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COMMUTANT OF COMPOSITE INTEGRAL OPERATORS

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ABSTRACT

In this paper we study the composite integral operators on L^p-spaces. The conditions for composite integral operators to be bounded are investigated. The commutants of composite integral operators and Volterra composition operators are computed.

Keywords: *Randon-Nikodym derivative, conditional expectation operator, Commutant, Contraction, Fixed point*

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1. INTRODUCTION AND PRELIMINARIES

Let (X, S,μ) be a σ-finite measure space and let $\phi: X\rightarrow X$ be a non-singular measurable transformation ($\mu(E) = 0 \Rightarrow \mu\phi^{-1}(E) =0$). Then a composition transformation $C_\phi : L^p(\mu) \rightarrow L^p(\mu)$ is defined by the equation

$$C_\phi f = fo\phi \text{ for every } f \in L^p(\mu) .$$

In case C_ϕ is continuous, we call it a composition operator induced by ϕ.

For each $f \in L^p(\mu)$, $1\leq p<\infty$, there exists a unique $\phi^{-1}(S)$ measurable function $E(f)$ such that

$$\int gfd\mu = \int gE(f)\, d\mu$$

for every $\phi^{-1}(S)$ measurable function g for which the left integral exits. The function E(f) is called conditional expectation of f with respect to the sub σ- algebra $\phi^{-1}(S)$. E has the property that for $f \in L^p(\mu)$, $E(f) = go\phi$ for exactly one S-measurable function g. We shall write $g = E(f)o\phi^{-1}$, which is well-defined measurable function. For more details about expectation operator, we refer to Parathasarthy [8].

Let $K: X \times X \rightarrow C$ be a measurable function. Then a linear transformation $I: L^p(\mu) \rightarrow L^p(\mu)$ defined by

$$(I f)(x) = \int K(x, y) f(y)\, d\mu(y) \qquad \text{for all } f \in L^p(\mu)$$

is known as integral operator. The composite integral operator I_ϕ is a bounded linear operator $I_\phi : L^p(\mu) \rightarrow L^p(\mu)$ defined by

$$(I_\phi f)(x) = \int K(x,y) f(\phi(y))\, d\mu(y) \qquad (1)$$

The equation (1) can also be written as

$$(I_\phi f)(x) = \int E(K_x)o\phi^{-1}(y)\, f_o(y)\, d\mu(y),$$

where $K_x(y) = K(x,y)$ and $f_o = \dfrac{d\mu\phi^{-1}}{d\mu}$, the Randon-Nikodym derivative of the measure $\mu\phi^{-1}$ with respect to the measure μ.

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The Volterra composition operator is a composition of Volterra integral operator V and a composition operator C_ϕ defined as

$$(V_\phi f)(x) = (V f) \circ \phi(x) = \int_0^x f(\phi(t))dt \quad \text{for every } f \in L^p[0,1],$$

where $\phi : [0,1] \to [0,1]$ is a measurable function.

An intensive study of composition operators is made over the past several decades. To worth mention, few of them are Singh ([10], [11]), Singh and Kumar [12], Singh and Komal [13], Singh and Manhas [14], Ridge [9] and Campbell [2]. The integral operators and composite integral operators in particular Volterra integral operator on $L^p(\mu)$ have received considerable attention in recent years. The theory of integral operators is the source of all modern functional analysis and operator theory. Mathematicians like Halmos and Sunder [6], Setpanov ([15], [16]), Bloom and Kermen [1] have done great deal of work on integral operators. Gupta and Komal ([3], [4], [5]) also studied composite integral operators. Whitley [17] established the Lyubic's conjecture [7] and generalized it to Volterra composition operators on $L^p[0,1]$.

In this paper we have make an effort to explore a commutant of composite integral operators.

2. BOUNDED COMPOSITE INTEGRAL OPERATORS

In this section we study bounded composite integral operators.

Theorem 2.1: Suppose $1 \le p, q < \infty$. Suppose $I_\phi : L^p(\mu) \to L^q(\mu)$ is a linear transformation. Then I_ϕ is continuous.

