

Bilateral Sequence Spaces  $c_0(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  and  $\ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  defined by Orlicz Function

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ABSTRACT

In this paper, we construct new sequence spaces  $c_0(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  and  $\ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  by using Orlicz function  $M$ . We also examine some of the properties like containment, linearity and completeness etc of these newly constructed sequence spaces.

**Keywords:** Bilateral Sequence, Sequence Space, Paranormed space, Orlicz function.

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1. INTRODUCTION

By a bilateral sequence, we mean a function whose domain is the set  $\mathbb{Z}$  of all integers with natural ordering. The utility of bilateral sequences can be found in [7] and [8]. We will denote a bilateral sequence by the symbol  $(a_k)_{-\infty}^{\infty}$  or  $\bar{a} = (a_k)_{-\infty}^{\infty}$ . As usual, by the convergence of the bilateral series  $\sum_{-\infty}^{\infty} a_k$  to  $s$  written as  $\sum_{-\infty}^{\infty} a_k = s$ , we shall mean the convergence of the sequence  $(S_n)_{n=1}^{\infty}$  to  $s$  where  $S_n = \sum_{-n}^n a_k$  is called  $n$ -th partial sum of the bilateral series  $\sum_{-\infty}^{\infty} a_k$ .

Again, let  $M$  be the Orlicz function. The definition of Orlicz function and Orlicz sequence spaces are as follows:

**Definition 1.1:** An Orlicz function  $M: [0, \infty) \rightarrow [0, \infty)$  is a continuous, non-decreasing and convex function defined for  $t \geq 0$  such that

- (i)  $M(x) > 0$  for  $x > 0$ ;
- (ii)  $M(0) = 0$  and
- (iii)  $\lim_{t \rightarrow \infty} M(t) = \infty$ .

An Orlicz function  $M$  can always be represented in the following integral form (see [1])

$$M(x) = \int_0^x p(t) dt$$

where  $p$  is known as the kernel of  $M$ , is right differentiable for  $t \geq 0$ ,  $p(0) = 0$ ,  $p(t) > 0$  for  $t > 0$ ,  $p$  is non-decreasing and  $p(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

**Definition 1.2:** Lindenstrauss and Tzafriri (see [1], [3], [4] and [5]) used the ideas of Orlicz function to construct the sequence space,

$$l_M = \left\{ x \in \omega: \sum_1^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

The space  $l_M$  with the norm

$$\|x\| = \inf \left\{ \rho > 0: \sum_1^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space.

Now let  $\bar{p} = (p_k)_{-\infty}^{\infty}$  and  $\bar{q} = (q_k)_{-\infty}^{\infty}$  be bilateral sequences of strictly positive real numbers and  $\bar{\lambda} = (\lambda_k)_{-\infty}^{\infty}$  and  $\bar{\mu} = (\mu_k)_{-\infty}^{\infty}$  be bilateral sequences of non-zero complex numbers and  $M$  be an Orlicz function. Now we introduce the following classes of Banach space  $X$ -valued bilateral sequences:

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$$c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p}) = \{ \bar{x} = (x_k)_{-\infty}^{\infty} : x_k \in X, k \in \mathbb{Z}, \text{ and } \left( M\left(\frac{\|\lambda_k x_k\|}{\rho}\right) \right)^{p_k} \rightarrow 0 \text{ as } k \rightarrow -\infty, \text{ as well as } k \rightarrow \infty \text{ for some } \rho > 0 \}.$$

$$\ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p}) = \{ \bar{x} = (x_k)_{-\infty}^{\infty} : x_k \in X, k \in \mathbb{Z}, \text{ and } \sum_{-\infty}^{\infty} \left( M\left(\frac{\|\lambda_k x_k\|}{\rho}\right) \right)^{p_k} < \infty, \text{ for some } \rho > 0 \}.$$

Throughout the paper we denote  $t_k = \left| \frac{\lambda_k}{\mu_k} \right|$ .

**Definition 1.3:** Let  $X$  be a linear space. A mapping  $g: X \rightarrow \mathbb{R}$  is called a paranorm if it satisfies

- (i)  $g(\theta) = 0$ ;
- (ii)  $g(x) = g(-x)$ ;
- (iii)  $g(x + y) \leq g(x) + g(y)$ ;
- (iv) if  $(\alpha_n)$  is a sequence of scalars with  $\alpha_n \rightarrow \alpha$  and  $(x_n)$  is a sequence in  $X$  with  $g(x_n - x) \rightarrow 0$  then  $g(\alpha_n x_n - \alpha x) \rightarrow 0$  (continuity of scalar multiplication). The paranorm is called total if
- (v)  $g(x) = 0$  implies  $x = 0$ , see [9].

In this paper our aim is to investigate results concerning the above defined classes with the help of Orlicz function  $M$ .

## 2. CONTAINMENT

**Lemma 2.1:**  $c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p}) \subset c_o(\mathbb{Z}, X, M, \bar{\mu}, \bar{p})$  if and only if

$$\lim_{k \rightarrow -\infty} \inf_k t_k > 0 \text{ and } \lim_{k \rightarrow \infty} \inf_k t_k > 0 \text{ with } l = \inf_k p_k \leq p_k.$$

**Proof:** For the sufficiency of the condition suppose that  $\lim_{k \rightarrow -\infty} \inf_k t_k > 0$  and  $\lim_{k \rightarrow \infty} \inf_k t_k > 0$  and  $\bar{x} = (x_k)_{-\infty}^{\infty} \in c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ . Then there exists a real number  $m > 0$  such that  $m < \left| \frac{\lambda_k}{\mu_k} \right|$  for all sufficiently large values of  $|k|$ . Thus  $m \|\mu_k x_k\| < \|\lambda_k x_k\|$ , for all sufficiently large values of  $|k|$ . Also  $\bar{x} \in c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  so we can find some  $\rho_1 > 0$  such that  $\left( M\left(\frac{\|\lambda_k x_k\|}{\rho_1}\right) \right)^{p_k} \rightarrow 0$ . Let us choose  $\rho$  such that  $\rho_1 < m\rho$ . Since  $M$  is non-decreasing, we have

$$\left( M\left(\frac{\|\mu_k x_k\|}{\rho}\right) \right)^{p_k} \leq \left( M\left(\frac{\|\lambda_k x_k\|}{m\rho}\right) \right)^{p_k} < \left( M\left(\frac{\|\lambda_k x_k\|}{\rho_1}\right) \right)^{p_k} \rightarrow 0$$

and hence  $\bar{x} \in c_o(\mathbb{Z}, X, M, \bar{\mu}, \bar{p})$  and hence  $c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p}) \subset c_o(\mathbb{Z}, X, M, \bar{\mu}, \bar{p})$ .

