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COMMON COUPLED FIXED POINT THEOREMS OF CONTRACTIVE MAPPINGS IN Q-FUZZY METRIC SPACES

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ABSTRACT

In the present paper, we prove a coupled fixed point theorem under ϕ-contractive conditions in the setting of a Q-fuzzy metric space in the sense of G. Sun and K. Yang.

INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [15]. It was developed extensively by many authors and used in various fields. To use this concept in topology and analysis, several researchers have defined fuzzy metric spaces in various ways. Mustafa and Sims [7] introduced a new notion of generalized metric space called a G-metric space. Mustafa, Sims and others studied fixed point theorems for mappings satisfying different contractive conditions (see [7-13]). V. Lakshmikantham et al. in [1, 6] introduced the concept of a coupled coincidence point of a mapping F from X × X into X and a mapping g from X into X, and studied fixed point theorems in partially ordered metric spaces. Recently, Hu[5] proved a common fixed point theorem for mappings under ϕ-contractive conditions in fuzzy metric spaces. Sun and Yang[14] introduced the concept of Q-fuzzy metric space and obtained some properties related to this space. Using approach of Q-fuzzy metric space , we prove a coupled fixed point theorem under ϕ-contractive conditions.

We first give some definitions and results that will be needed in the sequel.

In 2003, Mustafa and Sims [7] introduced the new approach of generalized metric spaces named as G-metric spaces as follows:

Definition 1.1[7]: Let X be a nonempty set. Let G: X × X × X → R be a function satisfying the following axioms:
(G1) G(x, y, z) = 0 if x = y = z,
(G2) 0 < G(x, x, y) for all x, y ∈ X with x ≠ y,
(G3) G(x, x, y) ≤ G(x, y, z), for all x, y, z ∈ X, with z ≠ y,
(G4) G(x, y, z) = G(x, z, y) = G(y, z, x) = · · · (symmetry in all three variables),
(G5) G(x, y, z) ≤ G(x, a, a) + G(a, y, z), for all x, y, z, a ∈ X, (rectangle inequality).

Then the function G is called a generalized metric or more specifically a G-metric on X and the pair (X, G) is called a G-metric space.

Let (X, G) be a G-metric space. Then a sequence $\{x_n\}$ is

- G-convergent to x if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is, for any $\varepsilon > 0$, there exists N ∈ N(natural number) such that $G(x, x_n, x_m) < \varepsilon$, for all n, m ≥ N(natural number). We call x the limit of the sequence and write $x_n \to x$ or $\lim x_n = x$.
- said to be G-Cauchy if for each $\varepsilon > 0$, there is N ∈ N(natural number) such that $G(x_n, x_m, x_l) < \varepsilon$, for all n, m, l ≥ N that is if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.
- said to be G-complete(or a complete G-metric space) if every G-Cauchy sequence in (X,G) is G-convergent in (X,G).

A G-metric space (X, G) is called a symmetric G-metric space if G(x, y, y) = G(y, x, x) for all x, y ∈ X.

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Proposition 1.2.[7]. Let (X, G) be a G-metric space. Then the following are equivalent:
(1) $\{x_n\}$ is G-convergent to x,
(2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,
(3) $G(x_n, x, x) \to 0$ as $n \to \infty$,
(4) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Proposition 1.3[7]. In a G-metric space (X, G) the following are equivalent:
(1) The sequence $\{x_n\}$ is G-Cauchy.
(2) For every $\varepsilon > 0$, there exists $N \in$ N (natural number) such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

Proposition 1.4[7]. Let (X, G) be a G-metric space. Then the function G(x, y, z) is jointly continuous in all three of its variables.

Proposition 1.5[7]. Let (X, G) be a G-metric space. Then, for any x, y, z, a $\in$ X it follows that:
(1) If G(x, y, z) = 0, then x = y = z,
(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(3) $G(x, y, y) \leq 2G(y, x, x)$,
(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
(5) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,
(6) $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$.

Definition1.6 [4]. Let sup $\sup_{0<t<1} \Delta(t,t) = 1$0<*t*<1. A *t*-norm Δ is said to be of H-type if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at *t* = 1,

where $\Delta^1(t) = t\Delta t$, $\Delta^{m+1}(t) = t\Delta(\Delta^m(t))$, m = 1, 2, ..., $t \in [0, 1]$ (1.1)

The *t*-norm $\Delta_M = \min$ is an example of *t*-norm of H-type, but there are some other *t*-norms Δ of H-type.

Obviously, Δ is a H-type *t* norm if and only if for any $\lambda \in (0, 1)$, there exists $\delta(\lambda) \in (0, 1)$ such that $\Delta^m(t) > 1 - \lambda$ for all $m \in N$ when $t > 1 - \delta$.

Recently, Sun and Yang [16] introduced the concept of Q-fuzzy metric spaces and proved two common fixed-point theorems for four mappings.

