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GENERALIZED NEAR APPROXIMATIONS IN TOPOLOGICAL SPACES

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ABSTRACT

In this paper, we apply topological concepts to introduce definitions for generalized near approximations, generalized near boundary regions and generalized near accuracy. Moreover, proved results and implications are provided. Generalized near approximations are mathematical tools to modify the generalized approximations. Since the area of uncertainty is the boundary region, we introduced the generalized near boundary regions as different areas of uncertainty. The suggested methods of generalized near approximations open way for constructing new types of lower and upper approximations.

Keywords: *Topological space; Generalized near approximations.*

1. INTRODUCTION:

One of the most powerful notions in system analysis is the concept of topological structures [6] and their generalizations. Rough set theory, introduced by Pawlak in 1982 [13], is a mathematical tool that supports also the uncertainty reasoning but qualitatively. In this paper, we shall integrate some ideas in terms of concepts in topology. Topology is a branch of mathematics, whose concepts exist not only in almost all branches of mathematics, but also in many real life applications. We believe that topological structure will be an important base for modification of knowledge extraction and processing.

2. PRELIMINARIES:

A topological space [6] is a pair (X,τ) consisting of a set X and family τ of subsets of X satisfying the following conditions:

(T1) $\phi \in \tau$ and $X \in \tau$.
(T2) τ is closed under arbitrary union.
(T3) τ is closed under finite intersection.

Throughout this paper (X,τ) denotes a topological space, the elements of X are called points of the space, the subsets of X belonging to τ are called open sets in the space, the complement of the subsets of X belonging to τ are called closed sets in the space, and the family of all open sets of (X,τ) is denoted by τ and the family of all closed sets of (X,τ) is denoted by $C(X)$.

For a subset A of a space (X,τ), $Cl(A)$ denote the closure of A and is given by $Cl(A) = \cap\{F \subseteq X : A \subseteq F \text{ and } F \in C(X)\}$. Evidently, $Cl(A)$ is the smallest closed subset of X which contains A. Note that A is closed iff $A = Cl(A)$. $Int(A)$ denote the interior of A and is given by $Int(A) = \cup\{G \subseteq X : G \subseteq A \text{ and } G \in \tau\}$ Evidently, $Int(A)$ is the largest open subset of X which contained in A. Note that A is open iff $A = Int(A)$. The boundary of a subset $A \subseteq X$ is denoted by $BN(A)$ and is given by $BN(A) = Cl(A) - Int(A)$.

We shall recall some concepts about some near open sets which are essential for our present study.

Definition: 2.1 A subset A of a space (X,τ) is called:

(i) Semi-open [8] (briefly $s-$open) if $A \subseteq Cl(Int(A))$.
(ii) Pre-open [10] (briefly $p-$open) if $A \subseteq Int(Cl(A))$.

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(iii) α-open [11] if $A \subseteq Int(Cl(Int(A)))$.

(iv) β-open [1] (= semi-pre-open [2]) if $A \subseteq Cl(Int(Cl(A)))$.

The complement of a $s-$open (resp. $p-$open, α-open and β-open) set is called $s-$closed (resp. $p-$closed , α-closed and β-closed) set .
The family of all $s-$open (resp. $p-$open, α-open and β-open) sets of (X,τ) is denoted by $SO(X)$ (resp. $PO(X)$, $\alpha O(X)$ and $\beta O(X)$).
The family of all $s-$closed (resp. $p-$closed, α- closed and β-closed) sets of (X,τ) is denoted by $SC(X)$ (resp. $PC(X)$, $\alpha C(X)$ and $\beta C(X)$).
The semi-closure (resp. α-closure, pre-closure, semi-pre-closure) of a subset A of (X,τ), denoted by ${}_sCl(A)$ (resp. ${}_\alpha Cl(A)$, ${}_pCl(A)$, ${}_{sp}Cl(A)$) and defined to be the intersection of all semi-closed (resp. α-closed, p-closed, sp-closed) sets containing A . The semi-interior (resp. α-interior, pre-interior, semi-pre-interior) of a subset A of (X,τ), denoted by ${}_sInt(A)$ (resp. ${}_\alpha Int(A)$, ${}_pInt(A)$, ${}_{sp}Int(A)$) and defined to be the union of all semi-open (resp. α-open, p-open, sp-open) sets contained in A .