For the necessity, let  $c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p}) \subset c_o(\mathbb{Z}, X, M, \bar{\mu}, \bar{p})$  but either  $\lim_{k \rightarrow -\infty} \inf t_k = 0$  or  $\lim_{k \rightarrow \infty} \inf t_k = 0$ . Let us take  $\lim_{k \rightarrow -\infty} \inf t_k = 0$ . Then there exists a sequence  $(k(n))$  such that  $k(n+1) > k(n) \geq 1, n \geq 1$ , for which  $n^2 |\lambda_{k(n)}| < |\mu_{k(n)}|$ . Now the bilateral sequence  $\bar{x} = (x_k)_{-\infty}^{\infty}$  defined by

$$x_k = \begin{cases} \lambda_{k(n)}^{-1} n^{-1} z & \text{if } k = k(n), n \geq 1, \text{ and} \\ \theta, & \text{otherwise} \end{cases}$$

where  $z \in X$  and  $\|z\| = 1$ . Then  $\|\lambda_{k(n)} x_{k(n)}\| = \frac{1}{n}$ . Which implies that  $\|\lambda_{k(n)} x_{k(n)}\| \rightarrow 0$  as  $n \rightarrow \infty$ .

Therefore  $\left( M\left(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}\right) \right)^l \rightarrow 0$  as  $n \rightarrow \infty$  for any fixed  $l$ .

Hence

$$\left( M\left(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}\right) \right)^{p_{k(n)}} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ since } l \leq p_k$$

i.e.,  $\bar{x} \in c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ . But  $\|\mu_{k(n)} x_{k(n)}\| > n$ , implies that

$$\left( M\left(\frac{\|\mu_{k(n)} x_{k(n)}\|}{\rho}\right) \right)^{p_{k(n)}/l} > 1 \text{ for all } n \geq 1 \text{ and for some fixed } l \leq p_k,$$

or  $\left( M\left(\frac{\|\mu_{k(n)} x_{k(n)}\|}{\rho}\right) \right)^{p_{k(n)}} > 1$  for all  $n \geq 1$  and for some  $\rho$ .

Which implies  $\bar{x} \notin c_o(\mathbb{Z}, X, M, \bar{\mu}, \bar{p})$ , a contradiction.

Similar proof can be given in the case when  $\lim_{k \rightarrow -\infty} \inf_k t_k > 0$ . This completes the proof.

**Lemma 2.2:**  $c_o(\mathbb{Z}, X, M, \bar{\mu}, \bar{p}) \subset c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  if and only if  $\lim_{k \rightarrow -\infty} \sup_k t_k < \infty$  and  $\lim_{k \rightarrow \infty} \sup_k t_k < \infty$  with  $l = \inf_k p_k \leq p_k$ .

**Proof:** Sufficiency is straightforward. On the other hand for the necessity suppose that  $(\mathbb{Z}, X, M, \bar{\mu}, \bar{p}) \subset c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  but either  $\lim_{k \rightarrow -\infty} \sup_k t_k = \infty$  or  $\lim_{k \rightarrow \infty} \sup_k t_k = \infty$ . Let  $\lim_{k \rightarrow \infty} \sup_k t_k = \infty$ . Then there exists a sequence  $(k(n))$ ,  $k(n) \geq 1$  such that for each  $n \geq 1$ ,  $|\lambda_{k(n)}| > n^2 |\mu_{k(n)}|$ . Now define the bilateral sequence  $\bar{x} = (x_k)_{-\infty}^{\infty}$  by

$$x_k = \begin{cases} \mu_{k(n)}^{-1} n^{-1} z & \text{if } k = k(n), n \geq 1 \text{ and,} \\ \theta, & \text{otherwise} \end{cases}$$

where  $z \in X$  and  $\|z\| = 1$ . Then  $\|\mu_{k(n)} x_{k(n)}\| = \frac{1}{n}$   $n \geq 1$  and  $\|\mu_k x_k\| = 0$ , otherwise. This implies that  $\frac{\|\mu_{k(n)} x_{k(n)}\|}{\rho} \rightarrow 0$  as  $n \rightarrow \infty$  for some  $\rho$ . Thus by the property of Orlicz function, we have  $\left(M\left(\frac{\|\mu_{k(n)} x_{k(n)}\|}{\rho}\right)\right)^l \rightarrow 0$  as  $n \rightarrow \infty$  for some  $\rho$  and for any fixed  $l$ .

Therefore  $\left(M\left(\frac{\|\mu_{k(n)} x_{k(n)}\|}{\rho}\right)\right)^{p_k} \rightarrow 0$  as  $n \rightarrow \infty$  since  $l = \inf p_k < p_k$ . This shows that  $\bar{x} \in c_o(\mathbb{Z}, X, M, \bar{\mu}, \bar{p})$ . But  $\|\lambda_{k(n)} x_{k(n)}\| > n$  implies that  $\|\lambda_{k(n)} x_{k(n)}\| \rightarrow \infty$  as  $n \rightarrow \infty$   $\left(M\left(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}\right)\right)^{p_k/l} \rightarrow \infty$  for arbitrary large  $n$  and  $l \leq p_k$ . This shows that  $\left(M\left(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}\right)\right)^{p_k} \rightarrow \infty$  as  $n \rightarrow \infty$ ; i.e.,  $\bar{x} \notin c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ , which is a contradiction.