Definition 1.7[14]: A 3-tuple (X, Q,*) is called a Q-fuzzy metric space if X is an arbitrary (non-empty) set, * is a continuous t-norm, and Q is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each x, y, z, a $\in$ X and t, s>0:
(i) Q(x, x, y, t)>0 and $Q(x, x, y, t) \geq Q(x, y, z, t)$ for all x, y, z $\in$ X with $z \neq y$
(ii) Q(x, y, z, t) = 1 if and only if x = y = z
(iii) Q(x, y, z, t) = Q(p(x, y, z),t), (symmetry) where p is a permutation function,
(iv) $Q(x, a, a, t) * Q(a, y, z, s) \leq Q(x, y, z, t+s)$,
(v) $Q(x, y, z, .): (0, \infty) \to [0,1]$ is continuous

A Q-fuzzy metric space is said to be symmetric if Q(x, y, y, t) = Q(x, x, y, t) for all x, y $\in$ X.

Example 1.8[14]: Let X is a nonempty set and G is the G-metric on X. Denote a*b = a.b for all a,b$\in$[0.1]. For each t>0:

$$Q(x, y, z, t) = \frac{t}{t + G(x,y,z)}$$

Then (X,Q,*) is a Q-fuzzy metric space.

Let (X,Q,*) be a Q-fuzzy metric space.For t>0, the open ball $B_Q(x, r, t)$ with center $x \in X$ and radius 0<*r*<1 is defined by: $B_Q(x, r, t) = \{y \in X: Q(x, y, y, t) > 1-r\}$

A subset A of X is called open set if for each $x \in A$ there exist t>0 and 0<*r*<1 such that $B_Q(x,r,t) \subseteq A$.

A sequence $\{x_n\}$ in X converges to x if and only if Q $(x_m, x_n, x, t) \to 1$ as $n \to \infty$, for each t>0. It is called a Cauchy sequence if for each $0<\varepsilon<1$ and t>0,there exist $n_0 \in N$ such that $Q(x_m, x_n, x_l)>1-\varepsilon$ for each l, n, $m \geq n_0$.The Q-fuzzy metric space is called to be complete if every Cauchy sequence is convergent. Following similar argument in G-metric space, the sequence $\{x_n)$ in X also converges to x if and only if $Q(x_n, x_n, x, t) \to 1$ as $n \to \infty$,for each t>0 and it is a Cauchy sequence if for each $0<\varepsilon<1$ and t>0, there exist $n_0 \in N$ such that $Q(x_m, x_n, x_n)>1-\varepsilon$ for each n, $m \geq n_0$.

Lemma 1.9[14]: If (X,Q,*) be a Q-fuzzy metric space, then Q(x, y, z, t) is non-decreasing with respect to t for all x, y, z ∈ X.

Lemma 1.10 (see [14]). Let (X, Q, ∗) be a Q-fuzzy metric space. Then, Q is a continuous function on $X^3 \times (0, \infty)$.

Definition 1.11[14]: Let (X,Q,*) be a Q-fuzzy metric space. The following conditions are satisfied:

$\lim_{n\to\infty} Q(x_n, y_n, z_n, t_n) = Q(x, y, z, t)$ whenever, $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} z_n = z$

and $\lim_{n\to\infty} Q(x, y, z, t_n) = Q(x, y, z, t)$; then Q is called continuous function on $X^3 \times (0,\infty)$.

Definition 1.12: Let (X, Q, ∗) be a Q-fuzzy metric space. Q is said to satisfy the *n*-property on $X^3 \times (0, \infty)$ if

$$\lim_{n\to\infty} \left[Q\left(x, y, z, k^n t\right)\right]^{n^p} = 1$$

whenever $x, y, z \in X$, $k > 1$ and $p > 0$.

Lemma 1.13: *Let* (X, Q, ∗) be a Q-fuzzy metric space *and Q satisfies the n-property; then*

$$\lim_{t\to+\infty} Q(x, y, z, t) = 1, \ \forall x, y \in X.$$

Proof. If not, since Q(x, y, z, t) is non-decreasing and $0 \leq Q(x, y, z, .) \leq 1$, there exists $x_0, y_0 \in X$ such that $\lim_{t\to+\infty} Q(x_0, y_0, z_0, t) = \lambda \prec 1$, then for $k > 1$,$k^n t \to +\infty$ when $n \to \infty$as $t > 0$ and we get $\lim_{n\to\infty} \left[Q\left(x, y, z, k^n t\right)\right]^{n^p} = 0,$ which is a contraction.

Define $\Phi = \{\varphi : R^+ \to R^+\}$, where $R^+ = (0,+\infty)$ and each $\varphi \in \Phi$ satisfies the following conditions:
(φ-1) φ is nondecreasing;
(φ-2) φ is upper semicontinuous from the right;
(φ-3) $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$, where $\phi^{n+1}(t) = \phi(\phi^n(t)), n \in N$.

