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A GENERALIZATION OF BANACH CONTRACTION PRINCIPLE IN FUZZY NORMED SPACES

Savita Rathee*
Department of Mathematics, Maharshi Dayanand University, Rohtak, India

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ABSTRACT

Chugh and Rathi [3] introduced the concept of Fuzzy normed space. In this paper, we prove the Banach contraction principle in Fuzzy normed space. Moreover, at the end we also give a generalization of Banach contraction principle in fuzzy normed space.

Keyword: *Fixed point, Fuzzy normed space.*

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INTRODUCTION

In 1942, K. Menger [9] introduced the notion of PM-space by generalizing the concept of metric space to the situation when we do not know the distance between the points. It is suitable to look upon the distance concept as a statistical or probabilistic rather than deterministic one, because the advantage of a probabilistic approach is that it permits from the initial formulation a greater flexibility rather than that offered by a deterministic approach.

The idea thus appears that, instead of a single positive number, we should associate a distribution function with the point pairs. Thus, for any p, q elements in the space, we have a distribution function F (p, q; x) and interpret F (p, q; x) as the probability that distance between p and q is less than x.

In 1963, Serstnev [11] generalized the concept of ordinary normed space to random normed space. In fact, a random normed space is a menger space if we set $G_{x,y} = F_{x-y}$. Fixed point theorems for contraction mappings in RN-spaces were first investigated by Boscan [1]. Thus many fixed point theorems for metric space have an immediate analogue in Random normed spaces. For topological preliminaries in RN – spaces, Schweizer and Sklar [10] and Serstnev [11] are excellent readings.

In 1965, Zadeh [14] introduced the concept of fuzzy set. Since then, a lot of work has been developed by many authors regarding the theory of fuzzy sets and applications. Especially, Erceg [4], Kaleva and Seikala [7], Kramosil and Michalek [8] have introduced the concept of fuzzy metric space in different ways. Grabiec [6] followed Kramosil and Michalek [8] and obtained the fuzzy version of Banach contraction principle. Grabiec [5] results were further generalized by Subrahmanyam [12] for a pair of commuting mappings. Moreover, George and Veeramani [5] modified the concept of fuzzy metric space, introduced by Kramosil and Michalek [8] and introduced the concept of Hausdorff topology on fuzzy metric spaces and showed that every metric induces a fuzzy metric

In this paper, we introduce the concept of fuzzy normed space and prove the Banach contraction principle in this newly defined space. Moreover, at the end we also give a generalization of Banach contraction principle in fuzzy normed space.
A fuzzy normed space is a fuzzy metric space if G(x, y, t) = M(x–y, t). We introduce the concept of fuzzy Normed space as follows:

Definition 1.1. A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 1.2. A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if * satisfies the following conditions:

1. * is associative and commutative,
2. * is continuous,
3. $a * 1 = a$ for all $a \in [0, 1]$,
4. $a * b \le c * d$ whenever $a \le c$ and $b \le d$, $(a, b, c, d \in [0, 1])$.

Corresponding author: *Savita Rathee**
Department of Mathematics, Maharshi Dayanand University, Rohtak, India

Example 1.1. a*b = a b, a *b = min {a, b} .

Definition 1.3. A triplet (X, M, *) is called a fuzzy normed space (briefly FN – space) if X is a real vector space, * is a continuous t-norm and M is a fuzzy set on $X \times [0, \infty)$ satisfying the following conditions

(FN – 1) M(x, 0) = 0,

(FN – 2) M(x, t) = 1 for all t > 0 if and only if x = 0 ,

(FN–3) $M(\alpha x, t) = M(x, \frac{t}{|\alpha|})$ for all $\alpha \in R$, $\alpha \neq 0$,

(FN–4) $M(x + y, t + s) \geq M(x, t) * M(y, s)$ for all $x, y \in X$ and $t, s \in R+$

(FN–5) $M(x, .) : [0, \infty) \to [0, 1]$ is left continuous for all $x \in X$

(FN–6) $\lim_{t\to\infty} M(x, t) = 1$ for all x in X and $t \in R+$.

A fuzzy normed space is a fuzzy metric space if we set G(x, y, t) = M(x – y, t).

Remark 1. M(x, t) can be thought of as the degree of nearness of norm of x with respect to t.

