u, v, w) = \frac{n!}{i!j!k!} u^i v^j w^k$

For all $i, j, k \in N_0$
$i + j + k = n$
$u + v + w = 1$

Now given points $b_{ijk} \epsilon R^3$ a triangular Bézier surface of degree n is defined by

$$q(u, v, w) = \sum_{i+j+k=n} B_{ijk}^n(u, v, w) b_{ijk}$$

Where $u + v + w = 1$ and $u, v, w \geq 0$; these conditions characterize the barycentric coordinates of any point P of the closed triangle r, s, t.

In analogy with Bézier curves, it is possible to build up smooth, complex surfaces from a no. of rectangular or triangular Bézier patches [5]. This leads to spline surfaces.

BÉZIER SURFACES IN 3-DIMENSIONS

The Bézier surface is formed as the Cartesian product of the blending functions of two orthogonal Bézier curves.

$$B(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} P_{i,j} \frac{m!}{i!\,(m-i)!} u^i (1-u)^{m-i} \frac{n!}{j!\,(n-j)!} v^j (1-v)^{n-j}$$

Where $0 \le u \le 1$, $0 \le v \le 1$ and P_{ij} is the i^{th} , j^{th} control points. There are $N_{i+1}\ and\ N_{j+1}$ control points in the $i\ and\ j$ directions resp.

The corresponding properties of the Bézier curve apply to the Bézier surface.

The surface does not in general pass through the control points except for the corners of the control pint grid.The surface is contained within the convex hull of the control points.Along the edges of the grid patch the Bézier surface matches that of a Bézier curve through the control points along that edge.

Closed surfaces can be formed by setting the last control point equal to the first. If the tangents also match between the first two and last two control points then the closed surface will have first order continuity.

B-splines can be evaluated in a numerically stable way by the de Boor algorithm. Simplified potentially faster variants of the de Boor algorithm have been created but they suffer from comparatively lower stability.

In the computer science subfields of computer aided design and computer graphics, the term B-spline frequently refers to a spline curve parameterized by spline functions that are expressed as linear combination of B-splines. A B-spline is simply a generalization of a Bézier curve.

B-SPLINE CURVES

The main disadvantage of the Bézier curves is the global influence of the control points on the whole curve. The B-splines are, just as the Bézier -splines, approximating curves, but the Bernstein-polynomials are replaced by the B-spline-polynomials $B_{k,d}$. These limit the number of control points, which influence any curve point, to d. important property of the B-spline weighting function is this, that the every curve point is the weightage average of the control points [6]. That is to say:

$$\sum_{k=0}^{n} B_{k,d}\,(u) = 1$$

B-SPLINE SURFACE

Extending the idea of B-spline curve, we obtain a Cartesian product B-spline surface.

$$Q(u,w) = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} B_{i,j}\, N_{i,k}(u) M_{j,l}(w)$$

Here, $u_{min} \le u \le u_{max}$, $w_{min} \le w \le w_{max}$ $2 \le k \le n+1$, $2 \le k \le m+1$

where $k, l =$ degrees of polynomial in respective parameters[3]. There can be 2 to the number of control points. If $k\ and\ l$ set to 1, then we can only plot with the control points.

B_{ij} is the input set of $(n+1)*(m+1)$ control points (POLYGON NET VERTICES).

Parameters u, w now depend on how we choose the other parameters (no longer locked to 0-1) $N_{i,k}\ and\ M_{j,l}$ are blending functions.

These are the polynomials of degree $k-1$, $l-n$ in each parameter and at each interval
$x_i \le u \le x_{i+1}$, $y_j \le w \le y_{j+1}$

B-Spline Blending Function: Blending function are defined by Cox-de Boor recursion formula are:

$$N_{i,1}(u)=\begin{Bmatrix}1 & if \quad x_i \le u \le x_{i+1} \\ 0 & otherwise\end{Bmatrix}$$

$$M_{j,1}(w)=\begin{Bmatrix}1 & if \quad y_i \le w \le y_{i+1} \\ 0 & otherwise\end{Bmatrix}$$

And

$$N_{i,k}(u)=\frac{(u-x_i)N_{i,k-1}(u)}{x_{i+k-1}-x_i}+\frac{(x_{i+k}-u)N_{i+1,k-1}(u)}{x_{i+k}-x_{i+1}}$$

$$M_{j,l}(w)=\frac{(w-y_j)M_{j,l-1}(w)}{y_{j+l-1}-y_j}+\frac{(y_{j+l}-w)M_{j+1,l-1}(w)}{y_{j+l}-y_{j+1}}$$

where x_i's and y_j's are the elements of knot vector $[x_i]$ and $[y_j]$ respectively, the restriction on x_i ,y_j is that the values of x_i and y_j's must be monotonically increasing . The convention $\frac{0}{0}=0$ by Cox de-Boor is adopted.

Thus Similar to Bézier surfaces, B-spline surfaces using tensor product and triangular patch are also defined. The nature of B-Spline surface will depends on type of knot vectors open, uniform and nonuniform.

Following figure shows Bicubic B-spline surface consisting of 3 x 3 patches together with its control net.