It is easy to prove that, if $\varphi \in \Phi$, then $\varphi(t) < t$ for all $t > 0$.

Lemma 1.14: *Let* (X, Q, ∗) be a Q-fuzzy metric space , *where ∗ is a continuous t-norm of H-type. If there exists* $\varphi \in \Phi$ *such that if*
$Q(x, y, z, \phi(t)) \geq Q(x, y, z, t)$ *for all* $t > 0$, *then* $x = y = z$.

Proof: *By definition 1.7 and lemma [2], we have the result.*

Definition 1.15[6]. *Let X be a* nonempty set. An element (x, y)∈ $X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if x =F(x,y) and y = F(y,x).

Definition 1.16 (see [6]). Let X be a nonem pty set. An element (x, y) ∈ X × X is called
(i) a coupled coincidence point of $F : X \times X \to X$ and $g : X \to X$ if gx =F(x, y) and gy = F(y, x).
(ii) a common coupled fixed point of $F : X \times X \to X$ and $g : X \to X$ if x = gx = F(x, y) and y=gy=F(y, x).

Definition 1.17: Let $F : X \times X \to X$ and $g : X \to X$ be mappings from a Q-fuzzy metric space (X, Q, ∗) into itself. Then the mappings are said to be compatible if

$$\lim_{n\to\infty} Q(gF(x_n, y_n), F(g(x_n), g(y_n)), F(g(x_n), g(y_n)), t) = 1$$

$$\lim_{n\to\infty} Q(gF(y_n, x_n), F(g(y_n), g(x_n)), F(g(y_n), g(x_n)), t) = 1 \ \forall t > 0$$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in *X* such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x, \quad \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$$

For all $x, y \in X$.

Definition 1.18: Let $F : X \times X \to X$ and $g : X \to X$ be mappings from a Q-fuzzy metric space (X, Q, ∗) into itself. Then the mappings are said to be semicompatible if

$$\lim_{n\to\infty} Q(F(g(x_n), g(y_n)), g(x), g(x), t) = 1,$$

$$\lim_{n\to\infty} Q(F(g(y_n), g(x_n)), g(y), g(y), t) = 1, \ \forall t > 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in *X* such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x, \quad \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y, \text{ for all } x, y \in X.$$

It follows that if (F, g) is semicompatible and F*(y, x) =gy*, F*(x, y) =gx*, then F(gy, gx) = *g(F(y, x)) and* F(gx, gy) = *g(F(x, y)).*

2. MAIN RESULTS

For convenience, we denote

$$[Q(x,y,z,t)]^n = \underbrace{Q(x,y,z,t) * Q(x,y,z,t) * . \quad . * Q(x,y,z,t)}_{n}, \qquad \text{for all } n \in \mathbb{N}. \qquad (2.1)$$

Theorem 1. *Let* (X, Q, ∗) be a Q-fuzzy metric space, *where* ∗ *is a continuous t-norm of H-type satisfying* (1.1).

Let $F : X \times X \to X$ *and* $g : X \to X$ *be two mappings and there exists* $\varphi \in \Phi$ *such that*

$$Q(F(x,y), F(u,v), F(u,v), \phi(t)) \ge Q(gx, gu, gu, t) * Q(gy, gv, gv, t) \text{ for all } x, y, u, v \in X, t > 0. \qquad (2.2)$$

Suppose that $F(X \times X) \subseteq g(X)$ *, and g(X) is complete subspace of X, F and g are semi-compatible. Then there exist* $x, y \in X$ *such that x = g(x)= F(x, x), that is, F and g have a unique common fixed point in X.*

Proof. Let $x_0, y_0 \in X$ and denote $z_n = F(x_n, y_n) = gx_{n+1}, p_n = F(y_n, x_n) = gy_{n+1}$, n = 0, 1, 2, (2.3)

The proof is divided into 4 steps.

Step 1. Prove that $\{z_n\}$ and $\{p_n\}$ are Cauchy sequences.

Since ∗ is a *t*-norm of H-type, for any $\lambda > 0$, there exists a $\mu > 0$ such that

$$\underbrace{(1-\mu) * (1-\mu) * .... * (1-\mu)}_{k} \ge 1 - \lambda \qquad \text{for all } k \in \mathbb{N}. \qquad (2.4)$$

Since $Q(x, y, z, \cdot)$ is continuous and $\lim_{t \to \infty} Q(x, y, z, t) = 1$ for all $x, y, z \in X$, there exists $t_0 > 0$ such that

$Q(z_0, z_1, z_1, t_0) \geq 1 - \mu$, $Q(p_0, p_1, p_1, t_0) \geq 1 - \mu$. (2.5)