Similar proof can be given in the case when  $\lim_{k \rightarrow -\infty} \sup_k t_k = \infty$ . This completes the proof.

On combining Lemma 2.1 and Lemma 2.2, we get the following theorem:

**Theorem 2.3:** For  $l = \inf p_k \leq p_k$ ,  $c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p}) = c_o(\mathbb{Z}, X, M, \bar{\mu}, \bar{p})$  if and only if

$$\begin{aligned} 0 < \lim_{k \rightarrow -\infty} \inf t_k \leq \lim_{k \rightarrow -\infty} \sup t_k < \infty \text{ and,} \\ 0 < \lim_{k \rightarrow \infty} \inf t_k \leq \lim_{k \rightarrow \infty} \sup t_k < \infty. \end{aligned}$$

**Corollary 2.4:** For  $l = \inf p_k \leq p_k$ ,

(i)  $c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p}) \subset c_o(\mathbb{Z}, X, M, \bar{\mu}, \bar{p})$  if and only if  $\lim_{k \rightarrow -\infty} \inf |\lambda_k|^{p_k} > 0$  and  $\lim_{k \rightarrow \infty} \inf |\lambda_k|^{p_k} > 0$ ;

(ii)  $c_o(\mathbb{Z}, X, M, \bar{\mu}, \bar{p}) \subset c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  if and only if  $\lim_{k \rightarrow -\infty} \sup |\lambda_k|^{p_k} < \infty$  and  $\lim_{k \rightarrow \infty} \sup |\lambda_k|^{p_k} < \infty$ ;

(iii)  $c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p}) = c_o(\mathbb{Z}, X, M, \bar{\mu}, \bar{p})$  if and only if  $0 < \lim_{k \rightarrow -\infty} \inf |\lambda_k|^{p_k} \leq \lim_{k \rightarrow -\infty} \sup |\lambda_k|^{p_k} < \infty$  and  $0 < \lim_{k \rightarrow \infty} \inf |\lambda_k|^{p_k} \leq \lim_{k \rightarrow \infty} \sup |\lambda_k|^{p_k} < \infty$ .

**Proof:**

(i) Take  $\mu_k = 1$ , for all  $k$  in Lemma 2.1,

(ii) Take  $\mu_k = 1$ , for all  $k$  in Lemma 2.2,

(iii) Take  $\mu_k = 1$ , for all  $k$  in Theorem 2.3.

**Lemma 2.5:**  $c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p}) \subset c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{q})$  if and only if

$$\lim_{k \rightarrow -\infty} \inf \frac{q_k}{p_k} > 0 \text{ and } \lim_{k \rightarrow \infty} \inf \frac{q_k}{p_k} > 0 \text{ with } l = \inf_k p_k \leq p_k.$$

**Proof:** For the sufficiency condition, suppose  $\lim_{k \rightarrow -\infty} \inf \frac{q_k}{p_k} > 0$  and

$\lim_{k \rightarrow \infty} \inf \frac{q_k}{p_k} > 0$  and  $\bar{x} = (x_k)_{-\infty}^{\infty} \in c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ . Then there exists a real number  $m > 0$  such that  $q_k > mp_k$  for all sufficiently large values of  $|k|$ . Further since  $\bar{x} \in c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  we have  $\left(M\left(\frac{\|\lambda_k x_k\|}{\rho}\right)\right)^{p_k} < 1$ , for all sufficiently

large values of  $|k|$  and hence  $(M(\frac{\|\lambda_k x_k\|}{\rho}))^{q_k} < [(M(\frac{\|\lambda_k x_k\|}{\rho}))^{p_k}]^m < 1$ , for all sufficiently large values of  $|k|$ . This implies that  $\bar{x} \in c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{q})$  and hence  $c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p}) \subset c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{q})$ .

For the necessity of the condition, suppose that inclusion holds but either  $\lim_{k \rightarrow -\infty} \inf \frac{q_k}{p_k} = 0$  or  $\lim_{k \rightarrow \infty} \inf \frac{q_k}{p_k} = 0$ . Here we prove the result for the case when  $\lim_{k \rightarrow \infty} \inf \frac{q_k}{p_k} = 0$ , then there exists a sequence  $(k(n)), k(n) \geq 1$  such that for each  $n \geq 1$   $nq_{k(n)} < p_{k(n)}$ . Now taking  $z \in X$  and  $\|z\| = 1$  for the bilateral sequence  $\bar{x} = (x_k)_{-\infty}^{\infty}$  defined by

$$x_k = \begin{cases} \lambda_{k(n)}^{-1} n^{-1} z & \text{if } k = k(n), n \geq 1 \text{ and,} \\ \theta, & \text{otherwise} \end{cases}$$

Then  $\bar{x} \in c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  as  $\|\lambda_{k(n)} x_{k(n)}\| = \frac{1}{n}$ . This implies that  $\|\lambda_{k(n)} x_{k(n)}\| \rightarrow 0$  as  $n \rightarrow \infty$ . Thus by the definition of Orlicz function, we have

$$M(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for some } \rho > 0$$

or  $(M(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}))^l \rightarrow 0$  as  $n \rightarrow \infty$  for some  $\rho$  and some fixed  $l$ .

or  $(M(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}))^{p_{k(n)}} \leq (M(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}))^l$  since  $l \leq p_{k(n)}$ .