Proof: Let $f_n \to f$ in $L^p(\mu)$ and $I_\phi f_n \to g$ in $L^q(\mu)$. Then there exists a dominated subsequence $\{ f'_n \}$ of $\{f_n\}$ such that

$$f'_n(x) \to f(x) \quad \text{a.e.} \tag{i}$$

Again since $I_\phi f_n \to g$ in $L^q(\mu)$, we can select a dominated subsequence $\{ f''_n \}$ of $\{ f'_n \}$ such that

$$(I_\phi f''_n)(x) \to g(x) \quad \text{a.e.}$$

or that

$$\int K_\phi(x, y)\, f''_n(y)\, d\mu(y) \to g(x) \quad \text{a.e.}$$

Also $|f''_n| \le h$ for some $h \in L^p(\mu)$.

It follows from (i) that

$$K_\phi(x, y)\, f''_n(y) \to K_\phi(x, y) f(y) \quad \text{a.e.} \tag{ii}$$

and

$$| K_\phi(x,y)\, f''_n(y) | \le | K_\phi(x, y) h(y)| \quad \text{for almost every y.}$$

But the dominated subsequence $\{K_\phi f''_n\}$ converges to $\{K_\phi f\}$ almost everywhere. By the Lebesgue dominated convergence theorem,

$$\int K_\phi(x, y)\, f''_n(y)\, d\mu(y) \to \int K_\phi(x, y) f(y)\, d\mu(y) \tag{iii}$$

From (ii) and (iii), we conclude that

$$(I_\phi f)(x) = \int K_\phi(x, y) f(y)\, d\mu(y) = g(x)$$

which proves that the graph of I_ϕ is closed. Hence, by the closed graph theorem, I_ϕ is continuous.

In the following theorem, we take r such that and $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$ and

$S(x) = \|= K_\phi(x, .)\|_{r/q}$

Theorem 2.2: For $1 \le p, q < \infty$, le $S \in L^{r/q}(\mu)$. Then $I_\phi: L^p(\mu) \to L^q(\mu)$ is a bounded composite integral operator.

Proof: For $f \in L^p(\mu)$, consider

$$\| I_\phi f \|^q = \int_X | I_\phi f |^q \, dx$$
$$= \int_X \left| \int_X K_\phi(x, y) f(y) \, dy \right|^q dx$$
$$\le \int_X \left\{ \left(\int | K_\phi(x, y)|^{r/q} dy\right)^{q/r} \left(\int |f(y)|^{p/q} dy\right)^{q/p} \right\} dx$$
$$= \| S(x) \|^q_{r/q} \,.\, \| f \|^p$$

This proves that I_ϕ is bounded composite integral operator.

In the next result we make an attempt to use composite integral operators to solve the integral equations.

Theorem 2.3: If $K_\phi \in L^2 (\mu \times \mu)$ and $g \in L^2 [0, 1]$, then the integral equation

$$f(x) = g(x) + \lambda \int K_\phi(x, y) f(y) d\mu(y) \qquad (1)$$

has unique solution for sufficiently small values of scalar λ .

Proof: Define $I_\phi : L^2 [0, 1] \to L^2[0,1]$ as $I_\phi f = h$

where $h(x) = g(x) + \lambda \int_0^1 K_\phi(x, y) f(y) d\mu(y)$.

We first show that

$$\psi(x) = \int K_\phi(x, y) f(y) \, d\mu(y) \quad \text{for every } f \in L^2[0,1] .$$

Consider

$$\left| \int_0^1 K_\phi(x, y) f(y) \, d\mu(y) \right| \le \left(\int_0^1 | K_\phi (x, y) |^2 d\mu(y)\right)^{1/2} \left(\int_0^1 |(f(y) |^2 d\mu(y)\right)^{1/2} \text{ (by using Holder's inequality)}$$

Therefore,

$$\int_0^1 | \psi(x) |^2 dx \le \int_0^1 \left(\int_0^1 | K_\phi (x, y) |^2 d\mu(y) \right) d\mu(x) \int_0^1 \left(\int_0^1 |(f(y) |^2 d\mu(y) \, d\mu(x)\right.$$
$$< \infty.$$