On the other hand, since $\varphi \in \Phi$, by condition (φ-3) we have $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$. Then for any $t > 0$, there exists $n_0 \in$ N such that $t > \sum_{k=n_0}^{\infty} \phi^k(t_0)$ (2.6)

From condition (2.2), we have

$$\begin{aligned} Q(z_1, z_2, z_2, \phi(t_0)) &= Q(F(x_1,y_1), F(x_2, y_2), F(x_2, y_2), \phi(t_0)) \\ &\geq Q(gx_1, gx_2, gx_2, t_0) * Q(gy_1, gy_2, gy_2, t_0) \\ &\geq Q(z_0, z_1, z_1, t_0) * Q(p_0, p_1, p_1, t_0) \end{aligned} \tag{2.7}$$

Also $$\begin{aligned} Q(p_1, p_2, p_2, \phi(t_0)) &= Q(F(y_1,x_1), F(y_2,x_2), F(y_2, x_2), \phi(t_0)) \\ &\geq Q(gy_1, gy_2, gy_2, t_0) * Q(gx_1, gx_2, gx_2, t_0) \\ &\geq Q(p_0, p_1, p_1, t_0) * Q(z_0, z_1, z_1, t_0) \end{aligned}$$

Similarly, we can also get

$$\begin{aligned} Q(z_2, z_3, z_3, \phi^2(t_0)) &= Q(F(x_2,y_2), F(x_3, y_3), F(x_3, y_3), \phi^2(t_0)) \\ &\geq Q(gx_2, gx_3, gx_3, \phi(t_0)) * Q(gy_2, gy_3, gy_3, \phi(t_0)) \\ &\geq Q(z_1, z_2, z_2, \phi(t_0)) * Q(p_1, p_2, p_2, \phi(t_0)) \\ &\geq [Q(z_0, z_1, z_1, t_0)]^2 * [Q(p_0, p_1, p_1, t_0)]^2, \end{aligned} \tag{2.8}$$

Also

$$\begin{aligned} Q(p_1, p_2, p_2, \phi(t_0)) &= Q(F(y_1,x_1), F(y_2,x_2), F(y_2, x_2), \phi(t_0)) \\ &\geq Q(gy_2, gy_3, gy_3, \phi(t_0) * Q(gx_2, gx_3, gx_3, \phi(t_0)) \\ &\geq Q(p_1, p_2, p_2, \phi(t_0)) * Q(z_1, z_2, z_2, \phi(t_0)) \\ &\geq [Q(p_0, p_1, p_1, t_0)]^2 * [Q(z_0, z_1, z_1, t_0)]^2 \end{aligned}$$

Continuing in same way we can get

$$Q(z_n, z_{n+1}, z_{n+1}, \phi^n(t_0)) \geq [Q(z_0, z_1, z_1, t_0)]^{2^{n-1}} * [Q(p_0, p_1, p_1, t_0)]^{2^{n-1}}, \tag{2.9}$$

$$Q(p_n, p_{n+1}, p_{n+1}, \phi^n(t_0)) \geq [Q(p_0, p_1, p_1, t_0)]^{2^{n-1}} * [Q(z_0, z_1, z_1, t_0)]^{2^{n-1}}$$

So, from (2.5) and (2.6), for $m > n \geq n_0$, we have

$$\begin{aligned} Q(z_n, z_m, z_m, t) &\geq Q(z_n, z_m, z_m, \sum_{k=n_0}^{\infty} \phi^k(t_0)) \\ &\geq Q(z_n, z_m, z_m, \sum_{k=n}^{m-1} \phi^k(t_0)) \\ &\geq Q(z_n, z_{n+1}, z_{n+1}, \phi^n(t_0)) * Q(z_n, z_{n+2}, z_{n+2}, \phi^{n+1}(t_0)) * \ldots * Q(z_{m-1}, z_m, z_m, \phi^{m-1}(t_0)) \\ &\geq [Q(z_0, z_1, z_1, t_0)]^{2^{n-1}} * [Q(p_0, p_1, p_1, t_0)]^{2^{n-1}} * [Q(z_0, z_1, z_1, t_0)]^{2^n} * [Q(p_0, p_1, p_1, t_0)]^{2^n} * \ldots \\ &\quad * [Q(z_0, z_1, z_1, t_0)]^{2^{m-2}} * [Q(p_0, p_1, p_1, t_0)]^{2^{m-2}} \\ &= [Q(z_0, z_1, z_1, t_0)]^{2^{(m-n)(m+n-3)}} * [Q(p_0, p_1, p_1, t_0)]^{2^{(m-n)(m+n-3)}} \\ &\geq \underbrace{(1-\mu) * (1-\mu) * \ldots * (1-\mu)}_{2^{2(m-n)(m+n-3)}} \geq 1 - \lambda \end{aligned} \tag{2.10}$$

which implies that

$Q(z_n, z_m, z_m, t) > 1 - \lambda,$ (2.11)

for all $m, n \in N$ with $m > n \geq n_0$ and $t > 0$. So $\{z_n\}$ is a Cauchy sequence.