Definition: 2.2 A subset A of a space (X,τ) is said to be:

(i) generalized closed[7] (briefly, g-closed) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ),
(ii) generalized semi-closed[3] (briefly, gs-closed) if ${}_sCl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ),
(iii) generalized semi-preclosed[4] (briefly, gsp-closed) if ${}_{sp}Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ),
(iv) α-generalized closed[9] (briefly, αg -closed) if ${}_\alpha Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ),
(v) generalized preclosed[12] (briefly, gp-closed) if ${}_pCl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ).

The complement of a g-closed (resp. gs-closed, gsp-closed, gp-closed and αg -closed) set is called g-open (resp. gs-open, gsp-open, gp-open and αg -open). The family of all g-open (resp. gs-open, gp-open, αg -open and gsp-open) sets of (X,τ) is denoted by $gO(X)$ (resp. $gSO(X)$, $gPO(X)$, $\alpha gO(X)$ and $gspO(X)$). The family of all g-closed (resp. gs-closed, gp-closed, αg -closed and gsp-closed) sets of (X,τ) is denoted by $gC(X)$ (resp. $gSC(X)$, $gPC(X)$, $\alpha gC(X)$, and $gspC(X)$).
The generalized interior (briefly g-interior) of A is denoted by ${}_gInt(A)$ and is defined by ${}_gInt(A) = \cup\{G \subseteq X : G \subseteq A, G$ is a g-open$\}$, and the generalized near interior (briefly gj-interior) of A is denoted by ${}_{gj}Int(A)$ for all $j \in \{s, p, \alpha, \beta\}$ and is defined by ${}_{gj}Int(A) = \cup\{G \subseteq X : G \subseteq A, G$ is a gj-open$\}$.
The generalized closure (briefly g-closure) of A is denoted by ${}_gCl(A)$ and is defined by ${}_gCl(A) = \cap\{F \subseteq X : A \subseteq F, F$ is a g-closed set$\}$, and the generalized near closure (briefly gj-closure) of A is denoted by ${}_{gj}Cl(A)$ for all $j \in \{s, p, \alpha, \beta\}$ and is defined by $gjCl(A) = \cap\{F \subseteq X : A \subseteq F, F$ is a gj closed set$\}$.
The generalized boundary (briefly g-boundary) region of A is denoted by ${}_gBN(A)$ and is defined by ${}_gBN(A) = {}_gCl(A) - {}_gInt(A)$ and the generalized near boundary (briefly gj-boundary) region of A is denoted by ${}_{gj}BN(A)$ for all $j \in \{s, p, \alpha, \beta\}$ and is defined by ${}_{gj}BN(A) = {}_{gj}Cl(A) - {}_{gj}Int(A)$.
The generalized exterior (briefly g-exterior) of A is denoted by ${}_gExt(A)$ and is defined by ${}_gExt(A) = X - {}_gCl(A)$ and the generalized near exterior (briefly gj-exterior) of A is denoted by ${}_{gj}Ext(A)$ for all $j \in \{s, p, \alpha, \beta\}$ and is defined by

$$_{gj}Ext(A) = X - {}_{gj}Cl(A)$$

3. GENERALIZATION OF PAWLAK APPROXIMATION SPACE:

Pawlak [14] noted that the approximation space $K = (X, R)$ with equivalence relation R defines a uniquely topological space (X, τ_k) where τ_k is the family of all clopen sets in (X, τ_k) and $X \setminus R$ is a base of τ_k. Moreover the lower (resp. upper) approximation of any subset $A \subseteq X$ is exactly the interior (resp. closure) of the subset A.

4.1. GENERALIZED NEAR LOWER AND GENERALIZED NEAR UPPER APPROXIMATIONS:

Definition: 4.1.1 [5]. Let $K = (X, R)$ be an approximation space with general relation R and τ_k is the topology associated to K. Then the triple (X, R, τ_k) is called a topologized approximation space.

