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ON THE STABILITY OF A TYPICAL THREE SPECIES SYN-ECO-SYSTEM

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ABSTRACT

The present paper deals with an investigation on a typical three species syn eco-system. The system comprises of a commensal (S_1), two hosts S_2 and S_3 ie., S_2 and S_3 both benefit S_1, without getting themselves effected either positively or adversely. Further S_2 is a commensal of S_3 and S_3 is a host of both S_1, S_2. Here all three species are having limited resources quantized by the respective carrying capacities. The mathematical model equations constitute a set of three first order non-linear simultaneous coupled differential equations in the strengths N_1, N_2, N_3 of S_1, S_2, S_3 respectively. In all, eight equilibrium points of the model are identified. The system would be stable, if all the characteristic roots are negative, in case they are real and have negative real parts, in case they are complex. Trajectories of the perturbations over the equilibrium points are illustrated. Further we establish global stability of a four spices syn-eco system by constructing a suitable Liapunov's function.

Keywords: *Commensal, Equilibrium Point, Host, Stable, Unstable.*

AMS Classification: *92D25, 92D40.*

1. INTRODUCTION:

Ecology is a branch of life and environment sciences dealing with the existence of diverse species in the same environment and habitat. It is natural that two or more species living in a common habitat interact in different ways. Mathematical modeling has been playing an important role for the last half a century in explaining several phenomena concerned with individuals and groups of populations in nature. Significant researches in the area of theoretical ecology has been initiated by Lotka[23] and by Volterra [29]. Since then, several mathematicians and ecologists contributed to the growth of this area of knowledge. The Ecological interactions can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation and Parasitism.

The general concept of modeling has been presented in the treatises of Meyer [24], Kushing[19], Paul[25], Kapur[20]. Srinivas[28] studied competitive ecosystem of two species and three species with limited and unlimited resources. Later, Lakshminarayan [21], Laxminarayan and Pattabhi Ramacharyulu [22] studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Stability analysis of competitive species was carried out by Archana Reddy, Pattabhi Ramacharyulu and Krishna Gandhi [5] and by Bhaskara Rama Sarma and Pattabhi Ramacharyulu [6], while Ravindra Reddy [27] investigated mutualism between two species. Acharyulu K.V.L.N and Pattabhi Ramacharyulu [1-4] obtained fruitful results on some mathematical models of ecological Ammensalism.Further Phani Kumar [26] studied some mathematical models of ecological commensalism. The present authors Hari Prasad and Pattabhi Ramacharyulu [7-18] discussed on the stability of a four species syn-ecosystem.

The present investigation is on an analytical study of a typical three species (S_1, S_2, S_3) syn-eco system. The system comprises of a commensal (S_1), two hosts S_2 and S_3 ie, S_2 and S_3 both benefit S_1, without getting themselves effected either positively or adversely. Further S_2 is a commensal of S_3 and S_3 is a host of both S_1, S_2. Figure 1 shows a schematic diagram of the interaction under study. Commensalism is a symbiotic interaction between two populations where one population (S_1) gets benefit from (S_2) while the other (S_2) is neither harmed nor benefited due to the interaction with (S_1). The benefited species (S_1) is called the commensal and the other, the helping one (S_2) is called the host species. A common example is an animal using a tree for shelter-tree (host) does not get any benefit from the animal (commensal).

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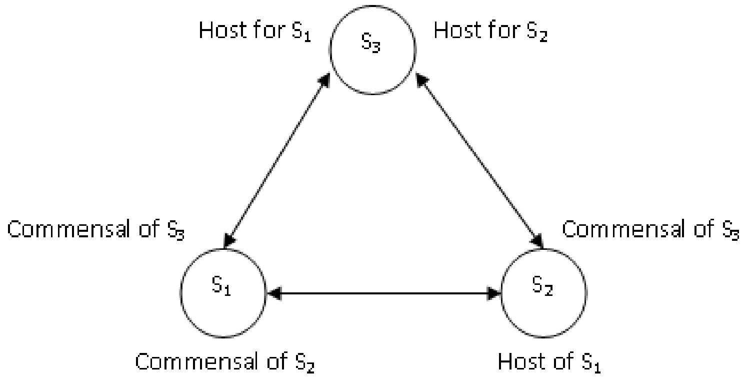

Figure: 1. Schematic Sketch of the Syn Eco-System.

2. BASIC EQUATIONS OF THE MODEL

The model equations for a typical three species ecosystem is given by the following system of first order non-linear ordinary differential equations employing the following notation.

S_1	:	Commensal of S_2 and S_3
S_2	:	Host of S_1 and commensal of S_3
S_3	:	Host of S_1 and S_2
$N_i(t)$	:	The population strength of S_i at time t, i = 1,2,3.
t	:	Time instant.
a_i	:	Natural growth rate of S_i , i = 1,2, 3.
a_{ii}	:	Self inhibition coefficients of S_i, i=1, 2, 3.
a_{12}, a_{13}	:	Interaction coefficients of S_1 due to S_2 and S_1 due to S_3.
a_{23}	:	Interaction coefficient of S_2 due to S_3
$k_i = \frac{a_i}{a_{ii}}$	:	Carrying capacities of S_i, i = 1, 2, 3.
t^*	:	The dominance reversal time.

Further the variables N_1, N_2, N_3 are non-negative and the model parameters a_1, a_2, a_3, a_{11}, a_{22}, a_{33}, a_{13}, a_{23} are assumed to be non-negative constants.

The model equations for the growth rates of S_1, S_2, S_3 are

$$\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^2 + a_{12} N_1 N_2 + a_{13} \; N_1 N_3 \qquad (2.1)$$

$$\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^2 + a_{23} N_2 N_3 \qquad (2.2)$$

$$\frac{dN_3}{dt} = a_3 N_3 - a_{33} N_3^2 \qquad (2.3)$$

3. EQUILIBRIUM STATES:

The system under investigation has 8 equilibrium states given by $\frac{dN_i}{dt} = 0, i = 1, 2, 3$ (3.1)

(i) Fully washed out state

$$E_1 : \overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = 0$$

(ii) States in which two of the tree species are washed out and third is not.

$$E_2 : \overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = k_3$$

$$E_3 : \overline{N}_1 = 0, \overline{N}_2 = k_3, \overline{N}_3 = 0$$

$$E_4 : \overline{N}_1 = k_1, \overline{N}_2 = 0, \overline{N}_3 = 0$$

(iii) Only one of the three species is washed out while the other two are not.

$$E_5 : \overline{N}_1 = 0,\ \overline{N}_2 = k_2 + \frac{a_{23}k_3}{a_{22}},\ \overline{N}_3 = k_3$$

$$E_6 : \overline{N}_1 = k_1 + \frac{a_{13}k_3}{a_{11}},\ \overline{N}_2 = 0,\ \overline{N}_3 = k_3$$

$$E_7 : \overline{N}_1 = k_1 + \frac{a_{12}k_2}{a_{11}},\ \overline{N}_2 = k_2,\ \overline{N}_3 = 0$$

iv) The co-existent state or normal steady state.

$$E_8 : \overline{N}_1 = k_1 + \frac{a_{12}}{a_{11}}\left(k_2 + \frac{a_{23}k_3}{a_{22}} \right) + \frac{a_{13}k_3}{a_{11}},\ \overline{N}_2 = k_2 + \frac{a_{23}k_3}{a_{11}},\ \overline{N}_3 = k_3$$

4. STABILITY OF THE EQUILIBRIUM STATES

Let N = (N_1, N_2, N_3) = $\overline{N} + U$ (4.1)

where $U = (u_1, u_2, u_3)^T$ is a small perturbation over the equilibrium state $\overline{N} = (\overline{N}_1, \overline{N}_2, \overline{N}_3)$.

The basic equations (2.1), (2.2) and (2.3) are quasi-linearized to obtain the equations for the perturbed state as

$$\frac{dU}{dt} = AU \tag{4.2}$$

$$\text{with } A = \begin{bmatrix} a_1 - 2a_{11}\overline{N}_1 + a_{12}\overline{N}_2 + a_{13}\overline{N}_3 & a_{12}\overline{N}_1 & a_{13}\overline{N}_1 \\ 0 & a_2 - 2a_{22}\overline{N}_2 + a_{23}\overline{N}_3 & a_{23}\overline{N}_2 \\ 0 & 0 & a_3 - 2a_{33}\overline{N}_3 \end{bmatrix} \tag{4.3}$$

The characteristic equation for the system is det [A – λI] = 0 (4.4)

The equilibrium state is stable, if all the roots of the equation (4.4) are negative, in case they are real or have negative real parts, in case they are complex.

There would arise in all 36 cases depending upon the ordering of the magnitudes of the growth rates a_1, a_2, a_3 and the initial values of the perturbations $u_{10}(t)$, $u_{20}(t)$, $u_{30}(t)$ of the species S_1, S_2, S_3. Of these 36 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations.

