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ON THE LOCATION OF ZEROS OF POLYNOMIALS

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ABSTRACT

In this paper we obtain certain generalizations and refinements of well known Enestrom – Kakeya Theorem for a polynomial under much less restrictions on its coefficients.

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1. INTRODUCTION AND STATEMENT OF RESULT:

Let P(z) be a polynomial of degree, such that

$$a_n \ge a_{n-1} \ge \dots \ge a_1 \ge a_0 \ge 0 \tag{1}$$

then according to a famous result due to Enestrom and Kakeya [8] the polynomial P(z) does not vanish in the closed disk $|z| > 1$

Applying this result to the polynomial $P\left(\frac{z}{a}\right)$, we obtain the following more general result:

THEOREM: A If
$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$,
is a polynomial of degree n, such that for some $a > 0$

$$a_n \ge a a_{n-1} \ge \dots \ge a^{n-1} a_1 \ge a^n a_0 > 0 \tag{2}$$

then P(z) does not vanish in $|z| > \frac{1}{a}$. This is a very elegant result but it is equally limited in scope. The hypothesis is very restrictive and does not seem always useful for applications. In the literature ([1], [3]-[7], [9]) there already exist some extensions and generalizations of this result. In connection with Theorem A it was asked by Govil and Rahman [5] that, can we drop the restriction that the coefficients are all positive and instead assume (2) to hold for the moduli of the coefficients? As an answer to this question they proved:

THEOREM: B Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

be a polynomial of degree n with complex coefficients such that for some $a > 0$

$$|a_n| \ge a|a_{n-1}| \ge a^2|a_{n-1}| \ge \dots \ge a^{n-1}|a_1| \ge a^n|a_0| \tag{3}$$

then P(z) has all its zeros in

$$|z| \le \left(\frac{1}{a}\right)k_1$$

where k_1 is the greatest positive root of the trinomial equation.

$$k^{n+1} - 2k^n + 1 = 0$$

Recently Aziz and Zarger [2], have relaxed the hypothesis of Enestrom-Kakeya Theorem in several ways and proved the following results:

THEOREM: C Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

be a polynomial of degree n such that for some $k \ge 1$.

$$k a_n \ge a_{n-1} \ge \dots \ge a_1 \ge a_0 \ge 0, \tag{4}$$

then P(z) has all its zeros in

$$|z + k - 1| \le k$$

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THEOREM: D Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$$

be a polynomial of degree n such that either

$a_n \geq a_{n-2} \geq ... \geq a_3 \geq a_1 \geq 0$

and $a_{n-1} \geq .a_{n-3} \geq ... \geq a_2 \geq a_0 \geq 0$, if n is odd

or

$a_n \geq a_{n-2} \geq ... \geq a_2 \geq a_0 \geq 0$

and $a_{n-1} \geq .a_{n-3} \geq ... \geq a_3 \geq a_1 \geq 0$, if n is even

then all the zeros of P(z) lie in the circle

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq 1 + \frac{a_{n-1}}{a_n}.$$

The following result immediately follows from Theorem C.

THEOREM: E Let P(z) be a polynomial of degree n satisfying (4), then P(z) has all its zeros in

$$|z| \leq 2k - 1$$

We first prove the following result which generalises Theorem B to lacunary polynomials and inparticular show that its conclusion remains valid under much weaker hypothesis:

THEOREM: 1.1 Let

$$P(z) = a_n z^n + a_p z^p + a_{p-1} z^{p-1} + ... + a_1 z + a_0$$

$0 \leq p \leq n-1$, be a polynomial of degree n with complex coefficients such that for some $t > 0$

$$t|a_n| \geq |a_j|, \; j = 0,1,..., p, \qquad 0 \leq p \leq n-1, \tag{5}$$

then P(z) has all its zeros in the closed disk

$$|z| \leq k_1,$$

where k_1 is the greatest positive root of the quadrinomial equation

$$k^{n+1} - k^n - tk^{p+1} + t = 0 \tag{6}$$

the bound (6) is best possible and is attained for the polynomial.

$$P(z) = z^n - t(t^p + z^{p-1} + ... + z + 1)$$

The following result immediately follows from Theorem 1.1, if we taken $p = n - 1$

COROLLARY 1.1 Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0,$$

be a polynomial of degree n with complex coefficients such that for some $t > 0$

$$t|a_n| \geq |a_j|, \qquad j = 0,1,2,..., n-1,$$

then P(z) has all its zeros in

$$|z| \leq k_2$$

where k_2 is the greatest positive root of the trinomial equation

$$k^{n+1} - (t+1)k^n + t = 0$$

Applying Corollary 1.1 to the polynomial $P\left(\frac{z}{a}\right)$, we get the following result which shows Theorem B remains valid under much weaker hypothesis:

COROLLARY: 1.2. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$$

be a polynomial of degree n with complex coefficients, such that for some $a > 0$ and $t > 0$,

$$t|a_n| \geq a^{n-j}|a_j|, \; j = 0,1,..., n-1,$$

then P(z) has all its zeros in

$$|z| \leq \frac{1}{a} k_2,$$

where k_2 is the greatest positive root of the trinomial equation.

$$k^{n+1} - (t+1)k^n + t = 0$$

The following result easily follows from Corollary 1.2 as a special case

COROLLARY 1.3 Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$$

be a polynomial of degree n, with complex coefficients, if for some $a > o$ and $t > 0$

$$t|a_n| \geq a|a_{n-1}| \geq ... \geq a^n|a_0|, \tag{9}$$

then P(z) has all its zeros in $|z| \leq \frac{1}{a} k_2$

where k_2 is the greatest positive root of the trinomial equation.

$$k^{n+1}-(t+1)k^{n}+t=0 \qquad (10)$$

For t = 1, Corollary 1.3 reduces to Theorem B.

Next, we shall present the following generalization of Theorem D.

THEOREM M: 1.2 Let

$$P(z)=a_n z^n+a_{n-1}z^{n-1}+...+a_1 z+a_0$$

be a polynomial of degree n, such that for some k $\geq$ 1, either

$ka_n \geq a_{n-1} \geq ... \geq a_3 \geq a_1 \geq 0$

and $a_{n-1} \geq a_{n-3} \geq ... \geq a_2 \geq a_0 \geq 0$, if n is odd

or

$ka_n \geq a_{n-2} \geq ... \geq a_2 \geq a_0 \geq 0$

and $a_{n-1} \geq a_{n-3} \geq ... \geq a_3 \geq a_1 \geq 0$, if n is even,

then all the zeros of P(z) lie in

$$\left|z+\frac{a_{n-1}}{a_n}\right| \leq (2k-1)+\frac{a_{n-1}}{a_n} \qquad (12)$$

REMARK: 2 For k = 1 Theorem 1.2 reduces to Theorem D. Finally, if we apply Theorem 1.2 to the polynomial P (tz), we get the following more general results:

THEOREM: 1.3. Let

$$P(z)=a_n z^n+a_{n-1}z^{n-1}+...+a_1 z+a_0$$

be a polynomial of degree, such that for some t > 0 and k$\geq$1, either

$$kt^n a_n \geq t^{n-2}a_{n-2} \geq ... \geq a_3 t^3 \geq a_1 t \geq 0$$

and $t^{n-1}a_{n-1} \geq t^{n-3}a_{n-3} \geq ... \geq t^2 a_2 \geq a_0 \geq 0$,

if n is odd or

$$kt^n a_n \geq t^{n-2}a_{n-2} \geq ... \geq a_2 t^2 \geq a_0 t \geq 0$$

and $t^{n-1}a_{n-1} \geq t^{n-3}a_{n-3} \geq ... \geq a^3 t_3 \geq ta_1 \geq 0$,

if n is even then all the zeros of P(z) lie in the circle

$$\left|z+\frac{a_{n-1}}{a_n}\right| \leq t(2k-1)+\frac{a_{n-1}}{a_n}$$

REMARK: 3. Taking k = 1 and t = 1 in Theorem 1.3, we get Theorem D.

PROOFS OF THE THEOREMS:

PROOF OF THEOREM 1.1: We shall prove that

$$P(z)=a_n z^n+a_{n-1}z^{n-1}+...+a_1 z+a_0$$

has all its zeros in $|z| \leq k_1$, where k_1 the greatest positive root of the equation defined by (6). To show this, it is sufficient to consider the case when $(p+1)t>1$. For (p+1) t $\leq$ 1, then on $|z|=R>1$, we have

$$|P(z)|=\left|a_n z^n+a_p z^p+...+a_1 z+a_0\right|$$

$$\geq |a_n|\left\{|z|^n-\left(\left|\frac{a_p}{a_n}\right||z|^p+...+\left|\frac{a_1}{a_n}\right||z|+\left|\frac{a_0}{a_n}\right|\right)\right\}$$

$$\geq |a_n|\left\{R^n-t(R^n+...+R+1)\right\}$$

$$=|a_n|(R^n-t(p+1)R^p)$$

$$\geq |a_n|(R^n-R^p)>0,\ for\ 0\leq p\leq n-1$$

So we assume t (p+1)>1. In this case it can be easily seen that k_1>1, where k_1 is the greatest positive root of the equation defined by (6). Now for $|z|=R>1$, we have

$$|P(z)| \geq |a_n||z|^n\left\{1-\left|\frac{a_p}{a_n}\frac{1}{z^{n-p}}+\frac{a_{p-1}}{a_n}\frac{1}{z^{n-p+1}}+...+\frac{a_0}{a_n}\frac{1}{z^n}\right|\right\}$$