Similarly, we can get that $\{p_n\}$ is also a Cauchy sequence.

Step 2. Prove that g and F have a coupled coincidence point.

Since g(*X) is G*-complete, $\{z_n\}$ and $\{p_n\}$ converge to some α *and* β in g(X), respectively. Hence, there exist $x, y \in X$ such that

$$\alpha = gx, \beta = gy : Q(z_n, F(x, y), F(x, y), \phi(t)) = Q(F(x_n, y_n), F(x, y), F(x, y), \phi(t)) \geq Q(gx_n, gx, gx, t) * Q(gy_n, gy, gy, t) \geq Q(z_{n-1}, gx, gx, t) * Q(p_{n-1}, gy, gy, t) \quad (2.12)$$

Letting $n \to \infty$, we get

$Q(gx, F(x, y), F(x, y), \phi(t)) \geq 1 * 1 = 1$

Hence F(x, y) = gx. Similarly, it can be shown that F(y, x) = gy.

Since (F, g) is semicompatible and F*(y, x) =gy*, F*(x, y) =gx*, then F(gy, gx) = *g(F(y, x)) and* F(gx, gy) = *g(F(x, y))*.

This implies $g\alpha$ = ggx = *g(F(x, y))* = F(gx,gy) = $F(\alpha, \beta)$,

$g\beta$ = ggy = *g(F(y,x))* = F(gy, gx) = $F(\beta, \alpha)$

$$Q(z_n, g\alpha, g\alpha, \phi(t_0)) = Q(F(x_n, y_n), ggx, ggx, \phi(t_0)) \geq Q(F(x_n, y_n), g(F(x, y)), g(F(x, y)), \phi(t_0)) \geq Q(F(x_n, y_n), F(gx, gy), F(gx, gy), \phi(t_0)) \geq Q(gx_n, ggx, ggx, t_0) * Q(gy_n, ggy, ggy, t_0) \geq Q(z_{n-1}, g\alpha, g\alpha, t_0) * Q(p_{n-1}, g\beta, g\beta, t_0)$$

Letting $n \to \infty$, we get

$$Q(\alpha, g\alpha, g\alpha, \phi(t_0)) \geq Q(\alpha, g\alpha, g\alpha, t_0) * Q(\beta, g\beta, g\beta, t_0) \quad (2.13)$$

Similarly, we can show that

$$Q(\beta, g\beta, g\beta, \phi(t_0)) \geq Q(\alpha, g\alpha, g\alpha, t_0) * Q(\beta, g\beta, g\beta, t_0) \quad (2.14)$$

From (2.13) and (2.14), we have

$$Q(\alpha, g\alpha, g\alpha, \phi(t_0)) * Q(\beta, g\beta, g\beta, \phi(t_0)) \geq [Q(\alpha, g\alpha, g\alpha, t_0)]^2 * [Q(\beta, g\beta, g\beta, t_0)]^2 \quad (2.15)$$

By this way, we can get for all $n \in N$

$$Q(\alpha, g\alpha, g\alpha, \phi^n(t_0)) * Q(\beta, g\beta, g\beta, \phi^n(t_0)) \geq [Q(\alpha, g\alpha, g\alpha, \phi^{n-1}(t_0))]^2 * [Q(\beta, g\beta, g\beta, \phi^{n-1}(t_0))]^2$$

$$\geq [Q(\alpha, g\alpha, g\alpha, t_0)]^{2^n} * [Q(\beta, g\beta, g\beta, t_0)]^{2^n} \quad (2.16)$$

Then, we have

$$\begin{aligned} Q(\alpha, g\alpha, g\alpha, t) * Q(\beta, g\beta, g\beta, t) &\geq Q\left(\alpha, g\alpha, g\alpha, \sum_{k=n_0}^{\infty}\phi^k(t_0)\right) * Q\left(\beta, g\beta, g\beta, \sum_{k=n_0}^{\infty}\phi^k(t_0)\right) \\ &\geq Q(\alpha, g\alpha, g\alpha, \phi^{n_0}(t_0)) * Q(\beta, g\beta, g\beta, \phi^{n_0}(t_0)) \qquad (2.17) \\ &\geq [Q(\alpha, g\alpha, g\alpha, t_0)]^{2^{n_0}} * [Q(\beta, g\beta, g\beta, t_0)]^{2^{n_0}} \\ &\geq \underbrace{(1-\mu)*(1-\mu)*....*(1-\mu)}_{2^{2n_0}} \geq 1-\lambda \end{aligned}$$

So for any $\lambda > 0$ we have

$$Q(\alpha, g\alpha, g\alpha, t) * Q(\beta, g\beta, g\beta, t) \geq 1-\lambda \qquad (2.18)$$

for all $t > 0$.We can get that $g\alpha = \alpha$ and $g\beta = \beta$.