5. STABILITY OF THE FULLY WASHED OUT STATE

To discuss the stability of equilibrium point $E_1(0,0,0)$. Let us consider small deviations $u_1(t)$, $u_2(t)$, $u_3(t)$ from the steady state.

ie, $N_i(t) = \overline{N}_i + u_i(t),\ i = 1,2,3$ (5.1)

where $u_i(t)$ is a small perturbations in the species S_i.

The corresponding linearized equations for the perturbations u_1, u_2, u_3 are

$$\frac{du_i}{dt} = a_i u_i, \quad i = 1, 2, 3 \tag{5.2}$$

The characteristic equation for which is

$$\Pi(\lambda - a_i) = 0, \qquad i = 1, 2, 3 \tag{5.3}$$

The roots a_1, a_2, a_3 of which are all positive. Hence the fully washed out state is **unstable** and the solutions of the equations (5.2) are

$$u_i = u_{io}\, e^{a_i t},\ i = 1, 2, 3 \tag{5.4}$$

where u_{10}, u_{20}, u_{30} are the initial values of u_1, u_2, u_3 respectively.

The solution curves are illustrated in Figures (5.1) to (5.4) and the observations are presented below.

Observations

Case (i): when $u_{10} < u_{20} < u_{30}$ and $a_2 < a_1 < a_3$

The commensal (S_1) has the least initial population strength and the host (S_2) has the least natural growth rate. Initially the host (S_2) dominates over the commensal (S_1) till the time instant t_{12}^* and thereafter the dominance is reversed as shown in Figure 5.1.

$$\text{Here } t_{12}^* = \frac{1}{a_1 - a_2} \ln\left(\frac{u_{20}}{u_{10}}\right) \tag{5.5}$$

Case (ii): when $u_{20} < u_{30} < u_{10}$ and $a_1 < a_2 < a_3$

The host (S_2) has the least initial population strength and the commensal (S_1) has the least natural growth rate. Initially the commensal (S_1) dominates over the host (S_3), host (S_2) till the time instant $t_{13}^*,\ t_{12}^*$ respectively and thereafter the dominance is reversed. This is illustrated in Figure 5.2.

$$\text{Here } t_{13}^* = \frac{1}{a_1 - a_3} \ln\left(\frac{u_{30}}{u_{10}}\right) \tag{5.6}$$

Case (iii): When $u_{30} < u_{20} < u_{10}$ and $a_3 < a_1 < a_2$

The host (S_3) has the least initial population strength as well as the least natural growth rate. Initially the commensal (S_1) dominates over the host (S_2) till the time instant t_{12}^* and thereafter the dominance is reversed. It is shown in Figure 5.3.

Case (iv): When $u_{10} < u_{30} < u_{20}$ and $a_2 < a_3 < a_1$

The commensal (S_1) has the least initial population strength and the host (S_2) has the least natural growth rate. Initially the host (S_2) dominates over the host (S_3), commensal (S_1) till the time instant t_{23}^*, t_{12}^* respectively and thereafter the dominance is reversed. Also the host (S_3) dominates over the commensal (S_1) till the time instant t_{13}^* and the dominance gets reversed thereafter. This is seen in Figure 5.4.

$$\text{Here } t_{23}^* = \frac{1}{a_2 - a_3} \ln\left(\frac{u_{30}}{u_{20}}\right) \tag{5.7}$$

Trajectories of Perturbations

The trajectories in $u_1 - u_2$, $u_2 - u_3$, $u_1 - u_3$ planes are

$$\left(\frac{u_1}{u_{10}}\right)^{a_2} = \left(\frac{u_2}{u_{20}}\right)^{a_1}, \left(\frac{u_2}{u_{20}}\right)^{a_3} = \left(\frac{u_3}{u_{30}}\right)^{a_2}, \left(\frac{u_1}{u_{10}}\right)^{a_3} = \left(\frac{u_3}{u_{30}}\right)^{a_1} \text{ respectively.}$$

6. STABILITY OF EQUILIBRIUM STATES IN WHICH TWO OF THE THREE SPECIES ARE WASHED OUT AND THIRD IS NOT

6.1. Equilibrium point $E_2 : \overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = k_3$

The corresponding linearized equations for the perturbations u_1, u_2, u_3 are

$$\left.\begin{aligned} \frac{du_1}{dt} &= \alpha_1 u_1 \\ \frac{du_2}{dt} &= \alpha_2 u_2 \\ \frac{du_3}{dt} &= -a_3 u_3 \end{aligned}\right\} \tag{6.1}$$

where $\alpha_1 = a_1 + a_{13} k_3 > 0$ and $\alpha_2 = a_2 + a_{23} k_3$ (6.2)

The characteristic equation for which is $(\lambda-\alpha_1)(\lambda-\alpha_2)(\lambda+a_3) = 0$ (6.3)

The characteristic roots of (6.3) are α_1, α_2 and $-a_3$. Since two of these three roots are positive, hence the state is **unstable**. The equations (6.1) yield the solutions.

$$\left.\begin{aligned} u_1 &= u_{10}e^{\alpha_1 t} \\ u_2 &= u_{20}\, e^{\alpha_2 t} \\ u_3 &= u_{30}e^{-a_3 t} \end{aligned}\right\} \qquad (6.4)$$

The solution curves are illustrated in Figures (6.1) to (6.4) and the observations are presented below.

Observations

Case (i): When $u_{10} < u_{30} < u_{20}$ and $\alpha_1 < \alpha_2 < a_3$

The commensal (S_1) has the least initial population strength and the host (S_3) has the least natural growth rate. Initially the host (S_3) dominates over the commensal (S_1) till the time instant t_{13}^* and thereafter the dominance is reversed as shown in Figure 6.1

Here $$t_{13}^* = \frac{1}{\alpha_1 + a_3}\ln\left(\frac{u_{30}}{u_{10}}\right) \qquad (6.5)$$

Case (ii): When $u_{20} < u_{10} < u_{30}$ and $a_3 < \alpha_2 < \alpha_1$

The host (S_2) has the least initial population strength and the host (S_3) has the least natural growth rate. Initially the host (S_3) dominates over the commensal (S_1), host (S_2) till the time instant t_{13}^*, t_{23}^* respectively and thereafter the dominance is reversed as illustrated in Figure 6.2.

Here $$t_{23}^* = \frac{1}{\alpha_2 + a_3}\ln\left(\frac{u_{30}}{u_{20}}\right) \qquad (6.6)$$

Case (iii): when $u_{30} < u_{10} < u_{20}$ and $\alpha_2 < a_3 < \alpha_1$

The host (S_3) has the least initial population strength as well as the least natural growth rate. Initially the host (S_2) dominates over the commensal (S_1) till the time instant t_{12}^* and thereafter the dominance is reversed as seen in Figure 6.3.

Here $$t_{12}^* = \frac{1}{\alpha_1 + \alpha_2}\ln\left(\frac{u_{20}}{u_{10}}\right) \qquad (6.7)$$

Case (iv): When $u_{10} < u_{20} < u_{30}$ and $\alpha_2 < a_3 < \alpha_1$

The commensal (S_1) has the least initial population strength and host (S_3) has the least natural growth rate. Initially the host (S_3) dominates over the host (S_2), commensal (S_1) till the time instant t_{23}^*, t_{13}^* respectively and thereafter the dominance is reversed. Also the host (S_2) dominate over the commensal (S_1) till the time instant t_{12}^* and the dominance gets reversed thereafter. This is illustrated in Figure 6.4.

Trajectories of perturbations

The trajectories in $u_1 - u_2$, $u_2 - u_3$, $u_1 - u_3$ planes are

$$\left(\frac{u_1}{u_{10}}\right)^{\alpha_2} = \left(\frac{u_2}{u_{20}}\right)^{\alpha_1}, \left(\frac{u_2}{u_{20}}\right)^{-a_3} = \left(\frac{u_3}{u_{30}}\right)^{\alpha_2}, \left(\frac{u_1}{u_{10}}\right)^{-a_3} = \left(\frac{u_3}{u_{30}}\right)^{\alpha_1}$$ respectively.