Definition 1.4. The natural topology t(M) is said to be topological if for each x in X and any $\in > 0$

$U_x(\in) = \{y : M(x-y, \in) > 1-\in \}$ is a neighbourhood of x in t(M).

Definition 1.5. A sequence $\{x_n\}$ in a fuzzy normed space is said to be convergent if for each r, 0 < r < 1, and t > 0, there exists $n_0 \in N$ such that

$M(x_n-x, t) > 1-r$ for all $n \geq n_0$.

Definition 1.6. A sequence $\{x_n\}$ in a fuzzy normed space is said to be a Cauchy if for each r, 0 < r < 1 and t > 0, there exists $n_0 \in N$ such that

$M(x_n- x_m, t) > 1-r$ for all $n, m \geq n_0$.

Definition 1.7. A fuzzy normed space is said to be complete if every Cauchy sequence is convergent.

Example 1.2. Let X = R. Define a * b = ab and $M(x, t) = \left[\exp\left(\frac{|x|}{t}\right)\right]^{-1}$ for all $x \in X$ and $t \in [0, \infty)$. Then (X, M, *) is a fuzzy normed space.

Proof. (i) M(x, 0) = 0

(ii) M(x, t) = 1 implies $\left[\exp\left(\frac{|x|}{t}\right)\right]^{-1} = 1$

or $\exp\left(\frac{|x|}{t}\right) = 1$

or $\exp\left(\frac{|x|}{t}\right) = \exp(0)$

or $\left(\frac{|x|}{t}\right) = 0$

$\Rightarrow$ $|x| = 0 \Rightarrow x = 0$.

If x = 0 , then M(x, t) = 1 .

(iii) $M(\alpha x, t) = M(x, \frac{t}{|\alpha|})$ is obvious

(iv) To prove

$M(x+y, t+s) \geq M(x, t) * M(y, s)$

Since $\frac{|x+y|}{t+s} \leq \frac{|x|+|y|}{t+s}$

$$= \frac{|x|}{t+s} + \frac{|y|}{t+s}$$

$$\leq \frac{|x|}{t} + \frac{|y|}{s}$$

$$\exp\left(\frac{|x+y|}{t+s}\right) \leq \exp\left(\frac{|x|}{t} + \frac{|y|}{s}\right)$$

$$= \exp\left(\frac{|x|}{t}\right).\exp\left(\frac{|y|}{s}\right)$$

Taking inverse, we have

$$\left[\exp\left(\frac{|x+y|}{t+s}\right)\right]^{-1} \geq \left[\exp\left(\frac{|x|}{t}\right)\right]^{-1} . \left[\exp\left(\frac{|y|}{s}\right)\right]^{-1}$$

Hence $M(x + y, t + s) \geq M(x, t) * M(y, s)$

(v) $M(x, .) : [0, \infty) \to [0, 1]$ is left continuous

(vi) $\lim_{t\to\infty} M(x, t) = 1$

Hence (X, M, *) is a fuzzy normed space.

Example 1.3. Let M be a fuzzy set on $X \times [0, \infty)$ defined by $M(x, t) = \frac{t}{t+|x|}$ for all $x \in X$, $t > 0$ and * is a t-norm defined by $a*b = ab$. Then (X, M, *) is a fuzzy normed space.

Lemma 1.1. Let $\{y_n\}$ be a sequence in a fuzzy normed space (X, M, *), where * is continuous and satisfies $*(x, x) \geq x$ for every $x \in \{0, 1\}$. If there exists a constant $k \in (0, 1)$ such that

(1.1) $M(y_n - y_{n+1}, kx) \geq M(y_{n-1} - y_n, x)$ for all n, then $\{y_n\}$ is a Cauchy sequence in X.