Thus, $g\alpha = \alpha = F(\alpha, \beta)$, $g\beta = \beta = F(\beta, \alpha)$.

Hence (α, β) is a common coupled fixed point of F and g.

Suppose (α^1, β^1) is another common coupled fixed point of F and g.

Step 4. Prove that $\alpha^1 = \alpha$ and $\beta^1 = \beta$..

Since $*$ is a t-norm of H-type, for any $\lambda > 0$, there exists a $\mu > 0$ such that

$$\underbrace{(1-\mu)*(1-\mu)*....*(1-\mu)}_{k} \geq 1-\lambda \qquad (2.19)$$

for all $k \in$ N.

Since $Q(x, y, z, \cdot)$ is continuous and $\lim_{t\to\infty} Q(x,y,z,t) = 1$ for all α, $\alpha^1 \in X$, there exists $t_0 > 0$ such that

$$\lim_{t\to\infty} Q(\alpha, \alpha^1, \alpha^1, t_0) \geq 1-\mu.$$

On the other hand, since $\varphi \in \Phi$, by condition (φ-3) we have $\sum_{n=0}^{\infty}\phi^n(t) < +\infty$. Then for any $t > 0$, there exists $n_0 \in$ N such that $t > \sum_{k=n_0}^{\infty}\phi^k(t_0)$

Since

$$\begin{aligned} Q(\alpha, \alpha^1, \alpha^1, \phi(t_0)) &= Q(F(\alpha,\beta), F(\alpha^1,\beta^1), F(\alpha^1,\beta^1), \phi(t_0)) \\ &\geq Q(\alpha, \alpha^1, \alpha^1, t_0) * Q(\beta, \beta^1, \beta^1, t_0) \qquad (2.20) \end{aligned}$$

Similarly,

$$\begin{aligned} Q(\beta, \beta^1, \beta^1, \phi(t_0)) &= Q(F(\beta,\alpha), F(\beta^1,\alpha^1), F(\beta^1,\alpha^1), \phi(t_0)) \\ &\geq Q(\alpha, \alpha^1, \alpha^1, t_0) * Q(\beta, \beta^1, \beta^1, t_0) \qquad (2.21) \end{aligned}$$

$$Q(\alpha, \alpha^1, \alpha^1, \phi(t_0)) * Q(\beta, \beta^1, \beta^1, \phi(t_0)) \geq [Q(\alpha, \alpha^1, \alpha^1, t_0)]^2 * [Q(\beta, \beta^1, \beta^1, t_0)]^2 \qquad (2.22)$$

By this way, we can get for all $n \in$ N

$$\begin{aligned} Q(\alpha, \alpha^1, \alpha^1, \phi^n(t_0)) * Q(\beta, \beta^1, \beta^1, \phi^n(t_0)) &\geq [Q(\alpha, \alpha^1, \alpha^1, \phi^{n-1}(t_0))]^2 * [Q(\beta, \beta^1, \beta^1, \phi^n(t_0))]^2 \\ &\geq [Q(\alpha, \alpha^1, \alpha^1, t_0)]^{2^n} * [Q(\beta, \beta^1, \beta^1, t_0)]^{2^n} \qquad (2.23) \end{aligned}$$

Then, we have

$$Q(\alpha,\ \alpha^1,\ \alpha^1,t)*\ Q(\beta,\ \beta^1,\ \beta^1,t)\geq Q\left(\alpha,\alpha^1,\alpha^1,\sum_{k=n_0}^{\infty}\phi^k(t_0)\right)*Q\left(\beta,\beta^1,\beta^1,\sum_{k=n_0}^{\infty}\phi^k(t_0)\right)$$
$$\geq Q(\alpha,\alpha^1,\alpha^1,\phi^{n_0}(t_0))*\ Q(\beta,\beta^1,\beta^1,\phi^{n_0}(t_0)) \tag{2.24}$$
$$\geq [Q(\alpha,\alpha^1,\alpha^1,t_0)]^{2^{n_0}}*[Q(\beta,\beta^1,\beta^1,t_0)]^{2^{n_0}}$$
$$\geq \underbrace{(1-\mu)*(1-\mu)*....*(1-\mu)}_{2^{2n_0}}\geq 1-\lambda$$

So for any $\lambda > 0$ we have

$$Q(\alpha,\ \alpha^1,\ \alpha^1,t)*\ Q(\beta,\ \beta^1,\ \beta^1,t)\geq 1-\lambda \tag{2.25}$$

for all $t > 0$.We can get that $\alpha^1=\alpha$ and $\beta^1=\beta$.

Hence (α,β) is the unique common coupled fixed point of F and g.