6.2. Equilibrium point $E_3 : \overline{N}_1 = 0, \overline{N}_2 = k_2, \overline{N}_3 = 0$.

The corresponding linearized equations for the perturbations u_1, u_2, u_3 are

$$\left.\begin{aligned} \frac{du_1}{dt} &= \beta_1 u_1 \\ \frac{du_2}{dt} &= -a_2 u_2 + a_{23} k_2 u_3 \\ \frac{du_3}{dt} &= a_3 u_3 \end{aligned}\right\} \tag{6.8}$$

where $\beta_1 = a_1 + a_{12}\, k_2 > 0$ (6.9)

The characteristic equation for which is $(\lambda - \beta_1)(\lambda + a_2)(\lambda - a_3) = 0$ (6.10)

The characteristic roots of (6.10) are $\beta_1, -a_2, a_3$. Since two of these three roots are positive, hence the state is **unstable**. The equations (6.8) yield the solutions.

$$\left.\begin{aligned} u_1 &= u_{10} e^{\beta_1 t} \\ u_2 &= (u_{10} - \alpha_3)\, e^{-a_2 t} + \alpha_3 e^{a_3 t} \\ u_3 &= u_{30} e^{a_3 t} \end{aligned}\right\} \tag{6.11}$$

where $\alpha_3 = \dfrac{a_{23} k_2 u_{30}}{a_2 + a_3}$ (6.12)

The solution curves are illustrated in Figures (6.5) and (6.6) and the observations are presented below.

Observations

Case (i): When $u_{10} < u_{30} < u_{20}$ and $\beta_1 < a_2 < a_3$

The commensal (S_1) has the least initial population strength as well as the least natural growth rate. Initially the host (S_2) dominates over the host (S_3) till the time instant t_{23}^* and thereafter the dominance is reversed. It is shown in Figure 6.5.

Case (ii): When $u_{20} < u_{30} < u_{10}$ and $a_3 < \beta_1 < a_2$

The host (S_2) has the least initial population strength and the host (S_3) has the least natural growth rate. Initially the commensal (S_1) and the host (S_3) dominates over the host (S_2) till the time instant t_{12}^* and t_{23}^*, and thereafter the dominance is reversed as shown in Figure 6.6.

Trajectories of perturbations

The trajectories in $u_1 - u_2$, $u_1 - u_3$, $u_2 - u_3$ planes are

$$\frac{u_2}{u_{20}} = A\left(\frac{u_1}{u_{10}}\right)^{\frac{-a_2}{\beta_1}} + \frac{\alpha_3}{u_{20}}\left(\frac{u_1}{u_{10}}\right)^{\frac{a_3}{\beta_1}}, \left(\frac{u_1}{u_{10}}\right)^{a_3} = \left(\frac{u_3}{u_{30}}\right)^{\beta_1}$$

$$\frac{u_2}{u_{20}} = A\left(\frac{u_3}{u_{30}}\right)^{\frac{-a_2}{a_3}} + \frac{\alpha_3}{u_{20}}\left(\frac{u_3}{u_{30}}\right)$$ respectively.

where $A = \dfrac{u_{10} - \alpha_3}{u_{20}}$

6.3. Equilibrium point $E_4 : \overline{N}_1 = k_1,\ \overline{N}_2 = 0,\ \overline{N}_3 = 0$

The corresponding linearized equations for the perturbations u_1, u_2, u_3 are

$$\left.\begin{aligned} \frac{du_1}{dt} &= -a_1u_1 + a_{12}\,k_1u_2 + a_{13}k_1u_3 \\ \frac{du_2}{dt} &= a_2u_2 \\ \frac{du_3}{dt} &= a_3u_3 \end{aligned}\right\} \tag{6.13}$$

The characteristic equation for which is $(\lambda + a_1)(\lambda - a_2)(\lambda - a_3) = 0$ (6.14)

The characteristic roots of (6.14) are $-a_1$, a_2, a_3. Since two of these three roots are positive, hence the state is **unstable**.

The equations (6.13) yield the solutions.

$$\left.\begin{aligned} u_1 &= (u_{10} - \alpha - \beta)e^{-a_1t} + \alpha e^{a_2t} + \beta e^{a_3t} \\ u_2 &= u_{20}e^{a_2t} \\ u_3 &= u_{30}e^{a_3t} \end{aligned}\right\} \tag{6.15}$$

where $\alpha = \dfrac{a_{12}k_1u_{20}}{a_1 + a_2} > 0$ and $\beta = \dfrac{a_{13}k_1u_{30}}{a_1 + a_3} > 0$ (6.16)

The solution curves are illustrated in Figures (6.7) and (6.8) and the observations are presented below.

Observations

Case (i): When $u_{10} < u_{20} < u_{30}$ and $a_3 < a_1 < a_2$

The commensal (S_1) has the least initial population strength and the host (S_3) has the least natural growth rate. Initially the host (S_3) dominates over the host (S_2), commensal (S_1) till the time instant t^*_{23}, t^*_{13} respectively and thereafter the dominance is reversed as shown in Figure 6.7.

Case (ii): When $u_{30} < u_{10} < u_{20}$ and $a_1 < a_3 < a_2$

The host (S_3) has the least initial population strength and the commensal (S_1) has the least natural growth rate. Initially the commensal (S_1) dominates over the host (S_3) till the time instant t^*_{13} and thereafter the dominance is reversed as illustrated in Figure 6.8.

Trajectories of perturbations

The trajectories in $u_1 - u_2$, $u_2 - u_3$, $u_1 - u_3$ planes are

$$\frac{u_1}{u_{10}} = B\left(\frac{u_2}{u_{20}}\right)^{\frac{-a_1}{a_2}} + \frac{\alpha}{u_{10}}\left(\frac{u_2}{u_{20}}\right) + \frac{\beta}{u_{10}}\left(\frac{u_2}{u_{20}}\right)^{\frac{a_3}{a_2}},\ \left(\frac{u_2}{u_{20}}\right)^{a_3} = \left(\frac{u_3}{u_{30}}\right)^{a_2}$$

$$\frac{u_1}{u_{10}} = B\left(\frac{u_3}{u_{30}}\right)^{\frac{-a_1}{a_3}} + \frac{\alpha}{u_{10}}\left(\frac{u_3}{u_{30}}\right)^{\frac{a_2}{a_3}} + \frac{\beta}{u_{10}}\left(\frac{u_3}{u_{30}}\right) \text{ respectively.}$$

where $B = \dfrac{u_{10} - \alpha - \beta}{u_{10}}$

7. STABILITY OF EQUILIBRIUM STATES ONLY ONE OF THE THREE SPECIES IS WASHED OUT WHILE THE OTHER TWO ARE NOT

7.1 Equilibrium point $E_5 : \overline{N}_1 = 0,\ \overline{N}_2 = k_2 + \dfrac{a_{23}k_3}{a_{22}},\ \overline{N}_3 = k_3$

The corresponding linearized equations for the perturbations u_1, u_2, u_3 are

$$\left.\begin{aligned} \frac{du_1}{dt} &= \gamma_1 u_1 \\ \frac{du_2}{dt} &= -\gamma_2 u_2 + \gamma_3 u_3 \\ \frac{du_3}{dt} &= -a_3 u_3 \end{aligned}\right\} \tag{7.1}$$

where

$$\left.\begin{aligned} \gamma_1 &= \left(a_1 + a_{12}k_2 + \frac{a_{12}a_{23}k_3}{a_{22}} + a_{13}k_3\right) > 0,\ \gamma_2 = \left(a_2 + a_{23}k_3\right) > 0 \\ \gamma_3 &= \left(a_{23}k_2 + \frac{a_{23}^2 k_3}{a_{22}}\right) > 0 \end{aligned}\right\} \tag{7.2}$$

The characteristic equation for which is $\left(\lambda - \gamma_1\right)\left(\lambda + \gamma_2\right)\left(\lambda + a_3\right) = 0$ (7.3)

The characteristic roots of (7.3) are $\gamma_1, -\gamma_2, -a_3$. Since one of these three roots is positive, hence the state is **unstable**. The equations (7.1) yield the solutions.

$$\left.\begin{aligned} u_1 &= u_{10}e^{\gamma_1 t} \\ u_2 &= \left(u_{20} - \gamma\right)e^{-\gamma_2 t} + \gamma e^{-a_3 t} \\ u_3 &= u_{30}e^{-a_3 t} \end{aligned}\right\} \tag{7.4}$$

where $\gamma = \dfrac{\gamma_3 u_{30}}{\gamma_2 - a_3}$ (7.5)

The solution curves are illustrated in Figures (7.1) to (7.4) and the observations are presented below.

Observations

Case (i): When $u_{10} < u_{20} < u_{30}$ and $\gamma_2 < a_3 < \gamma_1$

The commensal (S_1) has the least initial population strength and the host (S_2) has the least natural growth rate. Initially the host (S_3), host (S_2) dominates over the commensal (S_1) till the time instant t_{12}^*, t_{13}^* respectively and thereafter the dominance is reversed. This is shown in Figure 7.1.