This implies that  $(M(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}))^{p_{k(n)}} \rightarrow 0$  as  $n \rightarrow \infty$  for some  $\rho$ . Therefore  $\bar{x} = (x_k)_{-\infty}^{\infty} \in c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ . But

$$\left(M(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho})\right)^{p_{k(n)}} < \left(M(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho})\right)^{nq_{k(n)}}$$

Now we can choose some  $\rho_1 > \rho$  such that  $(M(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho_1}))^{p_{k(n)}}$  does not converge to zero.

Therefore  $(M(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}))^{nq_{k(n)}}$  does not converge to zero for some  $\rho_1 > \rho$  which shows that  $\bar{x} \notin c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ , a contradiction.

Similar proof can be given for the case when  $\lim_{k \rightarrow -\infty} \inf \frac{q_k}{p_k} = 0$ . This completes the proof.

**Lemma 2.6:**  $c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{q}) \subset c_o(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  if and only if

$$\limsup_{k \rightarrow -\infty} \frac{q_k}{p_k} < \infty \text{ and } \limsup_{k \rightarrow \infty} \frac{q_k}{p_k} < \infty$$

with  $l = \inf_k q_k \leq q_k$ .

**Proof:** Sufficiency is straightforward. On the other hand for the necessity, let the inclusion holds but  $\lim_{k \rightarrow -\infty} \sup \frac{q_k}{p_k} = \infty$  or  $\lim_{k \rightarrow \infty} \sup \frac{q_k}{p_k} = \infty$ . Here we prove the result for the case when  $\lim_{k \rightarrow \infty} \sup \frac{q_k}{p_k} = \infty$  then there exists a sequence  $(k(n)), k(n) \geq 1$  such that  $q_{k(n)} > np_{k(n)}$  for all  $n \geq 1$ . Thus, the bilateral sequence  $\bar{x} = (x_k)_{-\infty}^{\infty}$  defined by

$$x_k = \begin{cases} \lambda_{k(n)}^{-1} n^{-1} z & \text{if } k = k(n), n \geq 1 \text{ and,} \\ \theta, & \text{otherwise} \end{cases}$$

where  $z \in X, \|z\| = 1$ . We see that,  $\|\lambda_{k(n)} x_{k(n)}\| \rightarrow 0$  as  $n \rightarrow \infty$ . Thus by the definition of Orlicz function we have

$(M(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}))^l \rightarrow 0$  as  $n \rightarrow \infty$  for some  $\rho$  and some fixed  $l$  and

$(M(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}))^{q_{k(n)}} \leq (M(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}))^l$ , since  $l \leq q_{k(n)}$ .

Therefore  $(M(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}))^{q_{k(n)}} \rightarrow 0$  for some  $\rho$  implies that  $\bar{x} \in c_o(M, X, \lambda, q)$ . But  $(M(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}))^{np_{k(n)}}$  will not surely converge to zero for each  $n \geq 1$  as

$$\left(M(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho})\right)^{q_{k(n)}} < \left(M(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho})\right)^{np_{k(n)}}$$

and we can choose some  $\rho_1 < \rho$  such that  $\left(M\left(\frac{\|\lambda_{k(n)}x_{k(n)}\|}{\rho_1}\right)\right)^{q_{k(n)}} \rightarrow \infty$ . Therefore

$$\left(M\left(\frac{\|\lambda_{k(n)}x_{k(n)}\|}{\rho}\right)\right)^{q_{k(n)}} < \left(M\left(\frac{\|\lambda_{k(n)}x_{k(n)}\|}{\rho_1}\right)\right)^{q_{k(n)}} < \left(M\left(\frac{\|\lambda_{k(n)}x_{k(n)}\|}{\rho_1}\right)\right)^{np_{k(n)}}$$

Implies  $\left(M\left(\frac{\|\lambda_{k(n)}x_{k(n)}\|}{\rho}\right)\right)^{p_{k(n)}}$  will not converge to zero for some  $\rho > \rho_1$ . Therefore  $\bar{x} \notin c_0(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ , which is a contradiction.

Similar proof can be given for the case when  $\lim_{k \rightarrow \infty} \sup \frac{q_k}{p_k} = \infty$ . This completes the proof.

On combining Lemma 2.5 and Lemma 2.6 we get the following theorem:

**Theorem 2.7:**  $c_0(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p}) = c_0(\mathbb{Z}, X, M, \bar{\lambda}, \bar{q})$

$0 < \lim_{k \rightarrow \infty} \inf \frac{q_k}{p_k} < \lim_{k \rightarrow \infty} \sup \frac{q_k}{p_k} < \infty$ , and  $0 < \lim_{k \rightarrow \infty} \inf \frac{q_k}{p_k} < \lim_{k \rightarrow \infty} \sup \frac{q_k}{p_k} < \infty$ .

**Lemma 2.8:**  $\ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p}) \subset \ell(\mathbb{Z}, X, M, \bar{\mu}, \bar{p})$  if and only if

$$\lim_{k \rightarrow \infty} \inf t_k > 0 \text{ and } \lim_{k \rightarrow \infty} \inf t_k > 0.$$

**Proof:** Suppose  $\lim_{k \rightarrow \infty} \inf t_k > 0$  and  $\lim_{k \rightarrow \infty} \inf t_k > 0$  and  $\bar{x} = (x_k)_{-\infty}^{\infty} \in \ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ . Then there exists a real number  $m > 0$  such that  $m|\mu_k| < |\lambda_k|$  for all sufficiently large values of  $|k|$ . Thus  $m\|\mu_k x_k\| < \|\lambda_k x_k\|$  for all sufficiently large values of  $|k|$ . Since  $\bar{x} \in \ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  so there exists  $\rho_1 > 0$  such that  $\sum_{-\infty}^{\infty} \left(M\left(\frac{\|\lambda_k x_k\|}{\rho_1}\right)\right)^{p_k} < \infty$ . Let us choose  $\rho > 0$  such that  $\rho_1 < m\rho$ . Since  $M$  is non-decreasing therefore

$$\sum_{-\infty}^{\infty} \left(M\left(\frac{\|\mu_k x_k\|}{\rho}\right)\right)^{p_k} < \sum_{-\infty}^{\infty} \left(M\left(\frac{\|\lambda_k x_k\|}{m\rho}\right)\right)^{p_k} < \sum_{-\infty}^{\infty} \left(M\left(\frac{\|\lambda_k x_k\|}{\rho_1}\right)\right)^{p_k} < \infty$$

for some  $\rho > 0$ . Hence  $\bar{x} \in \ell(\mathbb{Z}, X, M, \bar{\mu}, \bar{p})$  and this implies that

$$\ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p}) \subset \ell(\mathbb{Z}, X, M, \bar{\mu}, \bar{p}).$$