$$\geq |a_n||z|^n\left\{1-\left(\left|\frac{a_p}{a_n}\right|\frac{1}{|z|^{n-p}}+\left|\frac{a_{p-1}}{a_n}\right|\frac{1}{|z|^{n-p+1}}+...+\left|\frac{a_0}{a_n}\right|\frac{1}{|z|^n}\right)\right\}$$

$$\geq |a_n||z|^n\left\{1-\frac{t}{|z|^{n-p}}\left(1+\frac{1}{|z|}+...+\frac{1}{|z|^p}\right)\right\}$$

$$\geq |a_n||z|^n\left\{1-\frac{t}{R^{n-p}}\left(1+\frac{1}{R}+...+\frac{1}{R^p}\right)\right\}$$

$$=|a_n||z|^n\left\{1-\frac{t}{R^{n-p}}\frac{(R^{p+1}-1)}{R^p(R-1)}\right\}$$

$$=|a_n||z|^n\left\{1-\frac{t}{R^n}\frac{(R^{p+1}-1)}{(R-1)}\right\}$$

>0,

if

$$R^{n+1}-R^n-tR^{p+1}+t>0.$$

This implies that

$\left|P(z)\right| > 0,\ for\ \left|z\right| > k_1$, where k_1 (>1) is the greatest positive root of the equation.

$$k^{n+1} - k^n - tk^{p+1} + t = 0.$$

Hence all the zeros of P(z) whose modulus is greater than 1 lie in $\left|z\right| \le k_1$, where k_1 is the greatest positive root of the equation defined by (6). Since all those zeros whose modulus is less than or equal to 1 already lie in $\left|z\right| \le k_1$. The desired result follows immediately.

PROOF OF THEOREM: 1.2 Consider

$$F(z) = (1 - z^2)P(z)$$

$$= -a_n z^{n+2} - a_{n-1}z^{n+1} + (a_n - a_{n-2})z^n + (a_{n-1} - a_{n-3})z^{n-1} + \ldots + (a_3 - a_1)z^3 + (a_2 - a_0)z^2 + a_1 z + a_0$$

For $\left|z\right| > 1$, we have

$$\left|F(z)\right| = \Big| -a_n z^{n+2} - a_{n-1}z^{n+1} - ka_n z^n + a_n z^n + (ka_n - a_{n-2})z^n + (a_{n-1} - a_{n-3})z^{n-1} + \ldots + (a_3 - a_1)z^3 + (a_2 - a_0)z^2 + a_1 z + a_0 \Big|$$

$$\ge \left|z\right|^{n+1}\Big\{\left|a_n z + a_{n-1}\right| - \Big|(k-1)a_n\frac{1}{z} + (ka_n - a_{n-2})\frac{1}{z^2} + (a_{n-1} - a_{n-3})\frac{1}{z^3} + \ldots + (a_3 - a_1)\frac{1}{z^{n-3}} + (a_2 - a_0)\frac{1}{z^{n-1}} + a_1\frac{1}{z^n} + a_0\frac{1}{z^{n+1}}\Big|\Big\}$$

$$\ge \left|z\right|^{n+1}\left\{\left|a_n z + a_{n-1}\right| - \left\{\begin{array}{l} \left|k-1\right|a_n\frac{1}{\left|z\right|} + \left|ka_n - a_{n-2}\right|\frac{1}{\left|z\right|^2} \\ + \left|a_{n-1} - a_{n-3}\right|\frac{1}{\left|z\right|^3} + \ldots + \left|a_3 - a_1\right|\frac{1}{\left|z\right|^{n-3}} \end{array}\right. + \left|a_2 - a_0\right|\frac{1}{\left|z\right|^{n-1}} + a_1\frac{1}{\left|z\right|^n} + a_0\frac{1}{\left|z\right|^{n+1}}\right\}\Bigg\}$$

$$> \left|z\right|^{n+1}\left\{\left|a_n z + a_{n-1}\right| - \left(\begin{array}{l}(k-1)a_n + (ka_n - a_{n-2}) \\ + (a_{n-1} - a_{n-3}) + \ldots + (a_3 - a_1)\end{array}\right.\right. + (a_2 - a_0) + a_1 + a_0\Big)\Big\}$$

$$= \left|z\right|^{n+1}\left\{\left|a_n z + a_{n-1}\right| - \left((k-1)a_n + ka_n + a_{n-1}\right)\right\}$$

> 0,

if

$$\left|a_n z + a_{n-1}\right| > (2k-1)a_n + a_{n-1}$$

or

$$\left|z + \frac{a_{n-1}}{a_n}\right| > (2k-1) + \frac{a_{n-1}}{a_n}$$

therefore all the zeros of F(z) whose modulus is greater than 1 lie in

$$\left|z + \frac{a_{n-1}}{a_n}\right| \le (2k-1) + \frac{a_{n-1}}{a_n}$$

but those zeros of F(z) whose modulus is less or equal to 1 already satisfy the inequality. Since all the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in the circle defined by (12). This proves Theorem 2.1 completely.

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