Here $t_{13}^* = \dfrac{1}{\gamma_1 + a_3} \ln\left(\dfrac{u_{30}}{u_{10}}\right)$ (7.6)

Caste (ii): When $u_{20} < u_{30} < u_{10}$ and $a_3 < \gamma_1 < \gamma_2$

The host (S_2) has the least initial population strength and the host (S_3) has the least natural growth rate. Initially the host (S_3) dominates over the host (S_2) till the time instant t_{23}^* and thereafter the dominance is reversed as show in Figure

Hence $t_{23}^* = \dfrac{1}{a_3 - \gamma_2} \ln\left(\dfrac{u_{30} - \gamma}{u_{20} - \gamma}\right)$ (7.7)

Case (iii): When $u_{30} < u_{10} < u_{20}$ and $a_3 < \gamma_2 < \gamma_1$

The host (S_3) has the least initial population strength as well as the least natural growth rate. Initially the host (S_2) dominates over the commensal (S_1) till the time instant t_{12}^* and thereafter the dominance is reversed as illustrated in Figure 7.3.

Case (iv): When $u_{10} < u_{30} < u_{20}$ and $\gamma_1 < \gamma_2 < a_3$

The commensal (S_1) has the least initial population strength and the host (S_2) has the least natural growth rate. Initially the host (S_2) dominates over the host (S_3), commensal (S_1) till the time instant t_{23}^*, t_{12}^* respectively and thereafter the dominance is reversed. Also the host (S_3) dominates over the commensal (S_1) till the time instant t_{13}^* and the dominance gets reversed thereafter. This is shown in Figure 7.4.

Trajectories of perturbations

The trajectories in $u_1 - u_2$, $u_1 - u_3$, $u_2 - u_3$ planes are

$$\frac{u_2}{u_{20}} = (1-C)\left(\frac{u_1}{u_{10}}\right)^{\frac{-\gamma_2}{\gamma_1}} + C\left(\frac{u_1}{u_{10}}\right)^{\frac{-a_3}{\gamma_1}}, \left(\frac{u_1}{u_{10}}\right)^{-a_3} = \left(\frac{u_3}{u_{30}}\right)^{\gamma_1},$$

$$\frac{u_2}{u_{20}} = (1-C)\left(\frac{u_3}{u_{30}}\right)^{\frac{\gamma_3}{a_3}} + C\left(\frac{u_3}{u_{30}}\right) \text{ respectively.}$$

where $C = \dfrac{\gamma_3 u_{30}}{(\gamma_2 - a_3)u_{20}}$

7.2. Equilibrium point $E_6 : \overline{N}_1 = k_1 + \dfrac{a_{13}k_3}{a_{11}}, \overline{N}_2 = 0, \overline{N}_3 = k_3$

The corresponding linearized equations for the perturbations u_1, u_2, u_3 are

$$\left.\begin{aligned} \frac{du_1}{dt} &= -\delta_1 u_1 + \delta_2 u_2 + \delta_3 u_3 \\ \frac{du_2}{dt} &= \beta_2 u_2 \\ \frac{du_3}{dt} &= -a_3 u_3 \end{aligned}\right\} \tag{7.8}$$

where $\delta_1 = a_1 + a_{13}k_3 > 0, \delta_2 = a_{12}k_1 + \dfrac{a_{12}a_{13}k_3}{a_{11}} > 0$ (7.9)

$$\delta_3 = a_{13}k_1 + \frac{a_{13}^2\, k_3}{a_{11}} > 0, \beta_2 = a_2 + a_{23}k_3 > 0 \tag{7.10}$$

The characteristic equation for which is $(\lambda + \delta_1)(\lambda - \beta_2)(\lambda + a_3) = 0$ (7.11)

The characteristic roots of (7.11) are $-\delta_1, \beta_2, -a_3$. Since one of these three roots is positive, hence the state is **unstable**. The equations (7.8) yield the solutions.

$$\left.\begin{aligned} u_1 &= (u_{10} - \rho_1 - \rho_2)\, e^{-\delta_1 t} + \rho_1 e^{\beta_2 t} + \rho_2 e^{-a_3 t} \\ u_2 &= u_{20} e^{\beta_2 t} \\ u_3 &= u_{30} e^{-a_3 t} \end{aligned}\right\} \tag{7.12}$$

where $\rho_1 = \dfrac{\delta_2 u_{20}}{\delta_1 + \beta_2}, \rho_2 = \dfrac{\delta_3 u_{30}}{\delta_1 - a_3}$ (7.13)

The solution curves are illustrated in Figures (7.5) to (7.8) and the observations are presented below.

Observations

Case (i): When $u_{10} < u_{30} < u_{20}$ and $\delta_1 < \beta_2 < a_3$

The commensal (S_1) has the least initial population strength and the host (S_3) has the least natural growth rate. Initially the host (S_3) dominates over the commensal (S_1) till the time instant t_{13}^* and thereafter the dominance is reversed as shown in Figure 7.5.

Case (ii): When $u_{20} < u_{10} < u_{30}$ and $\beta_2 < a_3 < \delta_1$

The host (S_2) has the least initial population strength and the host (S_3) has the least natural growth rate. Initially the host (S_3) dominates over the commensal (S_1), host (S_2) till the time instant t_{13}^*, t_{23}^* respectively and thereafter the dominance is reversed as shown in Figure 7.6.

Here $$t_{23}^* = \frac{1}{\beta_2 + a_3} \ln\left(\frac{u_{30}}{u_{20}}\right) \tag{7.14}$$

Case (iii): When $u_{30} < u_{20} < u_{10}$ and $a_3 < \delta_1 < \beta_2$

The host (S_3) has the least initial population strength as well as the least natural growth rate. Initially the commensal (S_1) dominates over the host (S_2) till the time instant t_{12}^* and thereafter the dominance is reversed as illustrated in Figure 7.7.

Case (iv): When $u_{10} < u_{20} < u_{30}$ and $\beta_2 < \delta_1 < a_3$

The commensal (S_1) has the least initial population strength and the host (S_3) has the least natural growth rate. Initially the host (S_3) dominates over the host (S_2), commensal (S_1) till the time instant t_{23}^*, t_{13}^* respectively and thereafter the dominance is reversed. Also the host (S_2) dominates over the commensal (S_1) till the time instant t_{12}^* and the dominance gets reversed thereafter. It is shown in Figure 7.8.

Trajectories of perturbations

The trajectories in $u_1 - u_2$, $u_2 - u_3$, $u_1 - u_3$ planes are

$$\frac{u_1}{u_{10}} = D\left(\frac{u_2}{u_{20}}\right)^{\frac{-\delta_1}{\beta_2}} + \frac{\rho_1}{u_{10}}\left(\frac{u_2}{u_{20}}\right) + \frac{\rho_2}{u_{10}}\left(\frac{u_2}{u_{20}}\right)^{\frac{-a_3}{\beta_2}}, \left(\frac{u_2}{u_{20}}\right)^{-a_3} = \left(\frac{u_3}{u_{30}}\right)^{\beta_2},$$

$$\frac{u_1}{u_{10}} = D.\left(\frac{u_3}{u_{30}}\right)^{\frac{\delta_1}{a_3}} + \frac{\rho_1}{u_{10}}\left(\frac{u_3}{u_{30}}\right)^{\frac{-\beta_2}{a_3}} + \frac{\rho_2}{u_{10}}\left(\frac{u_3}{u_{30}}\right) \text{ respectively.}$$

where $D = \dfrac{u_{10} - \rho_1 - \rho_2}{u_{10}}$

7.3. Equilibrium point $E_7 : \overline{N}_1 = k_1 + \dfrac{a_{12}k_2}{a_{11}}, \overline{N}_2 = k_2, \overline{N}_3 = 0$

The corresponding linearized equations for the perturbations u_1, u_2, u_3 are

$$\left.\begin{aligned} \frac{du_1}{dt} &= -\mu_1 u_1 + \mu_2 u_2 + \mu_3 u_3 \\ \frac{du_2}{dt} &= -a_2 u_2 + a_{23} k_2 u_3 \\ \frac{du_3}{dt} &= a_3 u_3 \end{aligned}\right\} \tag{7.15}$$

$$\left.\begin{array}{l} \text{where } \mu_1 = a_1 + a_{12}k_2 > 0,\ \mu_2 = a_{12}\ k_1 + \dfrac{a_{12}^2 k_2}{a_{11}} > 0 \\ \text{and } \mu_3 = a_{13}\ k_1 + \dfrac{a_{12}a_{13}k_2}{a_{11}} > 0 \end{array}\right\} \tag{17.16}$$

The characteristic equation for which is $(\lambda + \mu_1)(\lambda + a_2)(\lambda - a_3) = 0$ (17.17)

The characteristic roots of (7.17) are $-\mu_1$, $-a_2$, a_3. Since one of these three roots is positive, hence the state is **unstable**.

The equations (7.15) yield the solutions.

$$\left.\begin{array}{l} u_1 = (u_{10} - \delta - \mu)\, e^{-\mu_1 t} + \delta e^{-a_2 t} + \mu e^{a_3 t} \\ u_2 = (u_{20} - \rho)\, e^{-a_2 t} + \rho e^{a_3 t} \\ u_3 = u_{30} e^{a_3 t} \end{array}\right\} \tag{7.18}$$

where $\delta = \dfrac{\mu_2 (u_{30} - \delta)}{\mu_1 - a_2},\ \mu = \dfrac{\rho\mu_2 + \mu_3 u_{30}}{a_3 + \mu_1},\ \rho = \dfrac{a_{23}k_2 u_{30}}{a_2 + a_3}$ (7.19)

The solution curves are illustrated in Figures (7.9) and (7.10) , the observations are presented below.