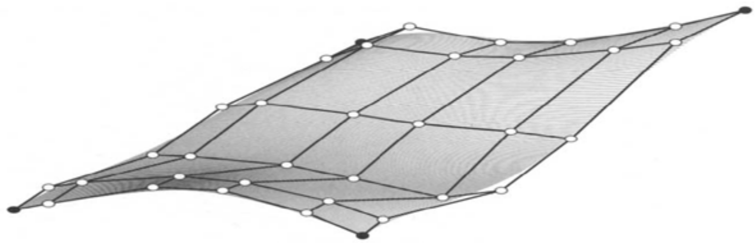

Fig: 3

PROPERTIES OF B-SPLINE SURFACES

1. The highest order in each parametric direction is limited to the number of defining polygon vertices in that direction.
2. The continuity of the surface in each parametric direction is k-2, l-2 respectively.
3. The surface is invariant to an affine transformation.
4. The variation diminishing property of B-spline surface is not well known.
5. The influence of any polygon net vertex is limited to $\pm k/2, \pm l/2$ spans in the respective parametric direction.
6. If the number of polygon net vertices is equal to the order of basis in that direction and if there are no interior knot values, then the B-spline surface reduces to a Bézier surface.

B-SPLINE SURFACE SUBDIVISION

1. A B-spline surface is subdivided by separately subdividing polygon grid lines in one or both parametric direction.
2. The flexibility of B-spline curves and surfaces is increased by raising the order of the basis function and hence the defining polygon/grid segments.
3. An alternative to degree raising is increasing the knot values in the knot vectors used.
4. The basic idea of degree raising or knot insertion is to achieve the flexibility without changing the shape of the curve or surface.
5. The nature of the knot vector is preserved (uniform, open) even after insertion of new knot values.

DE CASTELJAU & DE BOOR ALGORITHMS

In numerical analysis, de casteljau algorithm evaluates polynomial in Bernstein form or Bézier curves. It can be also used to split a single Bézier curve in to two Bézier curves at an arbitrary parameter value. Although the algorithm is slower, it is more numerically stable [1]. The Bézier curve B (of degree n) can be written in Bernstein form as follows

$$B(t)=\sum_{i=0}^{n}\beta_i\, b_{i,n}(t)$$

where b is a Bernstein basis polynomial

$$b_{i,n}(t) = \binom{n}{i}(1-t)^{n-i}t^{i}$$

The Bézier curve B can be split at point t_0 into two curves with respective control points:

$$\beta_0^{(0)}, \beta_0^{(1)}, \dots\dots\dots. \beta_0^{(n)}$$
$$\beta_0^{(n)}, \beta_1^{(n-1)}, \dots\dots\dots., \beta_n^{(0)}$$

next sentence when we split Bézier curve into three separate equations we get

$$B_1(t) = \sum_{i=0}^{n} x_i\, b_{i,n}(t) \ , \quad t \in [0,1]$$
$$B_2(t) = \sum_{i=0}^{n} y_i\, b_{i,n}(t) \ , \quad t \in [0,1]$$
$$B_3(t) = \sum_{i=0}^{n} z_i\, b_{i,n}(t) \ , \quad t \in [0,1]$$

which evaluated individually using De Casteljau's algorithm. Similarly, for Bézier Surfaces we can use de Casteljau's algorithm, based on the concept of isoparametric curves in the following manner:

$$p(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} B_{m,i}(u)B_{n,j}(v)p_{ij}$$

Can be written as the following

$$p(u,v) = \sum_{i=0}^{m} B_{m,i}(u)\left(\sum_{j=0}^{n} B_{n,j}(v)p_{ij}\right)$$

For $i = 0,1, \dots\dots, m$ define $q_i(v)$ as follows:

$$q_i(v) = \left(\sum_{j=0}^{n} B_{n,j}(v)p_{ij}\right)$$

For a fixed v, we have m+1 point $q_0(v), q_1(v), \dots\dots\dots.. q_m(v)$. Each $q_i(v)$ is a point on the Bézier curve defined by control points p_{i0} , p_{i1} , ... p_{in}. Plugging these back into the surface equation yields

$$p(u,v) = \sum_{i=0}^{m} B_{m,i}(u)q_i(v)$$

This means $p(u,v)$ is a point of the Bézier curve defined by m+1 control points $q_0(v), q_1(v), \dots\dots\dots.. q_m(v)$.

CONCLUSION

Thus, we have the following conclusion,

To find point $\boldsymbol{p(u,v)}$ on a Bézier surface, we can find m+1 control points $\boldsymbol{q_0(v)}, \boldsymbol{q_1(v)}, \dots\dots\dots.. \boldsymbol{q_m(v)}$ and then from these points find $\boldsymbol{p(u,v)}$.

In numerical analysis De Boor's algorithm is a generalization of the de Casteljau's algorithm for Bézier curves. It can be used on all types of curves and when it is applied to a Bézier curve, it reduces to de Casteljau's algorithm. Therefore, once we know de Casteljau's algorithm for Bézier surfaces, de Boor's algorithm for B-spline surface and its modification for NURBS surfaces, we will extend it to generalize de casteljau algorithm for B-spline surfaces after due modification this will be the focus of our research as still to this day, De Casteljau's algorithm for B-spline surfaces is not available.

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