Now, we will show that $\alpha=\beta$

$$Q(\alpha,\ \beta,\ \beta,\phi(t_0))=Q(F(\alpha,\beta),F(\beta,\alpha),F(\beta,\alpha),\phi(t_0))$$
$$\geq Q(\alpha,\beta,\beta,t_0)*Q(\beta,\alpha,\alpha,t_0) \tag{2.26}$$

Similarly,

$$Q(\beta,\alpha,\alpha,\phi(t_0))=Q(F(\beta,\alpha),F(\alpha,\beta),F(\alpha,\beta),\phi(t_0))$$
$$\geq Q(\beta,\alpha,\alpha,t_0)*Q(\alpha,\beta,\beta,t_0) \tag{2.27}$$

$$Q(\alpha,\ \beta,\ \beta,\phi(t_0))*\ Q(\beta,\alpha,\alpha,\phi(t_0))\geq [Q(\alpha,\beta,\beta,t_0)]^2*[Q(\beta,\alpha,\alpha,t_0)]^2 \tag{2.28}$$

By this way, we can get for all $n \in$ N

$$Q(\alpha,\beta,\beta,\phi^n(t_0))*\ Q(\beta,\alpha,\alpha,\phi^n(t_0))\geq [Q(\alpha,\beta,\beta,\phi^{n-1}(t_0))]^2*[Q(\beta,\alpha,\alpha,\phi^n(t_0))]^2 \tag{2.29}$$
$$\geq [Q(\alpha,\beta,\beta,t_0)]^{2^n}*[Q(\beta,\alpha,\alpha,t_0)]^{2^n}$$

Then, we have

$$Q(\alpha,\beta,\beta,t)*\ Q(\beta,\alpha,\alpha,t)\geq Q\left(\alpha,\beta,\beta,\sum_{k=n_0}^{\infty}\phi^k(t_0)\right)*Q\left(\beta,\alpha,\alpha,\sum_{k=n_0}^{\infty}\phi^k(t_0)\right)$$
$$\geq Q(\alpha,\beta,\beta,\phi^{n_0}(t_0))*\ Q(\beta,\alpha,\alpha,\phi^{n_0}(t_0)) \tag{2.30}$$
$$\geq [Q(\alpha,\beta,\beta,t_0)]^{2^{n_0}}*[Q(\beta,\alpha,\alpha,t_0)]^{2^{n_0}}$$
$$\geq \underbrace{(1-\mu)*(1-\mu)*....*(1-\mu)}_{2^{2n_0}}\geq 1-\lambda$$

So for any $\lambda > 0$ we have

$$Q(\alpha,\beta,\beta,t)*\ Q(\beta,\alpha,\alpha,t)\geq 1-\lambda \text{ for all } t>0. \tag{2.31}$$

We can get that $\beta=\alpha$ and $\alpha=\beta$.

Thus α is a common fixed point of F and g, that is, α = gα = F(α , α).

Suppose α^1 is another common fixed point of F and g

$$Q(\alpha^1,\alpha,\alpha,t) \geq Q(\alpha^1,\alpha,\alpha,\sum_{k=n_0}^{\infty}\phi^k(t_0))$$
$$\geq Q(\alpha^1,\alpha,\alpha,\sum_{k=n}^{m-1}\phi^k(t_0))$$
$$\geq Q(\alpha^1,\alpha,\alpha,\phi^n(t_0)) * Q(\alpha^1,\alpha,\alpha,\phi^{n+1}(t_0)) * \ldots * Q(\alpha^1,\alpha,\alpha,\phi^{m-1}(t_0)) \quad (2.32)$$
$$\geq [Q(\alpha^1,\alpha,\alpha,t_0)]^{2^{n-1}} * [Q(\alpha^1,\alpha,\alpha,t_0)]^{2^{n-1}} * [Q(\alpha^1,\alpha,\alpha,t_0)]^{2^n} * [Q(\alpha^1,\alpha,\alpha,t_0)]^{2^n} * \ldots$$
$$* [Q(\alpha^1,\alpha,\alpha,t_0)]^{2^{m-2}} * [Q(\alpha^1,\alpha,\alpha,t_0)]^{2^{m-2}}$$
$$= [Q(\alpha^1,\alpha,\alpha,t_0)]^{2^{(m-n)(m+n-3)}} * [Q(\alpha^1,\alpha,\alpha,t_0)]^{2^{(m-n)(m+n-3)}}$$
$$\geq \underbrace{(1-\mu)*(1-\mu)*\ldots*(1-\mu)}_{2^{2(m-n)(m+n-3)}} \geq 1-\lambda$$

which implies that

$$Q(\alpha^1,\alpha,\alpha,t) > 1-\lambda, \quad (2.33)$$

Hence $\alpha^1 = \alpha$. Thus, F and g have a unique common coupled fixed point of the form (α,α).

This completes the proof of the Theorem 1.

Taking $g = I$ (the identity mapping) in Theorem 1, we get the following consequence.