Observations

Case (i): When $u_{10} < u_{30} < u_{20}$ and $\mu_1 < a_3 < a_2$

The commensal (S_1) has the least initial population strength as well as the least natural growth rate. And the host (S_2) dominates the host (S_3), commensal (S_1) in natural growth rate as well as in its population strength. This is illustrated in Figure 7.9.

Case (ii): When $u_{30} < u_{10} < u_{20}$ and $a_2 < a_3 < \mu_1$

The host (S_3) has the least initial population strength and the host (S_2) has the least natural growth rate. Initially the host (S_2) dominates over the commensal (S_1), host (S_3) till the time instant t_{12}^*, t_{23}^* respectively and thereafter the dominance is reversed as show in Figure 7.10.

Here $t_{23}^* = \dfrac{1}{a_2 + a_3} \ln\left(\dfrac{u_{20} - \rho}{u_{30} - \rho}\right)$ (7.20)

Trajectories of perturbations

The trajectories in u_1-u_3, u_2 - u_3 planes are

$$\frac{u_1}{u_{10}} = D\left(\frac{u_3}{u_{30}}\right)^{\frac{-\mu_1}{a_3}} + \delta\left(\frac{u_3}{u_{30}}\right)^{\frac{-a_2}{a_3}} + \mu\left(\frac{u_3}{u_{30}}\right),\ \frac{u_2}{u_{20}} = \left(\frac{u_{20} - \delta}{u_{20}}\right)\left(\frac{u_2}{u_{30}}\right)^{\frac{-a_2}{a_3}} + \rho\left(\frac{u_3}{u_{30}}\right) \text{ respectively.}$$

where $D = \dfrac{u_{10} - \delta - \mu}{u_{10}}$

8. STABILITY OF THE NORMAL STEADY STATE: $E_8(\overline{N}_1, \overline{N}_2, \overline{N}_3)$

The corresponding linearized equations for the perturbations u_1, u_2, u_3 are

$$\left.\begin{array}{l} \dfrac{du_1}{dt} = -\sigma_1 u_1 + a_{12}\eta_1 u_2 + a_{12}\eta_1 u_3 \\ \dfrac{du_2}{dt} = -\sigma_2 u_2 + \sigma_3 u_3 \\ \dfrac{du_3}{dt} = -a_3 u_3 \end{array}\right\} \tag{8.1}$$

where
$$\left.\begin{aligned} \sigma_1 &= \left(a_1 + a_{12}k_2 + \frac{a_{12}a_{23}k_3}{a_{22}} + a_{13}k_3 \right) > 0 \\ \sigma_2 &= \left(a_2 + a_{23}k_3\right) > 0,\ \sigma_3 = \left(a_{23}k_3 + \frac{a_{23}^2 k_3}{a_{22}} \right) > 0 \end{aligned}\right\} \tag{8.2}$$

The characteristic equation for which is $\left(\lambda+\sigma_1\right)\left(\lambda+\sigma_2\right)\left(\lambda+a_3\right)=0$ (8.3)

The characteristic roots of (8.3) are $-\sigma_1, -\sigma_2, -a_3$. Since all the three roots are negative, hence the normal steady state is **stable**. The equations (8.1) yield the solutions.

$$\left.\begin{aligned} u_1 &= \left[u_{10} - \left(\varphi_2+\varphi_3\right)\right] e^{-\sigma_1 t} + \varphi_2 e^{-\sigma_2 t} + \varphi_3 e^{-a_3 t} \\ u_2 &= \left(u_{20} - \varphi_1\right) e^{-\sigma_2 t} + \varphi_1 e^{-a_3 t} \\ u_3 &= u_{30} e^{-a_3 t} \end{aligned}\right\} \tag{8.4}$$

where $$\varphi_1 = \frac{\sigma_3 u_{30}}{\sigma_2 - a_3},\ \varphi_2 = \frac{a_{12}\eta_1\left(\varphi_1 - u_{20}\right)}{\sigma_2-\sigma_1},\ \varphi_3 = \frac{a_{12}\eta_1\varphi_1 + a_{13}\eta_1 u_{30}}{\sigma_3-\sigma_1} \tag{8.5}$$

It can be noticed that $u_1 \to 0$, $u_2 \to 0$ and $u_3 \to 0$ as $t \to \infty$

The solution curves are illustrated in Figures (8.1) to (8.4) and observations are presented below.

Observations

Case (i): When $u_{10} < u_{20} < u_{30}$ and $\sigma_2 < \sigma_1 < a_3$

The commensal (S_1) has the least initial population strength and the host (S_2) has the least natural growth rate. Initially the host (S_2) dominates over the commensal (S_1) till the time instant t_{12}^* and thereafter the dominance is reversed as shown in Figure 8.1.

Case (ii): When $u_{20} < u_{30} < u_{10}$ and $\sigma_1 < \sigma_2 < a_3$

The host (S_2) has the least initial population strength and the commensal (S_1) has the least natural growth rate. Initially the commensal (S_1) dominates over the host (S_3), host (S_2) till the time instant t_{13}^*, t_{12}^* respectively and thereafter the dominance is reversed as illustrated in Figure 8.2.

Case (iii): When $u_{30} < u_{10} < u_{20}$ and $a_3 < \sigma_2 < \sigma_1$

The host (S_3) has the least initial population strength as well as the least natural growth rate. Initially the host (S_2) dominates over the commensal (S_1) till the time instant t_{12}^* and thereafter the dominance is reversed. This is illustrated in Figure 8.3.

Case (iv): When $u_{10} < u_{30} < u_{20}$ and $\sigma_2 < a_3 < \sigma_1$

The commensal (S_1) has the least initial populations strength and the host (S_2) has the least natural growth rate. Initially the host (S_2) dominates over the host (S_3), commensal (S_1) till the time instant t_{23}^*, t_{12}^* respectively and thereafter the dominance is reversed. Also the host (S_3) dominates over the commensal (S_1) till the time instant t_{13}^* and the dominance gets reversed thereafter. This is shown in Figure 8.4.

Case (v): When $u_{20} < u_{10} < u_{30}$ and $\sigma_2 < \sigma_1 < a_3$

The host (S_2) has the least initial population strength as well as the least natural growth rate. And the host (S_3) dominates the commensal (S_1), host (S_2) in natural growth rate as well as in its population strength. Further all the three species converge asymptotically to the equilibrium point as shown in Figure 8.5.

Trajectories of Perturbations

Trajectories in $u_1 - u_3$, $u_2 - u_3$ planes are

$$\frac{u_1}{u_{10}} = E\left(\frac{u_3}{u_{30}}\right)^{\frac{\sigma_1}{a_3}} + \phi_2\left(\frac{u_3}{u_{30}}\right)^{\frac{\sigma_2}{a_3}} + \phi_3\left(\frac{u_3}{u_{30}}\right), \frac{u_2}{u_{20}} = F\left(\frac{u_3}{u_{30}}\right)^{\frac{\sigma_2}{a_3}} + \phi_1\left(\frac{u_3}{u_{30}}\right) \text{ respectively.}$$

where $E = \dfrac{u_{10} - (\varphi_2 + \varphi_3)}{u_{10}}, F = \dfrac{u_{20} - \varphi_1}{u_{20}}$