Definition: 4.1.2 [5]. Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the lower approximation (resp. upper approximation) of A is defined by

$$\underline{R}A = Int(A) \text{ (resp. } \overline{R}A = Cl(A)\text{)}.$$

Definition: 4.1.3. Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized lower approximation (briefly g -lower approximation) of A is denoted by $\underline{R}_g A$ and is defined by $\underline{R}_g A = {}_g Int(A)$

Definition: 4.1.4. Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized near lower approximation (briefly gj -lower approximation) of A is denoted by $\underline{R}_{gj}A$ and is defined by

$$\underline{R}_{gj}A = {}_{gj}Int(A), \text{ where } j \in \{s, p, \alpha, \beta\}.$$

Definition: 4.1.5. Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized upper approximation (briefly g -upper approximation) of A is denoted by $\overline{R}_g A$ and is defined by $\overline{R}_g A = {}_g Cl(A)$

Definition: 4.1.6. Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized near upper approximation (briefly gj -upper approximation) of A is denoted by $\overline{R}_{gj}A$ and is defined by

$$\overline{R}_{gj}A = {}_{gj}Cl(A), \text{ where } j \in \{s, p, \alpha, \beta\}.$$

Theorem: 4.1.1 Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then $\underline{R}A \subseteq \underline{R}_g A \subseteq A \subseteq \overline{R}_g A \subseteq \overline{R}A$.

Proof: $\underline{R}A = Int(A) = \cup\{G \in \tau : G \subseteq A\} \subseteq \cup\{G \in gO(X) : G \subseteq A\} = {}_g Int(A) = \underline{R}_g A \subseteq A$, since every open set is a generalize open.

$$\to (1)$$

$$\overline{R}A = Cl(A) = \cap\{F \in C(X) : A \subseteq F\} \supseteq \cap\{F \in gC(X) : A \subseteq F\} = {}_g Cl(A) = \overline{R}_g A \supseteq A$$

$$\to (2)$$

From (1) and (2) we get $\underline{R}A \subseteq \underline{R}_g A \subseteq A \subseteq \overline{R}_g A \subseteq \overline{R}A$

Theorem: 4.1.2. Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then $\underline{R}A \subseteq \underline{R}_{gj}A \subseteq A \subseteq \overline{R}_{gj}A \subseteq \overline{R}A$ for all $j \in \{s, p, \alpha, \beta\}$.

Proof: We shall prove the proposition in the case of $j=\alpha$ and the other cases can be proved similarly. Now, $\underline{R}A = Int(A) = \cup\{G \in \tau : G \subseteq A\} \subseteq \cup\{G \in g\alpha O(X) : G \subseteq A\} = {}_{g\alpha}Int(A) = \underline{R}_{g\alpha}A \subseteq A$, since every open set is a generalize α -open.

$\rightarrow (1)$

$$\overline{R}A = Cl(A) = \cap\{F \in C(X) : A \subseteq F\} \supseteq \cap\{F \in g\alpha C(X) : A \subseteq F\} = {}_{g\alpha}Cl(A) = \overline{R}_{g\alpha}A \supseteq A$$

$\rightarrow (2)$

From (1) and (2) we get $\underline{R}A \subseteq \underline{R}_{g\alpha}A \subseteq A \subseteq \overline{R}_{g\alpha}A \subseteq \overline{R}A$.

Theorem: 4.1.3. Let $K=(X,R,\tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the implications between lower approximation and gj -lower approximations of A are given by the following diagram for all $j \in \{s, p, \alpha, \beta\}$.

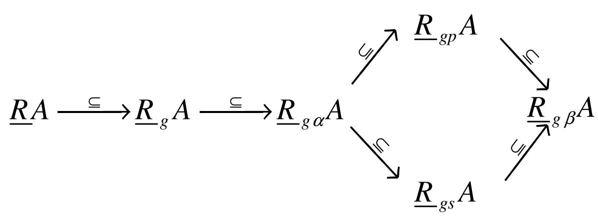


Figure 4.1.1.
Relation between lower approximation and gj -lower approximations

Proof:

(i) $\underline{R}A = Int(A) = \cup\{G \in \tau : G \subseteq A\} \subseteq \cup\{G \in gO(X) : G \subseteq A\}$

$\underline{R}A = Int(A) \subseteq {}_{g}Int(A) = \underline{R}_{g}A$

(ii) $\underline{R}_{g}A = {}_{g}Int(A) = \cup\{G \in gO(X) : G \subseteq A\} \subseteq \cup\{G \in g\alpha O(X) : G \subseteq A\}$