Conversely, let the inclusion holds but either  $\lim_{k \rightarrow \infty} \inf t_k = 0$  or  $\lim_{k \rightarrow \infty} \inf t_k = 0$ . Here we take  $\lim_{k \rightarrow \infty} \inf t_k = 0$ , then there exists a sequence  $(k(n))$ ,  $k(n) \geq 1$  such that  $n^2|\lambda_{k(n)}| < |\mu_{k(n)}|$  for all  $n \geq 1$ . Now we see that  $\bar{x} = (x_k)_{-\infty}^{\infty}$  defined by

$$x_k = \begin{cases} \lambda_{k(n)}^{-1} n^{-1} z & \text{if } k = k(n), n \geq 1 \text{ and,} \\ \theta, & \text{otherwise} \end{cases}$$

where  $z \in X$ ,  $\|z\| = 1$  is in  $\ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  but not in  $\ell(\mathbb{Z}, X, M, \bar{\mu}, \bar{p})$  as  $\|\lambda_{k(n)}x_{k(n)}\| = \frac{1}{n}$ . Therefore  $\left(M\left(\frac{\|\lambda_{k(n)}x_{k(n)}\|}{\rho}\right)\right)^l \rightarrow 0$  as  $n \rightarrow \infty$  for some  $\rho$  and for any fixed  $l$ . Hence  $\sum_{-\infty}^{\infty} \left(M\left(\frac{\|\lambda_k x_k\|}{\rho}\right)\right)^{p_k} < \infty$  for  $\ell = \inf_k p_k \leq p_k$ . But  $\|\mu_{k(n)}x_{k(n)}\| = \left|\frac{\mu_{k(n)}}{\lambda_{k(n)}}\right| \frac{1}{n} > n$ , implies that  $\left(M\left(\frac{\|\mu_{k(n)}x_{k(n)}\|}{\rho}\right)\right)^{p_{k(n)}/l} > 1$  for  $\rho > 0$  and for some fixed  $l \leq p_k$ , or  $\sum_{-\infty}^{\infty} \left(M\left(\frac{\|\mu_{k(n)}x_{k(n)}\|}{\rho}\right)\right)^{p_{k(n)}} > 1$

Hence  $\bar{x} \notin \ell(\mathbb{Z}, X, M, \bar{\mu}, \bar{p})$ , which is a contradiction.

Similar proof can be given for the case when we take  $\lim_{k \rightarrow \infty} \inf t_k = 0$ . This completes the proof.

**Lemma 2.9:**  $\ell(\mathbb{Z}, X, M, \bar{\mu}, \bar{p}) \subset \ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  if and only if

$$\lim_{k \rightarrow \infty} \sup_k t_k < \infty \text{ and } \lim_{k \rightarrow \infty} \sup_k t_k < \infty \text{ with } l = \inf_k p_k \leq p_k.$$

**Proof:** Sufficiency is straightforward. On the other hand suppose that

$\ell(\mathbb{Z}, X, M, \bar{\mu}, \bar{p}) \subset \ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  but either  $\lim_{k \rightarrow -\infty} \sup_k t_k = \infty$  or  $\lim_{k \rightarrow \infty} \sup_k t_k = \infty$ . Let  $\lim_{k \rightarrow \infty} \sup_k t_k = \infty$ . Then there exists a sequence  $(k(n))$ ,  $k(n) \geq 1$  such that for each  $n \geq 1$ ,  $|\lambda_{k(n)}| > n|\mu_{k(n)}|$ . Now define the bilateral sequence  $\bar{x} = (x_k)_{-\infty}^{\infty}$  by

$$x_k = \begin{cases} \mu_{k(n)}^{-1} n^{-2} z & \text{if } k = k(n), \quad n \geq 1 \text{ and,} \\ \theta, & \text{otherwise} \end{cases}$$

where  $z \in X$  and  $\|z\| = 1$ . Then  $\bar{x} \in \ell(\mathbb{Z}, X, M, \bar{\mu}, \bar{p})$  since  $\|\mu_{k(n)} x_{k(n)}\| = \frac{1}{n^2}$  i.e.,  $\|\mu_{k(n)} x_{k(n)}\| \rightarrow 0$  as  $n \rightarrow \infty$ .

Therefore  $\left(M\left(\frac{\|\mu_{k(n)} x_{k(n)}\|}{\rho}\right)\right)^l \rightarrow 0$  as  $n \rightarrow \infty$  for any fixed  $l = \inf_k p_k$ . Hence

$$\sum_{-\infty}^{\infty} \left(M\left(\frac{\|\mu_k x_k\|}{\rho}\right)\right)^{p_k} < \infty \text{ for } l = \inf_k p_k \leq p_k.$$

But  $\|\lambda_{k(n)} x_{k(n)}\| = \left|\frac{\lambda_{k(n)}}{\mu_{k(n)}}\right| \cdot n^2 > n$ , implies that  $M\left(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}\right) > 1$ . Therefore

$$\sum_{-\infty}^{\infty} \left(M\left(\frac{\|\lambda_{k(n)} x_{k(n)}\|}{\rho}\right)\right)^{p_{k(n)}} > 1 \text{ for arbitrary large } n. \text{ Hence } \bar{x} \notin \ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p}), \text{ which is a contradiction.}$$

Similar proof can be given for the case when we take  $\lim_{k \rightarrow -\infty} \sup t_k = \infty$ . This completes the proof.