9. GRAPHICAL ILLUSTRATION OF THE PERTURBATIONS:

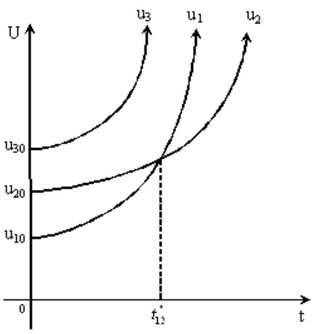

Figure 5.1 Graph of
$u_{10} < u_{20} < u_{30}$; $a_2 < a_1 < a_3$

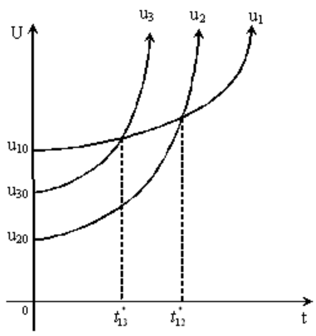

Figure 5.2 Graph of
$u_{20} < u_{30} < u_{10}$; $a_1 < a_2 < a_3$

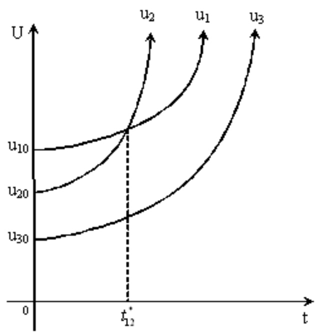

Figure 5.3 Graph of
$u_{30} < u_{20} < u_{10}$; $a_3 < a_1 < a_2$

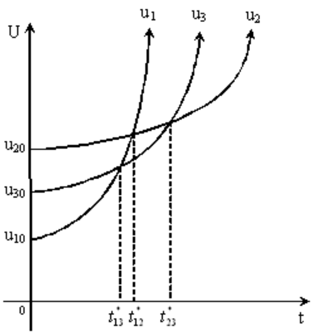

Figure 5.4 Graph of
$u_{10} < u_{30} < u_{20}$; $a_2 < a_3 < a_1$

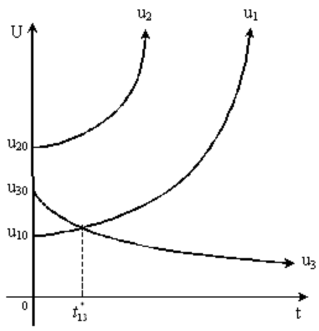

Figure 6.1 Graph of
$u_{10} < u_{30} < u_{20}$; $\alpha_1 < \alpha_2 < a_3$

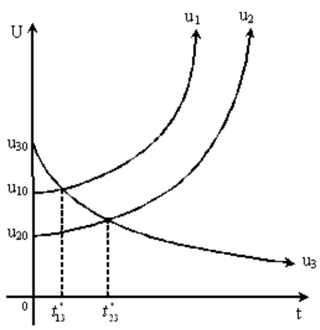

Figure 6.2 Graph of
$u_{20} < u_{10} < u_{30}$; $a_3 < \alpha_2 < \alpha_1$

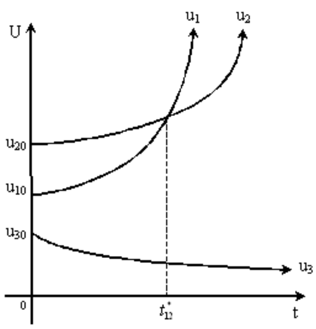

Figure 6.3 Graph of
$u_{30} < u_{10} < u_{20}$; $\alpha_2 < a_3 < \alpha_1$

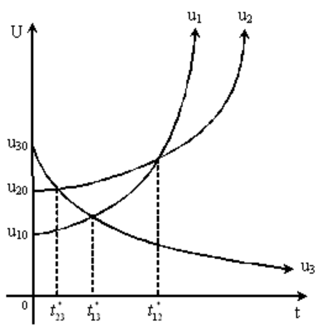

Figure 6.4 Graph of
$u_{10} < u_{20} < u_{30}$; $\alpha_2 < a_3 < \alpha_1$

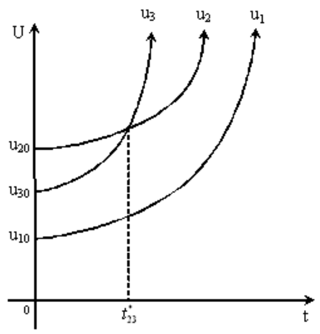

Figure 6.5 Graph of
$u_{10} < u_{30} < u_{20}$; $\beta_1 < a_2 < a_3$

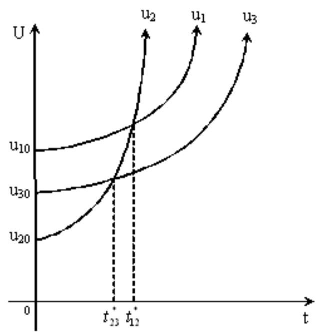

Figure 6.6 Graph of
$u_{20} < u_{30} < u_{10}$; $a_3 < \beta_1 < a_2$

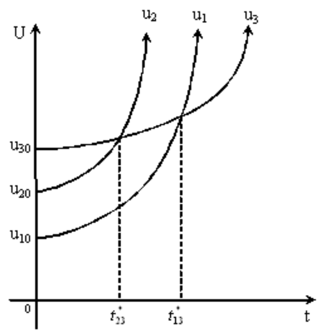

Figure 6.7 Graph of
$u_{10} < u_{20} < u_{30}$; $a_3 < a_1 < a_2$

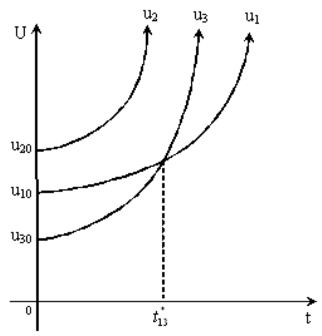

Figure 6.8 Graph of
$u_{30} < u_{10} < u_{20}$; $a_1 < a_3 < a_2$

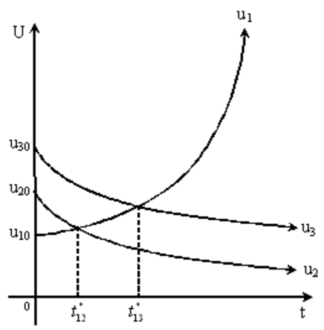

Figure 7.1 Graph of
$u_{10} < u_{20} < u_{30}$; $\gamma_2 < a_3 < \gamma_1$

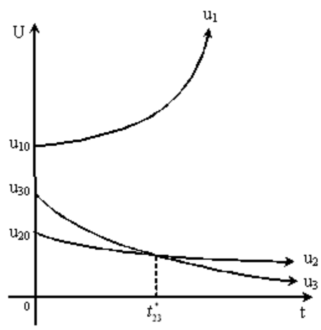

Figure 7.2 Graph of
$u_{20} < u_{10} < u_{30}$; $a_3 < \gamma_1 < \gamma_2$

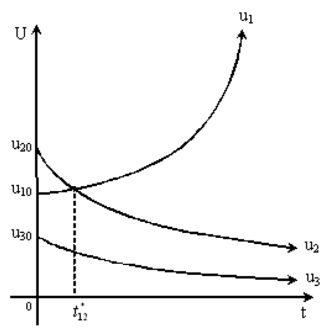

Figure 7.3 Graph of
$u_{30} < u_{10} < u_{20}$; $a_3 < \gamma_2 < \gamma_1$

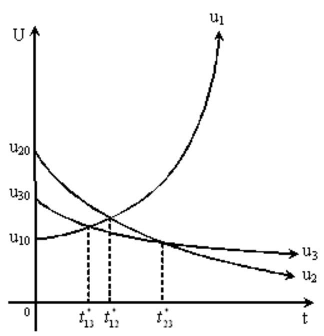

Figure 7.4 Graph of
$u_{10} < u_{30} < u_{20}$; $\gamma_1 < \gamma_2 < a_3$

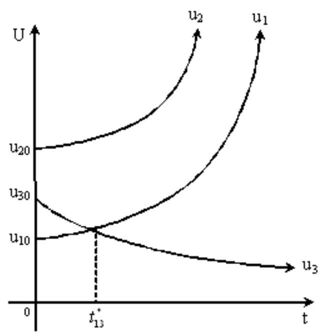

Figure 7.5 Graph of
$u_{10} < u_{30} < u_{20}$; $\delta_1 < \beta_2 < a_3$

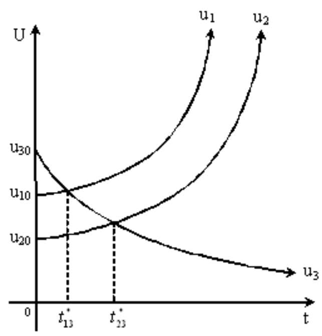

Figure 7.6 Graph of
$u_{20} < u_{10} < u_{30}$; $\beta_2 < a_3 < \delta_1$

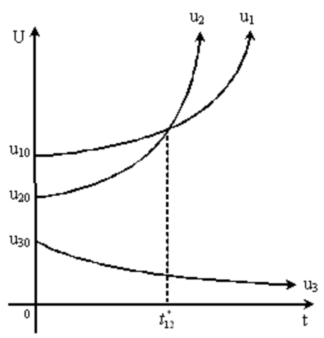

Figure 7.7 Graph of
$u_{30} < u_{20} < u_{10}$; $a_3 < \delta_1 < \beta_2$

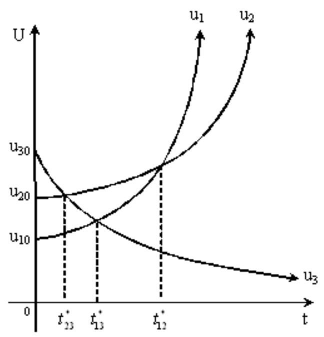

Figure 7.8 Graph of
$u_{10} < u_{20} < u_{30}$; $\beta_2 < \delta_1 < a_3$

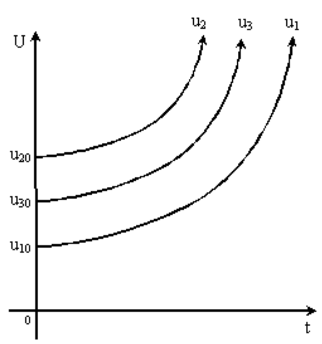

Figure 7.9 Graph of
$u_{10} < u_{30} < u_{20}$; $\mu_1 < a_3 < a_2$

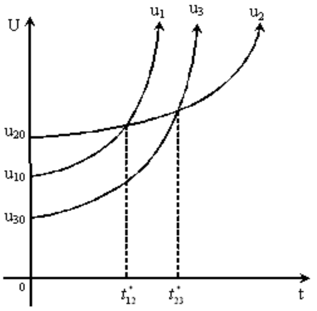

Figure 7.10 Graph of $u_{30} < u_{10} < u_{20}$; $a_2 < a_3 < \mu_1$

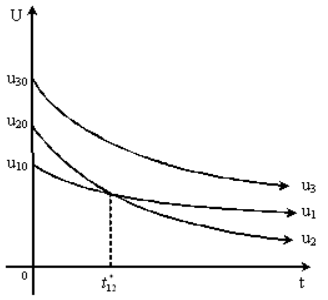

Figure 8.1 Graph of $u_{10} < u_{20} < u_{30}$; $\sigma_2 < \sigma_1 < a_3$

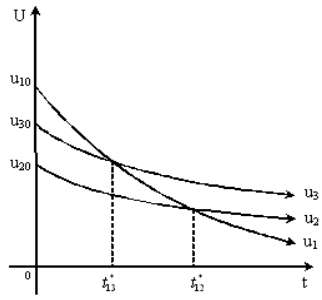

Figure 8.2 Graph of $u_{20} < u_{30} < u_{10}$; $\sigma_1 < \sigma_2 < a_3$

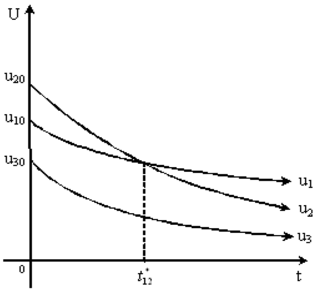

Figure 8.3 Graph of $u_{30} < u_{10} < u_{20}$; $a_3 < \sigma_2 < \sigma_1$

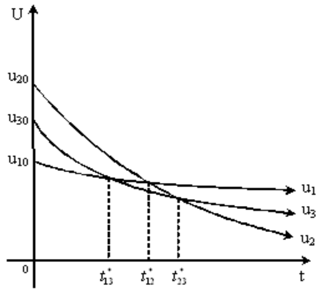

Figure 8.4 Graph of $u_{10} < u_{30} < u_{20}$; $\sigma_2 < a_3 < \sigma_1$

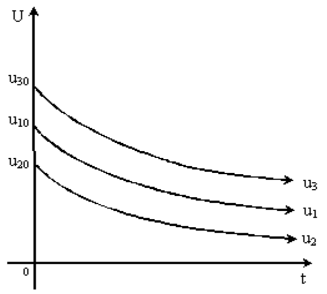

Figure 8.5 Graph of $u_{20} < u_{10} < u_{30}$; $\sigma_2 < \sigma_1 < a_3$

10. LIAPUNOV'S FUNCTION FOR GLOBAL STABILITY:

We discussed the local stability of all eight equilibrium states. From which the co-existent state $E_8\left(\bar{N}_1, \bar{N}_2, \bar{N}_3\right)$ is **stable** and rest of them are **unstable**. We now examine the global stability of dynamical system (2.1), (2.2) and (2.3) by suitable Liapunov's function at this co-existent state.

