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COEFFICIENT ESTIMATES FOR A CERTAIN CLASS OF ANALYTIC FUNCTIONS DEFINED USING THE GENERALIZED CARLSON SHAFFER OPERATOR

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ABSTRACT

For some real $\alpha(\alpha > 1)$ using the Generalized Carlson - Shaffer operator a subclass $M_\mu(a, c; \alpha)$ of analytic functions f with $f(0) = 0$ and $f'(0) = 1$ in U is introduced. The object of the present paper is to obtain the results concerning the coefficient estimates for the functions f belonging to the class $M_\mu(a, c; \alpha)$.

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1. INTRODUCTION AND DEFINITION

Let A denote the class of functions f of the form,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$.

Let $M(\alpha)$ be the subclass of A consisting of functions f which satisfy the inequality,

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} < \alpha \quad (z \in U) \tag{1.2}$$

for some $\alpha(\alpha > 1)$.

Let $N(\alpha)$ be the subclass of A consisting of functions f which satisfy the inequality,

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \alpha \quad (z \in U) \tag{1.3}$$

for some $\alpha(\alpha > 1)$.

Then, we observe that $f \in N(\alpha)$ if and only if $zf'(z) \in M(\alpha)$.

Remark 1.1. The classes $M(\alpha)$ and $N(\alpha)$ were introduced by Owa and Nishwaki [2].

Remark 1.2. The classes $M(\alpha)$ and $N(\alpha)$ for $1 < \alpha < \frac{4}{3}$ were introduced by Uralegaddi, Ganigi and Sarangi [5].

Remark 1.3. The classes $M(\alpha)$ and $N(\alpha)$ correspond to the case $k = 2$ of the classes $M_k(\alpha)$ and $N_k(\alpha)$ respectively which were investigated by Owa and Srivastava [3].
It can be seen that,
i. $f(z) = z(1 - z)^{2(\alpha-1)} \in M(\alpha)$
ii. $g(z) = \frac{1}{2\alpha-1}\{1 - (1 - z)^{2\alpha-1}\} \in N(\alpha)$

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Definition 1.4. Let $M_\mu(a,c;\alpha)$ denote the subclass of A consisting of functions f satisfying the inequality,

$$\Re\left\{\frac{z(L_\mu(a,c)f(z))'}{L_\mu(a,c)f(z)}\right\} < \alpha \quad (Z \in U) \tag{1.4}$$

where $\alpha(\alpha < 1)$ and $L_\mu(a,c)f$ is the Generalized Carlson-Shaffer operator defined as,

$$L_\mu(a,c)f(z) = \phi(a,c;z) * L_\mu f(z) \tag{1.5}$$

where

$$\phi(a,c;z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n$$

and

$$L_\mu f(z) = (1-\mu)f(z) + \mu f(z)$$

Equivalently,

$$L_\mu(a,c)f(z) = z + \sum_{n=2}^{\infty} \tau_n(a,c;\mu)a_n z^n \tag{1.6}$$

where

$$\tau_n(a,c;\mu) = \frac{(a)_{n-1}}{(c)_{n-1}}[1+\mu(n-1)]$$

Note that $M_0(1,1;\alpha) = M(\alpha)$ and $M_0(2,1;\alpha) = N(\alpha)$.

2. INCLUSION THEOREMS INVOLVING COEFFICIENT INEQUALITIES

Theorem: 2.1. *If $f \in A$ satisfies,*

$$\sum_{n=2}^{\infty} \{(n-k) + |n+k-2\alpha|\}\, \tau_n(a,c;\mu)|a_n| \le 2(\alpha-1) \tag{1.7}$$

for some $k(0 \le k \le 1)$ and some $\alpha(\alpha > 1)$, then $f \in M_\mu(a,c;\alpha)$.

Proof. Let us suppose that

$$\sum_{n=2}^{\infty} \{(n-k) + |n+k-2\alpha|\}\, \tau_n(a,c;\mu)|a_n| \le 2(\alpha-1)$$

for $f \in A$.

It suffices to show that,

$$\left| \frac{\dfrac{z(L_\mu(a,c)f(z))'}{L_\mu(a,c)f(z)} - k}{\dfrac{z(L_\mu(a,c)f(z))'}{L_\mu(a,c)f(z)} - (2\alpha-k)} \right| < 1 \quad (z \in U)$$

We note that,

$$\left| \frac{\dfrac{z(L_\mu(a,c)f(z))'}{L_\mu(a,c)f(z)} - k}{\dfrac{z(L_\mu(a,c)f(z))'}{L_\mu(a,c)f(z)} - (2\alpha-k)} \right| \le \left| \frac{(1-k) + \sum_{n=2}^{\infty}(n-k)\tau_n(a,c;\mu)a_n z^{n-1}}{(1+k-2\alpha) + \sum_{n=2}^{\infty}(n+k-2\alpha)\tau_n(a,c;\mu)a_n z^{n-1}} \right|$$

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$$\leq \frac{(1-k)+\sum_{n=2}^{\infty}(n-k)\tau_n(a,c;\mu)|a_n||z|^{n-1}}{(2\alpha-1-k)-\sum_{n=2}^{\infty}|(n+k-2\alpha)\tau_n(a,c;\mu)|\,|a_n||z|^{n-1}}$$

$$< \frac{(1-k)+\sum_{n=2}^{\infty}(n-k)\tau_n(a,c;\mu)|a_n|}{(2\alpha-1-k)-\sum_{n=2}^{\infty}|(n+k-2\alpha)\tau_n(a,c;\mu)|\,|a_n|}$$

This expression is bounded above by 1 if,

$$(1-k)+\sum_{n=2}^{\infty}(n-k)\tau_n(a,c;\mu)|a_n| < (2\alpha-1-k)-\sum_{n=2}^{\infty}|(n+k-2\alpha)\tau_n(a,c;\mu)|\,|a_n|$$

which is equivalent to condition 2.1. This completes the proof.