Corollary 1. *Let* (X, Q, $*$) be a Q-fuzzy metric space, *where $*$ is a continuous t-norm of H-type satisfying* (1.1).

Let $F: X \times X \to X$ and there exists $\varphi \in \Phi$ such that

$Q(F(x,y),F(u,v),F(u,v),\phi(t)) \geq Q(x,u,u,t) * Q(y,v,v,t)$ *for all x, y, u, v $\in X$, $t > 0$.*

Then there exist $x \in X$ such that $x = F(x, x)$, that is, F admits a unique fixed point in X.

Let $\varphi(t) = kt$, where $0 < k < 1$, the following by Lemma 1.13, we get the following.

Corollary 2. *Let $a * b \geq ab$ for all a, b $\in$ [0, 1] and* (X, Q, $*$) be a *complete* Q- *fuzzy metric space such that Q has n-property. Let $F : X \times X \to X$ and $g : X \to X$ be two functions such that*

$Q(F(x,y),F(u,v),F(u,v),\phi(t)) \geq Q(gx,gu,gu,t) * Q(gy,gv,gv,t)$ *for all x, y, u, v $\in X$, $t > 0$.*

Suppose that $F(X \times X) \subseteq g(X)$, and g(X) is complete subspace of X, F and g are w-compatible. Then there exist x, y $\in X$ such that $x = g(x) = F(x, x)$, that is, F and g have a unique common fixed point in X.

Example: Let X = [-1, 1] with usual G-metric G and Define

$$Q(x, y, z, t) = \frac{t}{t+|x-y|+|y-z|+|z-x|}$$

For all x, y, z $\in X$ *and* $t > 0$. Clearly (X,Q,*) is a complete Q-fuzzy metric space where * is defined by

a*b = a.b . Let $\phi(t) = \frac{t}{2}$, gx =x and $F : X \times X \to X$ defined as $F(x,y) = \frac{x^2}{8}+\frac{y^2}{8}-1$, for all x, y$\in X$. Then F satisfies all the conditions of theorem 1, and there exists a point $x = 2-2\sqrt{2}$ which is the unique common fixed point of g and F.

REFERENCES

1. Bhaskar, T.G and Lakshmikantham, G.V. Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis 65, 1379–1393, 2006.
2. Fang, J-X., Common fixed point theorems of compatible and weakly compatible maps in Menger spaces, Nonlinear Analysis. Theory, Methods &Applications, Vol. 71, no. 5-6,pp. 1833-1843,2009.
3. Grabiec, M., Fixed points in fuzzy metric spaces, *Fuzzy Sets and Systems*, vol. 27, no. 3, pp. 385–389, 1988.
4. Hadˇzi´c, O. and Pap, E., *Fixed Point Theory in Probabilistic Metric Spaces*, vol. 536 of *Mathematics and its Applications*, Kluwer Academic, Dordrecht, The Netherlands, 2001.
5. Hu, Xin-Qi, Common Coupled Fixed Point Theorems for Contractive Mappings in Fuzzy Metric Spaces, Fixed Point Theory Appl. 2011 (2011), Article ID 363716,14 pages, doi:10.1155/2011/363716.
6. Lakshmikantham, V. and ´Ciri´c, Lj.B. Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis 70, 4341–4349, 2009.
7. Mustafa, Z. and Sims, B. Some Remarks concerning D-metric spaces, in: Proc. Int. Conf. on Fixed Point Theor. Appl., Valencia (Spain), July 2003, pp. 189–198.
8. Mustafa, Z. and Sims, B. A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2), 289–297, 2006.
9. Mustafa, Z., Shatanawi, W. and Bataineh, M. Existence of fixed point results in G-metric spaces, International Journal of Mathematics and Mathematical Sciences Volume, Article ID 283028, 10 pages, 2009.
10. Mustafa, Z., Obiedat, H. and Awawdeh, F. Some common fixed point theorem for mapping on complete G-metric spaces, Fixed Point Theor. Appl. Article ID 189870, 12 pages, 2008.
11. Shatanawi, W. Common fixed point result for two self-maps in G-metric space, submitted.
12. Shatanawi, W. Some fixed point theorems in ordered G-metric spaces and applications, Abstr. Appl. Anal. 2011 (2011), Article ID 126205, 11 pages, doi:10.1155/2011/126205
13. Shatanawi, W. Fixed point theory for contractive mappings satisfying ϕ-maps in G- metric spaces, Fixed Point Theory Appl. 2010 (2010), Article ID 181650, 9 pages, doi:10.1155/2010/181650
14. Sun, G. and Yang, K. , Generalized fuzzy metric spaces with properties, *Research journal of Applied Sciences, Engineering and Technology*, vol. 2, no. 7, pp. 673–678, 2010.
15. Zadeh, L. A., Fuzzy sets, *Information and Computation*, vol. 8, pp. 338–353, 1965.

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