Theorem. *The equilibrium point* $E_8\left(\bar{N}_1, \bar{N}_2, \bar{N}_3\right)$ *is globally asymptotically stable.*

Proof: Let us consider the following Liapunov's function

$$\left.\begin{aligned} V\left(N_1, N_2, N_3\right) &= N_1 - \bar{N}_1 - \bar{N}_1 \ln\left(\frac{N_1}{\bar{N}_1}\right) + d_1\left[N_2 - \bar{N}_2 - \bar{N}_2 \ln\left(\frac{N_2}{\bar{N}_2}\right)\right] \\ &\quad + d_2\left[N_3 - \bar{N}_3 - \bar{N}_3 \ln\left(\frac{N_3}{\bar{N}_3}\right)\right] \end{aligned}\right\} \tag{10.1}$$

where d_1 and d_2 are suitable constants to be determined as in the subsequent steps.

Now, the time derivative of V, along with solutions of (2.1), (2.2) and (2.3) can be written as

$$\frac{dV}{dt} = \left(\frac{N_1 - \bar{N}_1}{N_1}\right)\frac{dN_1}{dt} + d_1\left(\frac{N_2 - \bar{N}_2}{N_2}\right)\frac{dN_2}{dt} + d_2\left(\frac{N_3 - \bar{N}_3}{N_3}\right)\frac{dN_3}{dt} \tag{10.2}$$

$$\begin{aligned} &= \left(N_1 - \bar{N}_1\right)\left(a_1 - a_{11}N_1 + a_{12}N_2 + a_{13}N_3\right) + d_1\left(N_2 - \bar{N}_2\right)\left(a_2 - a_{22}N_2 + a_{23}N_3\right) \\ &\quad + d_2\left(N_3 - \bar{N}_3\right)\left(a_3 - a_{33}N_3\right) \\ &= \left(N_1 - \bar{N}_1\right)\left(a_{11}\bar{N}_1 - a_{12}\bar{N}_2 - a_{13}\bar{N}_3 - a_{11}N_1 + a_{12}N_2 + a_{13}N_3\right) \\ &\quad + d_1\left(N_2 - \bar{N}_2\right)\left(a_{22}\bar{N}_2 - a_{23}\bar{N}_3 - a_{22}N_2 + a_{23}N_3\right) + d_2\left(N_3 - \bar{N}_3\right)\left(a_{33}\bar{N}_3 - a_{33}N_3\right) \end{aligned}$$

$$=-a_{11}\left(N_1-\bar{N}_1\right)^2+a_{12}\left(N_1-\bar{N}_1\right)\left(N_2-\bar{N}_2\right)+a_{13}\left(N_1-\bar{N}_1\right)\left(N_3-\bar{N}_3\right)$$
$$+d_1\left[-a_{22}\left(N_2-\bar{N}_2\right)^2+a_{23}\left(N_2-\bar{N}_2\right)\left(N_3-\bar{N}_3\right)\right]+d_2\left[-a_{33}\left(N_3-\bar{N}_3\right)^2\right]$$

$$\left.\begin{aligned}\frac{dV}{dt}&=-\left[\sqrt{a_{11}}\left(N_1-\bar{N}_1\right)+\sqrt{d_1a_{22}}\left(N_2-\bar{N}_2\right)+\sqrt{d_2a_{33}}\left(N_3-\bar{N}_3\right)\right]^2\\&+\left(2\sqrt{d_1a_{11}a_{22}}+a_{12}\right)\left(N_1-\bar{N}_1\right)\left(N_2-\bar{N}_2\right)+\left(2\sqrt{d_2a_{11}a_{33}}+a_{13}\right)\left(N_1-\bar{N}_1\right)\left(N_3-\bar{N}_3\right)\\&+\left(2\sqrt{d_1d_2a_{22}a_{33}}+d_1a_{23}\right)\left(N_2-\bar{N}_2\right)\left(N_3-\bar{N}_3\right)\end{aligned}\right\}\tag{10.3}$$

The positive constants d_1 and d_2 as so chosen that, the coefficients of $\left(N_1-\bar{N}_1\right)\left(N_2-\bar{N}_2\right)$, $\left(N_1-\bar{N}_1\right)\left(N_3-\bar{N}_3\right)$ and $\left(N_2-\bar{N}_2\right)\left(N_3-\bar{N}_3\right)$ in (10.3) vanish.

Then we have $d_1=\dfrac{a_{12}^2}{4a_{11}a_{12}}>0$ and $d_2=\dfrac{a_{13}^2}{4a_{11}a_{33}}>0$, with this choice of the constants d_1 and d_2

$$\left.\begin{aligned}V\left(N_1,N_2,N_3\right)&=N_1-\bar{N}_1-\bar{N}_1\ \ln\left(\frac{N_1}{\bar{N}_1}\right)+\frac{a_{12}^2}{4a_{11}a_{22}}\left[N_2-\bar{N}_2-\bar{N}_2\ln\left(\frac{N_2}{\bar{N}_2}\right)\right]\\&+\frac{a_{13}^2}{4a_{11}a_{33}}\left[N_3-\bar{N}_3-\bar{N}_3\ln\left(\frac{N_3}{\bar{N}_3}\right)\right]\end{aligned}\right\}\tag{10.4}$$

$$\frac{dV}{dt}=-\sqrt{a_{11}}\left[\left(N_1-\bar{N}_1\right)+\frac{a_{12}}{2a_{11}}\left(N_2-\bar{N}_2\right)+\frac{a_{13}}{2a_{11}}\left(N_3-\bar{N}_3\right)\right]^2\tag{10.5}$$

which is negative definite, when $2a_{13}a_{22}=a_{12}a_{23}$

Hence, the normal steady state is globally asymptotically stable.