$\underline{R}_{g}A = {}_{g}Int(A) \subseteq {}_{g\alpha}Int(A) = \underline{R}_{g\alpha}A$

(iii) $\underline{R}_{g\alpha}A = {}_{g\alpha}Int(A) = \cup\{G \in g\alpha O(X) : G \subseteq A\} \subseteq \cup\{G \in gsO(X) : G \subseteq A\}$

$\underline{R}_{g\alpha}A = {}_{g\alpha}Int(A) \subseteq {}_{gs}Int(A) = \underline{R}_{gs}A$

(iv) $\underline{R}_{gs}A = {}_{gs}Int(A) = \cup\{G \in gsO(X) : G \subseteq A\} \subseteq \cup\{G \in g\beta O(X) : G \subseteq A\}$

$\underline{R}_{gs}A = {}_{gs}Int(A) \subseteq {}_{g\beta}Int(A) = \underline{R}_{g\beta}A$

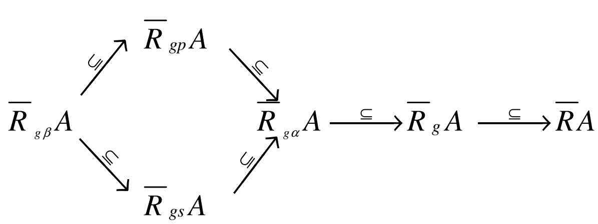
(v) $\underline{R}_{g\alpha}A = {}_{g\alpha}Int(A) = \cup\{G \in g\alpha O(X) : G \subseteq A\} \subseteq \cup\{G \in gpO(X) : G \subseteq A\}$

$\underline{R}_{g\alpha}A = {}_{g\alpha}Int(A) \subseteq {}_{gp}Int(A) = \underline{R}_{gp}A$

(vi) $\underline{R}_{gp}A = {}_{gp}Int(A) = \cup\{G \in gpO(X) : G \subseteq A\} \subseteq \cup\{G \in g\beta O(X) : G \subseteq A\}$

$\underline{R}_{gp}A = {}_{gp}Int(A) \subseteq {}_{g\beta}Int(A) = \underline{R}_{g\beta}A$

Theorem: 4.1.4. Let $K=(X,R,\tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the implications between upper approximation and gj -upper approximations of A are given by the following diagram for all $j \in \{s, p, \alpha, \beta\}$.



Relation between upper approximation and gj -upper approximations

Figure 4.1.2.

Proof:

(i) $\overline{R}A = Cl(A) = \cap\{F \in C(X) : A \subseteq F\} \supseteq \cap\{F \in gC(X) : A \subseteq F\}$ $\overline{R}A \supseteq {}_gCl(A) = \overline{R}_g A$

(ii) $\overline{R}_g A = {}_gCl(A) = \cap\{F \in gC(X) : A \subseteq F\} \supseteq \cap\{F \in g\alpha C(X) : A \subseteq F\}$

$\overline{R}_g A \supseteq {}_{g\alpha}Cl(A) = \overline{R}_{g\alpha} A$

(iii) $\overline{R}_{g\alpha} A = {}_{g\alpha}Cl(A) = \cap\{F \in g\alpha C(X) : A \subseteq F\} \supseteq \cap\{F \in gsC(X) : A \subseteq F\}$

$\overline{R}_{g\alpha} A \supseteq {}_{gs}Cl(A) = \overline{R}_{gs} A$

(iv) $\overline{R}_{gs} A = {}_{gs}Cl(A) = \cap\{F \in gsC(X) : A \subseteq F\} \supseteq \cap\{F \in g\beta C(X) : A \subseteq F\}$

$\overline{R}_{gs} A \supseteq {}_{g\beta}Cl(A) = \overline{R}_{g\beta} A$

(v) $\overline{R}_{g\alpha} A = {}_{g\alpha}Cl(A) = \cap\{F \in g\alpha C(X) : A \subseteq F\} \supseteq \cap\{F \in gpC(X) : A \subseteq F\}$

$\overline{R}_{g\alpha} A \supseteq {}_{gp}Cl(A) = \overline{R}_{gp} A$

(vi) $\overline{R}_{gp} A = {}_{gp}Cl(A) = \cap\{F \in gpC(X) : A \subseteq F\} \supseteq \cap\{F \in g\beta C(X) : A \subseteq F\}$

$\overline{R}_{gp} A \supseteq {}_{g\beta}Cl(A) = \overline{R}_{g\beta} A$

4.2. GENERALIZED NEAR BOUNDARY REGIONS:

In this section we obtain some rules to find generalized boundary regions and generalized accuracy in different ways in generalized approximation spaces with general relations.