Proof. Let $\in$, λ be positive reals. Then for $m \geq n$, we have by (FN–4).

$$M(y_n - y_m, \in) \geq M(y_n - y_{n+1}, \in - k\in) * M(y_{n+1} - y_m, k\in)$$
$$\geq M(y_0 - y_1, (\in - k\in)\ k^{-n}) * M(y_{n+1} - y_m, k\in) \quad \text{(by 1.1)}$$

Taking $(\in - k\in)\, k^{-n} = h$, we get

$$M(y_n - y_m, \in) \geq M(y_0 - y_1, h) * (M(y_{n+1} - y_{n+2}, k\in - k^2\in) * M(y_{n+2} - y_m, k^2\in))$$
$$\geq M(y_0 - y_1, h) * (M(y_0 - y_1, h) * M(y_{n+2} - y_m, k^2 \in)$$

Repeating these arguments

$$M(y_n - y_m, \in) \geq M(y_0 - y_1, h) * M(y_{m-1} - y_m, k^{m-n-1} \in)$$
$$\geq M(y_0 - y_1, h) * M(y_0 - y_1, k^{-n} \in)$$
$$\geq M(y_0 - y_1, h) * M(y_0 - y_1, h)$$
$$\geq M(y_0 - y_1, (\in - k\in)\, k^{-n})$$

Therefore, if N be so chosen that $M(y_0 - y_1, (\in - k\in)\, k^{-N}) > 1 - r$, it follows that

$$M(y_n - y_m, \in) \geq 1 - r \text{ for all } m > n \geq N.$$

Hence $\{y_n\}$ is a Cauchy sequence.

2. MAIN RESULTS

Theorem 2.1. (Banach contraction principle) Let T be a continuous self mapping of a complete fuzzy normed space (X, M, *), where * is a continuous t-norm such that

(2.1) $M(Tx - Ty, k\in) \geq M(x - y, \in)$ for all x, y in X with $0 < k < 1$ and $\in > 0$

Then T has a unique fixed point in X.

Proof: Let x_0 be an arbitrary point of X. Construct a sequence $\{x_n\}$ in X defined by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots.$$

If $x_n = x_{n+1}$, then x_n is a fixed point of T. So, we assume $x_n \neq x_{n+1}$, for every n.

Now, we prove that $\{x_n\}$ is a Cauchy sequence in X. Given $\in > 0$, consider

$$\begin{aligned} M(x_n - x_{n+1}, k\in) &= M(Tx_{n-1} - Tx_n, \in) \\ &\geq M(x_{n-1} - x_n, \in) \end{aligned}$$

Therefore, by lemma (1.1), $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, so there exists a point $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Also T is continuous, therefore,

$$z = T(\lim_{n\to\infty} x_n) = Tz, \text{ z is a fixed point of T.}$$

To prove the uniqueness, let $v(\neq u)$ be another fixed point of T. Then by condition (2.1),

$$\begin{aligned} M(u - v, k\in) &= M(Tu - Tv, k\in) \\ &\geq M(u - v, \in) \\ &\geq \text{--------} \geq M(u - v, \in/k^n) \to 1 \text{ as } n \to \infty \end{aligned}$$

Hence implies u = v. This completes the proof of the theorem.

Theorem 2.2: Let (X, M, *) be a fuzzy normed space with three class of functions M_1, M_2 and M_3 and * be a continuous t-norm satisfying the following conditions:

(2.2) $M_1(x - y, k\in) \geq M_2(x - y, k\in) \geq M_3(x - y, k\in)$

(2.3) $(X, M_2, *)$ is complete

(2.4) $T : X \to X$ is continuous w.r.t. $(X, M_1, *)$

(2.5) T satisfies the condition

$M_3(Tx - Ty, k\in) \geq M_3(x - y, \in)$ for all x, y in X with $0 < k < 1$ and $\in > 0$.
Then T has a unique fixed point in X.

Proof: let $x_0 \in X$ be an arbitrary point. Consider a sequence $\{x_n\}$ in X defined by $x_{n+1} = fx_n$, $x \geq 0$. Then proceeding as in Theorem 2.1 after simplification we obtain $M_3(x_n - x_{n+1}, k\in) \geq M_3(x_{n-1} - x_n, \in)$.

By Lemma 1.1., $\{x_n\}$ is a Cauchy sequence in X with respect to $(X, M_3, *)$

By hypothesis (2.2), $\{x_n\}$ is a Cauchy sequence with respect to $(X, M_2, *)$ and $(X, M_2, *)$ being complete, there is a point $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Again by hypothesis, sequence x_n converges to z w.r.t. to $(X, M_1, *)$. By continuity of T w.r.t. $(X, M_1, *)$, we have

$$z = \lim_{n\to\infty} Tx_{n+1} = \lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = Tz. \text{ i.e., z is a fixed point of T.}$$

Uniqueness of z follows from the hypothesis (2.5).

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