11. NUMERICAL APPROACH OF THE GROWTH RATE EQUATIONS

The numerical solutions of the growth rate equations (2.1), (2.2) and (2.3) computed employing the fourth order Runge - Kutta method for specific values of the parameters that characterize the model and the initial conditions. For this Mat-Lab has been used and the results are illustrated in Figures (11.1) to (11.4).

Consider the model parameter values
a_1=1.2, a_2=0.73, a_3=2.8, a_{12}=0.14, a_{13}=0.47, a_{23}=0.63, k_1=2.4, k_2 =1, k_3=3.5

Case (a): If $N_{i0}<\dfrac{k_i}{2}$, i = 1, 2, 3.

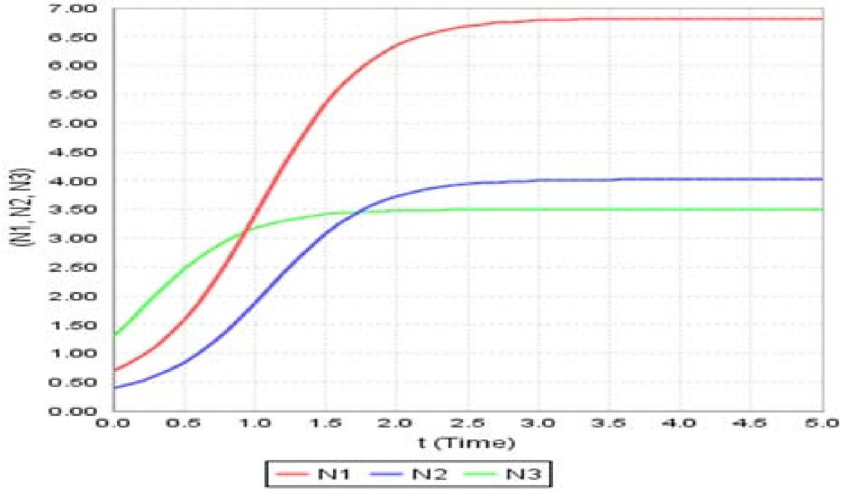

Figure 11.1: Variation of N_1, N_2 and N_3 against time (t) for N_{10}=0.7, N_{20}=0.4, N_{30}=1.3

Case (b): If $N_{i0} > k_i$, i = 1, 2, 3.

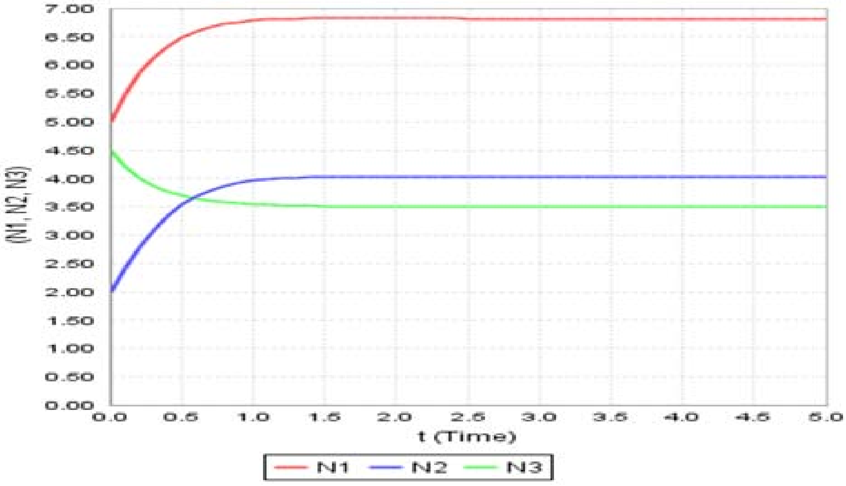

Figure 11.2. Variation of N_1, N_2 and N_3 against time(t) for N_{10}=5,N_{20}=2, N_{30}=4.5

Case (c): If $\frac{k_i}{2} < N_{i0} < k_i$, i = 1, 2, 3.

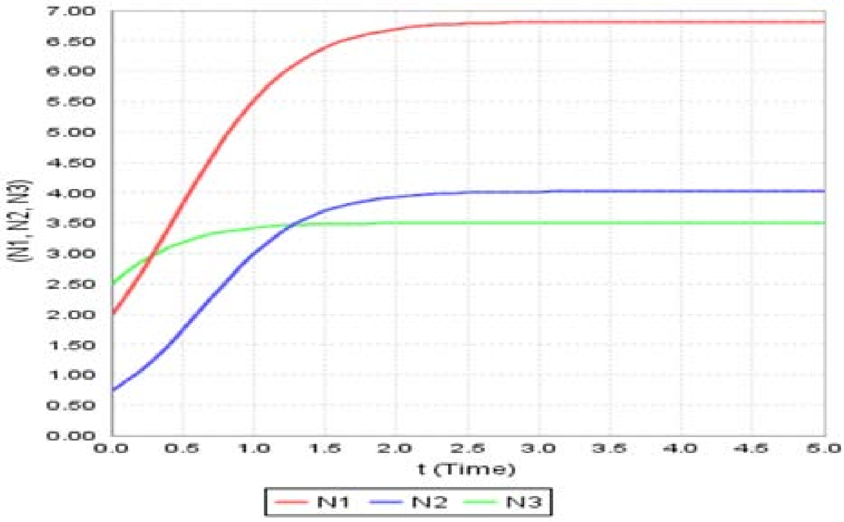

Figure 11.3. Variation of N_1, N_2 and N_3 against time(t) for N_{10}=2,N_{20}=0.75, N_{30}=2.5

Case (d): If $N_{i0} = k_i$, i = 1, 2, 3.

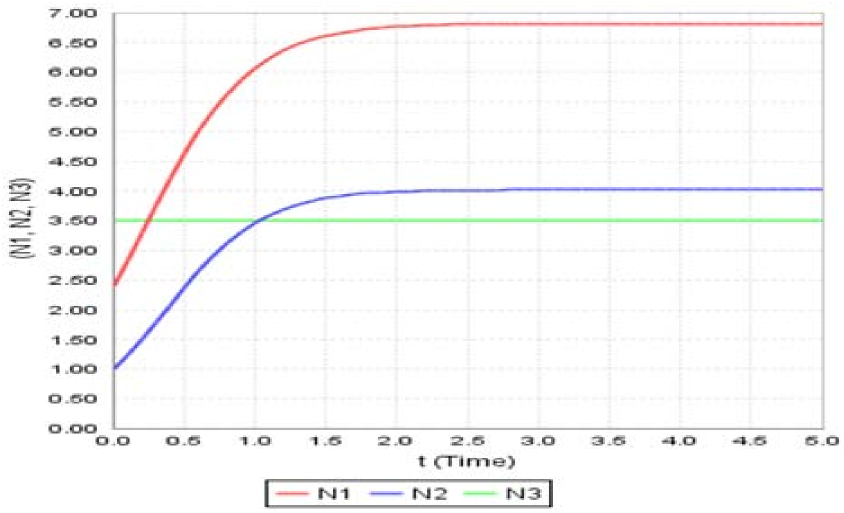

Figure 11.4. Variation of N_1, N_2 and N_3 against time (t) for N_{10}=2.4,N_{20}=1, N_{30}=3.5

12. CONCLUSION:

Investigate some relation-chains between the species such as Prey-Predation, Neutralism, Commensalism, Mutualism, Competition and Ammensalism between three species (S_1, S_2, S_3) with the population relations.

The present paper deals with an investigation on a typical three species syn eco-system. The system comprises of a commensal (S_1), two hosts S_2 and S_3 ie., S_2 and S_3 both benefit S_1, without getting themselves effected either positively or adversely. Further S_2 is a commensal of S_3 and S_3 is a host of both S_1, S_2. It is observed that, in all eight equilibrium states, only the coexistent state is stable and it is globally asymptotically stable when $2a_{13}a_{22} = a_{12}a_{23}$. Furher the numarical solutions for the growth rate equations are computed using Runge-Kutta foutrh order method in four cases.

(a): The initial values of the three species are less than half the respective their carrying capacities.
(b): The initial values of the three species are greater than their respective carrying capacities.
(c): The initial values of the three species are lie between half their respective carrying capacities and its carrying capacities.
(d): The initial values of the three species are equal their respective carrying capacities.

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