On combining above two Lemmas 2.8 and 2.9 we easily get:

**Theorem 2.10:**  $\ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p}) = \ell(\mathbb{Z}, X, M, \bar{\mu}, \bar{p})$  if and only if

$$0 < \liminf_{k \rightarrow -\infty} t_k < \limsup_{k \rightarrow -\infty} t_k < \infty \\ \text{and } 0 < \liminf_{k \rightarrow \infty} t_k < \limsup_{k \rightarrow \infty} t_k < \infty.$$

### 3. LINEARITY

As far as linear space structures of  $c_0(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  and  $\ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  are concerned, here also we take co-ordinate-wise addition and scalar multiplication in what follows for  $\bar{p} = (p_k)_{-\infty}^{\infty} \in \ell_{\infty}(\mathbb{Z})$  we shall use the notation  $H = \max(1, \sup_k p_k)$ .

**Theorem 3.1:**  $c_0(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  forms a linear space over the field  $\mathbb{C}$ .

**Proof:** Let  $\bar{x}, \bar{y} \in c_0(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  and  $\alpha, \beta \in \mathbb{C}$  therefore there exist some positive  $\rho_1$  and  $\rho_2$  such that  $\left(M\left(\frac{\|\lambda_k x_k\|}{\rho_1}\right)\right)^{p_k} \rightarrow 0$  as  $k \rightarrow -\infty$  as well as  $k \rightarrow \infty$  and  $\left(M\left(\frac{\|\lambda_k y_k\|}{\rho_2}\right)\right)^{p_k} \rightarrow 0$  as  $k \rightarrow -\infty$  as well as  $k \rightarrow \infty$ . In order to prove the result, we need to find some  $\rho_3 > 0$  such that,

$$\left(M\left(\frac{\|\alpha \lambda_k x_k + \beta \lambda_k y_k\|}{\rho_3}\right)\right)^{p_k} \rightarrow 0 \text{ as } k \rightarrow -\infty \text{ as well as } k \rightarrow \infty.$$

Consider  $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$  i.e.,  $\frac{|\alpha|}{\rho_3} \leq \frac{1}{2\rho_1}$  and  $\frac{|\beta|}{\rho_3} \leq \frac{1}{2\rho_2}$ . Then we have

$$\begin{aligned} \left(M\left(\frac{\|\alpha \lambda_k x_k + \beta \lambda_k y_k\|}{\rho_3}\right)\right)^{p_k} &\leq \left(M\left(\frac{\|\alpha \lambda_k x_k\|}{\rho_3} + \frac{\|\beta \lambda_k y_k\|}{\rho_3}\right)\right)^{p_k} \\ &\leq \left(M\left(\frac{\|\lambda_k x_k\|}{2\rho_1} + \frac{\|\lambda_k y_k\|}{2\rho_2}\right)\right)^{p_k} \\ &\leq \frac{1}{2^{p_k}} \left(M\left(\frac{\|\lambda_k x_k\|}{\rho_1}\right) + M\left(\frac{\|\lambda_k y_k\|}{\rho_2}\right)\right)^{p_k} \\ &< \left(M\left(\frac{\|\lambda_k x_k\|}{\rho_1}\right) + M\left(\frac{\|\lambda_k y_k\|}{\rho_2}\right)\right)^{p_k} \\ &\leq c \left(M\left(\frac{\|\lambda_k x_k\|}{\rho_1}\right)\right)^{p_k} + c \left(M\left(\frac{\|\lambda_k y_k\|}{\rho_2}\right)\right)^{p_k} \rightarrow 0 \end{aligned}$$

as  $k \rightarrow -\infty$  as well as  $k \rightarrow \infty$ ,

where  $c = \max(1, 2^{H-1})$ . This proves that  $c_0(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  forms a linear space over  $\mathbb{C}$ .

**Theorem 3.2:**  $\ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  forms a linear space over the field  $\mathbb{C}$ .

**Proof:** We can prove the theorem on the lines of Theorem 3.1.

#### 4. PARANORMED SPACE STRUCTURE

We define

$$(4.1) \quad P(\bar{x}) = \inf \left\{ \rho^{p_n/H} : \sup_k \left( M \left( \frac{\|\lambda_k x_k\|}{\rho} \right) \right)^{p_k/H} \leq 1, n \in \mathbb{Z}^+ \right\} \text{ and}$$

$$(4.2) \quad Q(\bar{x}) = \inf \left\{ \rho^{p_n/H} : \left( \sum_{-\infty}^{\infty} \left( M \left( \frac{\|\lambda_k x_k\|}{\rho} \right) \right)^{p_k} \right)^{\frac{1}{H}} \leq 1, n \in \mathbb{Z}^+ \right\}$$

where  $H = (1, \sup_k p_k)$

**Theorem 4.1:**  $c_0(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  is a total paranormed space with paranorm defined by (4.1).

**Proof:(i)** Clearly  $P(\bar{x}) = P(-\bar{x})$ .

**(ii)**  $P(\bar{x} + \bar{y}) \leq P(\bar{x}) + P(\bar{y})$  follows by putting  $\alpha = \beta = 1$  in Theorem 3.1 .