If we take $k=1$ and some $\alpha(1<\alpha\leq\frac{3}{2})$ in Theorem 2.1, then we have,

Corollary: 2.2 *If $f\in A$ satisfies,*

$$\sum_{n=2}^{\infty}(n-\alpha)\tau_n(a,c;\mu)|a_n|\leq\alpha-1$$

for some $\alpha(1<\alpha\leq\frac{3}{2})$, then $f\in M_\mu(a,c;\alpha)$.

Example 2.1. The function f given by

$$f(z)=z+\sum_{n=2}^{\infty}\frac{4(\alpha-1)}{n(n+1)\{(n-k)+|n+k-2\alpha|\}\tau_n(a,c;\mu)}z^m$$

belongs to the class $M_\mu(a,c;\alpha)$.

Remark: 2.3 For the parametric values $a=1$, $c=1$ and $\mu=0$ Theorem 2.1 yields Theorem 2.1 of [2] and Corollary 2.2 yields Corollary 2.2 of [2].

Remark: 2.4 For the parametric values $a=2$, $c=1$ and $\mu=0$ Theorem 2.1yields Theorem 2.3 of [2] and Corollary 2.2 yields Corollary 2.4 of [2].

The coefficient estimates of functions $f\in M(a,c;\alpha)$ is contained in the following:

Theorem: 2.5 *If $f\in M_\mu(a,c;\alpha)$, then*

$$|a_n|\leq\frac{\prod_{j=1}^{n}(j+2\alpha-4)}{\tau_n(a,c;\mu)(n-1)!} \tag{2.2}$$

Proof. Let us define the function $p(z)$ by,

$$p(z)=\frac{\alpha-\dfrac{z(L_\mu(a,c)f(z))'}{L_\mu(a,c)f(z)}}{\alpha-1}$$

for $f\in M_\mu(a,c;\alpha)$.

Then $p(z)$ is analytic in U, $p(0)=1$ and $\Re\{p(z)\}>0\;(z\in U)$.

If, $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ then $|p_n| \leq 2 \quad (n \geq 1)$.

Since,

$$\alpha L_\mu(a,c)f(z) - z(L_\mu(a,c)f(z))' = (\alpha - 1)p(z)L_\mu(a,c)f(z)$$

we obtain that,

$$(1-n)\tau_n(a,c;\mu)a_n = (\alpha-1)\{p_{n-1} + p_{n-2}\tau_2(a,c;\mu)a_2 + p_{n-3}\tau_3(a,c;\mu)a_3 + \\ \ldots\ldots\ldots + p_1\tau_{n-1}(a,c;\mu)a_{n-1}\}$$

If $n = 2$, then

$$\tau_2(a,c;\mu)a_2 \leq (\alpha - 1)p_1$$

implies that

$$|a_2| \leq \frac{(\alpha-1)|p_1|}{\tau_2(a,c;\mu)} \leq \frac{2(\alpha-1)}{\tau_2(a,c;\mu)}$$

Hence the coefficient estimate for (2.2) is true for $n = 2$.

Let us suppose that the coefficient estimate,

$$|a_k| \leq \frac{\prod_{j=2}^{k}(j+2\alpha-4)}{\tau_k(a,c;\mu)(k-1)!}$$

is true for all $k = 2, 3, 4, \ldots\ldots\ldots, n$.

Then we have,

-$na_{n+1} = (\alpha-1)\{p_n + p_{n-2}\tau_2(a,c;\mu)a_2 + p_{n-3}\tau_2(a,c;\mu)a_3 + \ldots\ldots\ldots\ldots + p_1\tau_n(a,c;\mu)a_n\}$

so that,

$$\begin{aligned} n\tau_{n+1}(a,c;\mu)|a_{n+1}| &\leq (2\alpha-2)\,(1 + \tau_2(a,c;\mu)|a_2| + \tau_3(a,c;\mu)|a_3| \\ &\quad + \ldots\ldots\ldots\ldots + \tau_n(a,c;\mu)|a_n|) \\ &\leq (2\alpha-2)\left(1 + (2\alpha-2) + \frac{(2\alpha-2)(2\alpha-1)}{2!} + \right. \\ &\qquad \left. \ldots\ldots\ldots\ldots\ldots + \frac{\prod_{j=2}^{n}(j+2\alpha-4)}{(n-1)!}\right) \\ &= (2\alpha-2)\left(\frac{(2\alpha-1)(2\alpha)(2\alpha+1)\ldots\ldots\ldots(2\alpha+n-4)}{(n-2)!}\right. \\ &\quad \left. + \frac{(2\alpha-2)(2\alpha-1)(2\alpha)\ldots\ldots\ldots(2\alpha+n-4)}{(n-1)!}\right) \\ &= \frac{\prod_{j=2}^{n+1}(j+2\alpha-4)}{(n-1)!} \end{aligned}$$

$$\Longrightarrow |a_{n+1}| \leq \frac{\prod_{j=2}^{n+1}(j+2\alpha-4)}{\tau_n(a,c;\mu)(n!)}$$

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Hence the coefficient estimate (2.2) holds true for the case of $k = n + 1$. Applying mathematical induction for the coefficient estimate (2.2), we complete the proof of Theorem 2.5.

Remark 2.6 The parametric substitutions $a = 1, c = 1, \mu = 0$ yield Theorem 2.6 and the substitutions $a = 2$, $c = 1$, $\mu = 0$ yield Theorem 2.7 of [2].

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