Now

$$\| I_\phi f - I_\phi f_1 \| = \| \lambda \{ \int_0^1 K_\phi(x, y)[f(y) - f_1(y)] \, d\mu(y) \| \}$$
$$\le | \lambda | \left(\int_0^1 \int_0^1 | K_\phi (x, y) |^2 d\mu(x) \, d\mu(y)\right)^{1/2} \left(\int_0^1 |f(y) - f_1(y)|^2 d\mu(y)\right)^{1/2}$$
$$\le M \| f - f_1 \|,$$

where $M = | \lambda | \left(\int_0^1 \int_0^1 | K_\phi (x, y) |^2 d\mu(x) d\mu(y)\right)^{1/2}$.

This proves that I_ϕ is a contraction and hence it has a unique fixed point, say f^*. Thus f^* is a unique solution of eq. (1).

3. COMMUTANT OF COMPOSITE INTEGRAL OPERATOR

In this section we have made an attempt to compute the commutant of composite integral operator.

Theorem 3.1: Let $I_\phi \in B(L^p(\mu))$. Then M_θ commutes with I_ϕ if and only if $\theta = \theta o \phi$ a.e.

Proof: For $f \in L^p(\mu)$,

$$(I_\phi M_\theta f)(x) = \int K(x, y)(M_\theta f)o\phi(y)\, d\mu(y)$$

$$= \int E(K_x o\phi^{-1})(y) f_o(y)\theta(y)f(y)\, d\mu(y) \qquad \text{(i)}$$

and

$$(M_\theta I_\phi f)(x) = \theta(x)(I_\phi f)(x)$$

$$= \theta(x) \int E(K_x o\phi^{-1})(y) f_o(y) f(y) d\mu(y) \qquad \text{(ii)}$$

In view of (i) and (ii)

$$(M_\theta I_\phi f)(x) - (I_\phi M_\theta f)(x) = \int f_o(y) E(K_x o\phi^{-1})(y)[\theta(y) - \theta(x)]\, f(y) d\mu(y).$$

Hence, the result.

In the next theorem we characterize multiplication operators which commute with Volterra composite operators.

Theorem 3.2: Let $M_\theta \in B(L^2(\mu))$. Suppose ϕ is an injective map. Then M_θ commutes with V_ϕ if and only if $\theta = \theta o\phi$ a.e.

Proof: For $f \in L^2(\mu)$, we have

$$(M_\theta V_\phi f)(x) = (\theta.V(fo\phi))(x)$$

$$= \theta(x) \int_0^x f(\phi(t))dt$$

$$= \theta(x) \int_0^x \chi_{[0,x]}(t) f(\phi(t))\, dt$$

$$= \theta(x), \qquad \text{for } f = \chi_{[0,x]}$$

Also we have

$$(V_\phi M_\theta f)(x) = V(M_\theta f)o\phi(x)$$

$$= \int_0^x \theta o\phi\, .fo\phi\, (t)dt$$

$$= \int_0^1 \chi_{[0,x]}(t)\, \theta o\phi(t)\, f(\phi(t))\, dt$$

$$(M_\theta V_\phi f)(x) - (V_\phi M_\theta f)(x) = \int_0^1 \chi_{[0,x]}(t)\, [\theta(x) - \theta o\phi(t)]\, f(\phi(t))\, dt$$

as ϕ is injective, C_ϕ has dense range

$$\chi_{[0,x]}(t)\, [\theta(x) - \theta o\phi(t)] = 0.$$

Hence the result follows using the given condition.

Corollary: There is a composition operator $C_\phi \in L^2(\mu)$ such that $V C_\phi = C_\phi V$

Proof: For $f \in L^2(\mu)$,we have

$$(VC_\phi f)(x) = \int_0^x C_\phi f(t)\, dt = \int_0^x fo\phi(t)$$

$$(C_\phi Vf)(x) = (Vf)o\phi(x) = \int_0^{\phi(x)} f(t)dt = \int_0^x fo\phi(t)$$

Hence, the result follows.

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