Definition: 4.2.1. Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized near boundary (briefly gj -boundary) region of A is denoted by $BN_{R_{gj}}(A)$ and is defined by

$$BN_{R_{gj}}(A) = \overline{R}_{gj}(A) - \underline{R}_{gj}(A), \text{ where } j \in \{s, p, \alpha, \beta\}.$$

Definition: 4.2.2. Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized near positive (briefly gj -positive) region of A is denoted by $POS_{Rgj}(A)$ and is defined by $POS_{Rgj}(A) = \underline{R}_{gj}A$, where $j \in \{s, p, \alpha, \beta\}$.

Definition: 4.2.3. Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized near negative (briefly gj -negative) region of A is denoted by $NEG_{Rgj}(A)$ and is defined by

$$NEG_{Rgj}(A) = X - \overline{R}_{gj}A, \text{ where } j \in \{s, p, \alpha, \beta\}.$$

Theorem: 4.2.1. $k = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then $BN_{Rgj}(A) \subseteq BN_R(A)$, for all $j \in \{s, p, \alpha, \beta\}$.

Proof: Obvious.

Theorem: 4.2.2. Let $k = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the implications between boundary and gj -boundary of A are given by the following diagram for all $j \in \{s, p, \alpha, \beta\}$.

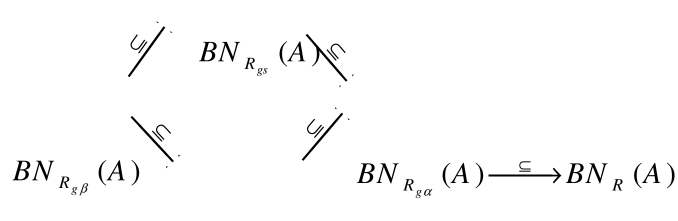


Figure 4.2.1.
Relations between boundary and gj -boundary of A

Proof: Obvious.

4.3. GENERALIZED NEAR ROUGH AND GENERALIZED NEAR EXACT SETS:

In this section, we used topological concepts to introduce definitions to generalized near rough and generalized near exact sets.

Definition: 4.3.1. Let (X, τ) be a topological space and $A \subseteq X$. Then

i) A is totally gj -definable (gj -exact) set if $_{gj}Int(A) = A = {}_{gj}Cl(A)$,

ii) A is internally gj -definable set if $A = {}_{gj}Int(A)$,

iii) A is externally gj -definable set if $A = {}_{gj}Cl(A)$,

iv) A is gj - indefinable set if $A \neq {}_{gj}Int(A), A \neq {}_{gj}Cl(A)$,

where $j \in \{s, p, \alpha, \beta\}$.

Theorem: 4.3.1. Let (X, τ) be a topological space and $A \subseteq X$. If A is an exact set then it is gj - exact for all $j \in \{s, p, \alpha, \beta\}$.

Proof: We shall prove this Theorem in the case of $j = \beta$, and the other cases similarly.

Let A be exact set, then $Cl(A) = A = Int(A)$. Now,

$$Cl(A) = \cap\{F \subseteq X : A \subseteq F, F \in C(X)\} \supseteq \cap\{F \subseteq X : A \subseteq F, F \in {}_{g\beta}C(X)\}$$

Also,

$$Int(A) = \cup\{G \subseteq X, G \in \tau\} \subseteq \cup\{G \subseteq X : G \subseteq A, G \in {}_{g\beta}O(X)\} = {}_{g\beta}Int(A).$$

Therefore, $Cl(A) \supseteq {}_{g\beta}Cl(A) \supseteq A \supseteq {}_{g\beta}Int(A) \supseteq Int(A)$. Since A is exact we get $_{g\beta}Cl(A) = A = {}_{g\beta}Int(A)$; Hence A is $g\beta$-exact.

5. CONCLUSIONS:

In this paper, we used topological concepts to introduce definitions to generalized near boundary regions and generalized near rough and generalized near exact sets.

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