**(iii)** If  $\bar{x} = \theta$  then  $P(\theta) = 0$  follows easily since

$$\sup_k \left( M \left( \frac{\|\lambda_k x_k\|}{\rho} \right) \right)^{p_k/H} = 0 \text{ for all } k$$

Conversely suppose  $P(\bar{x}) = 0$  then

$$\inf \left\{ \rho^{p_n/H} : \sup_k \left( M \left( \frac{\|\lambda_k x_k\|}{\rho} \right) \right)^{p_k/H} \leq 1, n \in \mathbb{Z}^+ \right\} = 0$$

In such a case, for given  $\epsilon > 0$ , there exists some  $\rho_\epsilon, 0 < \rho_\epsilon < \epsilon$  such that  $\sup_k \left( M \left( \frac{\|\lambda_k x_k\|}{\rho_\epsilon} \right) \right)^{p_k/H} \leq 1$ . Thus,

$$\sup_k \left( M \left( \frac{\|\lambda_k x_k\|}{\epsilon} \right) \right)^{p_k/H} \leq \sup_k \left( M \left( \frac{\|\lambda_k x_k\|}{\rho_\epsilon} \right) \right)^{p_k/H} \leq 1.$$

Suppose,  $x_{n_m} \neq 0$ , for some m. Let  $\epsilon \rightarrow 0$ , then  $\left( \frac{\|x_{n_m}\|}{\epsilon} \right) \rightarrow \infty$ . It follows that

$$\sup_m \left( M \left( \frac{\|\lambda_m x_{n_m}\|}{\epsilon} \right) \right)^{p_m/H} \rightarrow \infty$$

which is a contradiction. Therefore  $x_{n_m} = 0$  for each  $m$ .

**(iv)** Finally, we prove that scalar multiplication is continuous. Let  $\mu$  be any number. By definition,

$$P(\mu\bar{x}) = \inf \left\{ \rho^{p_n/H} : \sup_k \left( M \left( \frac{\|\mu\lambda_k x_k\|}{\rho} \right) \right)^{p_k/H} \leq 1, n \in \mathbb{Z}^+ \right\}.$$

$$\text{Then } P(\mu\bar{x}) = \inf \left\{ (\mu r)^{p_n/H} : \sup_k \left( M \left( \frac{\|\lambda_k x_k\|}{r} \right) \right)^{p_k/H} \leq 1, n \in \mathbb{Z}^+ \right\},$$

where  $r = \frac{\rho}{\mu}$ . Since  $|\mu|^{p_k} \leq \max(1, |\mu|^H)$ . Therefore  $|\mu|^{p_k/H} \leq (\max(1, |\mu|^H))^{1/H}$ .

$$\begin{aligned} \text{Hence, } P(\mu\bar{x}) &\leq (\max(1, |\mu|^H))^{1/H} \inf \left\{ (r)^{p_n/H} : \sup_k \left( M \left( \frac{\|\lambda_k x_k\|}{r} \right) \right)^{p_k/H} \leq 1, n \in \mathbb{Z}^+ \right\} \\ &= (\max(1, |\mu|^H))^{1/H} P(\bar{x}), \end{aligned}$$

which converges to zero as  $P(\bar{x})$  converges to zero in  $c_0(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ . Now suppose  $\mu_n \rightarrow 0$  and  $\bar{x} \in c_0(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ . For arbitrary  $\epsilon > 0$ , let  $N$  be a positive integer such that  $\left( M \left( \frac{\|\lambda_k x_k\|}{\rho} \right) \right)^{p_k} < \frac{\epsilon}{2}, k \in \mathbb{Z} \setminus \mathbb{Z}(-N, N)$

for some  $\rho > 0$ . This implies that  $\sup_k \left( M \left( \frac{\|\lambda_k x_k\|}{\rho} \right) \right)^{p_k/H} \leq \frac{\epsilon}{2}, k \in \mathbb{Z} \setminus \mathbb{Z}(-N, N)$ .

Let  $0 < |\mu| < 1$ , then by convexity of M, we get

$$\left( M \left( \frac{\|\mu\lambda_k x_k\|}{\rho} \right) \right)^{p_k} < \left( |\mu| M \left( \frac{\|\lambda_k x_k\|}{\rho} \right) \right)^{p_k} < \left( \frac{\epsilon}{2} \right)^H, k \in \mathbb{Z} \setminus \mathbb{Z}(-N, N).$$

Since  $M$  is continuous everywhere in  $[0, \infty)$ , then  $f(t) = \left(M\left(\frac{\|\lambda_k x_k\|}{\rho}\right)\right)$ ,  $k \in \mathbb{Z} \setminus \mathbb{Z}(-N, N)$  is continuous at 0. So there is  $1 > \delta > 0$  such that  $|f(t)| < \frac{\epsilon}{2}$ ,  $0 < t < \delta$ . Let  $K$  be such that  $|\mu_n| < \delta$  for all  $n > K$ , then for  $n > K$

$$\left(M\left(\frac{\|\mu_n \lambda_k x_k\|}{\rho}\right)\right)^{p_k/H} < \frac{\epsilon}{2}, \quad k \in \mathbb{Z}(-N, N),$$

Hence  $\sup_k \left(M\left(\frac{\|\mu_n \lambda_k x_k\|}{\rho}\right)\right)^{p_k/H} < \frac{\epsilon}{2}$ ,  $k \in \mathbb{Z}(-N, N)$ . Thus

$$\sup_k \left(M\left(\frac{\|\mu_n \lambda_k x_k\|}{\rho}\right)\right)^{p_k/H} < \epsilon, \quad \text{for } n > K, k \in \mathbb{Z}(-N, N).$$

This completes the proof.

**Theorem 4.2:** Let  $1 \leq p_k < \infty$ . Then  $c_0(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  is a complete paranormed space with paranorm

$$P(\bar{x}) = \inf \left\{ \rho^{p_n/H} : \sup_k \left(M\left(\frac{\|\lambda_k x_k\|}{\rho}\right)\right)^{p_k/H} \leq 1, \text{ for some } \rho \text{ and } n \in \mathbb{Z} \right\}$$

**Proof:** Let  $(\bar{x}^{(i)})$  be a Cauchy sequence in  $c_0(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ . Let  $r$  and  $x_0$  be fixed positive real numbers with  $M\left(\frac{rx_0}{2}\right) > 1$ . Then for each  $\frac{\epsilon}{rx_0} > 0$  there exists a positive integer  $N$  such that

$$(4.3) \quad P(\bar{x}^{(i)} - \bar{x}^{(j)}) < \frac{\epsilon}{rx_0} \quad \text{for all } i, j \geq N.$$

Using definition of paranorm, we get

$$(4.4) \quad \sup_k \left(M\left(\frac{\|\lambda_k x_k^{(i)} - \lambda_k x_k^{(j)}\|}{P(\bar{x}^{(i)} - \bar{x}^{(j)})}\right)\right)^{p_k/H} \leq 1 \quad \text{for all } i, j \geq N, \text{ and } k \in \mathbb{Z}.$$

Thus

$$\left(M\left(\frac{\|\lambda_k x_k^{(i)} - \lambda_k x_k^{(j)}\|}{P(\bar{x}^{(i)} - \bar{x}^{(j)})}\right)\right)^{p_k} \leq 1 \quad \text{for all } i, j \geq N \text{ and } k \in \mathbb{Z}.$$

Since  $1 \leq p_k < \infty$ , it implies that

$$\left(M\left(\frac{\|\lambda_k x_k^{(i)} - \lambda_k x_k^{(j)}\|}{P(\bar{x}^{(i)} - \bar{x}^{(j)})}\right)\right) \leq 1 \quad \text{for all } k \geq 1 \text{ and for all } i, j \geq N.$$

But  $M\left(\frac{rx_0}{2}\right) > 1$ . Therefore

$$M\left(\frac{\|\lambda_k x_k^{(i)} - \lambda_k x_k^{(j)}\|}{P(\bar{x}^{(i)} - \bar{x}^{(j)})}\right) < M\left(\frac{rx_0}{2}\right).$$

But  $M$  is non-decreasing therefore

$$\frac{\|\lambda_k x_k^{(i)} - \lambda_k x_k^{(j)}\|}{P(\bar{x}^{(i)} - \bar{x}^{(j)})} < \frac{rx_0}{2}$$

or,  $\|\lambda_k x_k^{(i)} - \lambda_k x_k^{(j)}\| < \frac{rx_0}{2} \cdot [P(\bar{x}^{(i)} - \bar{x}^{(j)})]$

or,  $\|\lambda_k x_k^{(i)} - \lambda_k x_k^{(j)}\| < \frac{rx_0}{2} \cdot \frac{\epsilon}{rx_0} = \frac{\epsilon}{2}$ .

Hence  $(x_k^{(i)})$  is a Cauchy sequence in  $X$  for all  $k \in \mathbb{Z}$  and therefore convergent. But  $X$  is complete, therefore  $x_k^{(i)} \rightarrow x_k$  (say) as  $i \rightarrow \infty$ . Let us choose  $\rho > 0$  such that  $P((\bar{x}^{(i)} - \bar{x}^{(j)})) < \rho < \epsilon$  for all  $i, j \geq N$ . Since  $M$  is non decreasing we have by (4.4)

$$\sup_k \left(M\left(\frac{\|\lambda_k x_k^{(i)} - \lambda_k \lim_{j \rightarrow \infty} x_k^{(j)}\|}{\rho}\right)\right)^{p_k/H} \leq \sup_k \left(M\left(\frac{\|\lambda_k x_k^{(i)} - \lambda_k x_k^{(j)}\|}{P(\bar{x}^{(i)} - \bar{x}^{(j)})}\right)\right)^{p_k/H} \leq 1,$$

for all  $i, j \geq N$



Letting  $j \rightarrow \infty$  and using continuity of  $M$ , we get

$$\sup_k \left( M \left( \frac{\|\lambda_k x_k^{(i)} - \lambda_k \lim_{j \rightarrow \infty} x_k^{(j)}\|}{\rho} \right) \right)^{p_k/H} \leq 1 \quad \text{for all } k \in \mathbb{Z}(-N, N).$$

Thus  $\sup_k \left( M \left( \frac{\|\lambda_k x_k^{(i)} - \lambda_k x_k\|}{\rho} \right) \right)^{p_k/H} \leq 1$  for all  $k \in \mathbb{Z}(-N, N)$ .

Taking infimum of such  $\rho$ 's, we get

$$P(\bar{x}^{(i)} - \bar{x}) = \inf \left\{ \rho^{p_n/H} : \sup_k \left( M \left( \frac{\|\lambda_k x_k^{(i)} - \lambda_k x_k\|}{\rho} \right) \right)^{p_k/H} \leq 1 \quad \text{for all } i \geq N \right\} \\ \leq \rho < \epsilon.$$

Hence  $P(\bar{x}^{(i)} - \bar{x}) < \epsilon$  for all  $i \geq N$ .

Since  $(\bar{x}^{(i)}) \in c_0(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  and  $M$  is continuous, it follows that  $\bar{x} \in c_0(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ . This completes the proof.

**Theorem 4.3:**  $\ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  is a total paranormed space with

$$Q(\bar{x}) = \inf \left\{ \rho^{p_n/H} : \left( \sum_{-\infty}^{\infty} \left( M \left( \frac{\|\lambda_k x_k\|}{\rho} \right) \right)^{p_k} \right)^{\frac{1}{H}} \leq 1, \quad n \in \mathbb{Z}^+ \right\}$$

where  $H = (1, \sup_k p_k)$ .

**Proof:** The theorem can be proved on the lines of Theorem 4.1

**Theorem 4.4:** Let  $1 \leq p_k < \infty$ . Then  $\ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$  is a complete paranormed space with respect to paranorm

$$Q(\bar{x}) = \inf \left\{ \rho^{p_n/H} : \left( \sum_{-\infty}^{\infty} \left( M \left( \frac{\|\lambda_k x_k\|}{\rho} \right) \right)^{p_k} \right)^{\frac{1}{H}} \leq 1, \quad n \in \mathbb{Z}^+ \right\}$$

**Proof:** We can prove this theorem on the lines of Theorem 4.2.

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