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ON ROUGH TOPOLOGICAL SPACES

Boby P. Mathew[a] **& Sunil Jacob John**[b,*]

[a]Department of Mathematics, St. Thomas College, Pala, 686 574, Kerala, India

[b]Department of Mathematics, National Institute of Technology, Calicut, 673 601, Kerala, India

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ABSTRACT

Rough set theory is a mathematical tool to deal with incomplete and imprecise data and topology is the study of invariance of a space under topological transformations known as homeomorphisms. In this paper an attempt is made to develop general topological structure on rough sets. We defined rough topology on a rough set and some basic topological properties of the resultant Rough Topological Spaces (RTS), such as rough open sets, rough closed sets, rough base and rough closure, etc. are studied.

Keywords: *Rough Sets, Rough Topological Spaces, Rough Open Sets, Rough Closed Sets, Rough Basis, Rough Closure*

Mathematics Subject Classification: *03Exx, 54A05.*

INTRODUCTION

The problem of imperfect knowledge has been tackled for a long time by philosophers, logicians and mathematicians. Recently it became also a crucial issue for computer scientists, particularly in the area of artificial intelligence. There are many approaches to the problem of how to understand and manipulate imperfect knowledge. Rough set theory by Z. Pawalak [6], is a new mathematical approach to vagueness or imperfect knowledge. Pawalak's Rough set theory expresses vagueness by employing a boundary region of a set. If the boundary region of a set is empty it means that the set is crisp or exact, otherwise the set is rough or inexact. Nonempty boundary region of a set means that our knowledge about the set is not sufficient to define the set precisely. A monograph is given in [7], and some applications of rough set are given in [8 - 10].

Topology is the study of invariance of a space under topological transformations known as homeomorphisms. A detailed study and historical notes can be seen in Willard [12]. Many theories and applications are presented in Munkres [4] and Joshy [1].

Rough set theory has found many interesting applications [8 - 10], and it has been extended in many ways. Some research work regarding the combination of rough set theory and topology is given in [2, 3, 5, 11, 13]. The topological space on rough set and corresponding topological properties on the topological rough space are done by QingE Wu *et.al.* [11]. In this paper they defined topology on a rough set using a metric and then extended it to the general topological approach. In our study, we observed that according to their definition of open sets $\phi = (\phi, \phi)$ is not always open, since $\phi \subseteq X_L \subseteq \phi \subseteq X_U$ is not true for a rough set $X = \left(X_L, X_U\right)$ with nonempty lower approximation X_L. Thus ϕ is not always open and therefore the basic definition of topology in [11] is not valid. In our work we are trying to rectify this problem and defined a new topological structure on rough sets so as to create a rough topological space (*RTS*). Also the basic topological properties of the resulting *RTS* are studied.

2. PRELIMINARIES

Suppose we are given a non empty set of objects *U* called the universe and an equivalence relation called indiscerniibility relation *R* on *U*, then the pair (*U*, *R*) is known as the approximation space [6]. Let *X* be a subset of *U*.

Corresponding author: *Sunil Jacob John*[b,*]
[b]Department of Mathematics, National Institute of Technology, Calicut, 673 601, Kerala, India

In order to characterize X with respect to R, we associate two crisp sets to X, called its lower and upper approximations.

Definition 2.1. [6] The equivalence class of R containing an element x will be denoted by $R(x)$ and is called granules of knowledge generated by R, which represents elementary portion of knowledge we are able to perceive due to R.

Definition 2.2. [6] The lower approximation of X denoted by X_L is the union of all granules which are entirely included in the set. That is $X_L=\{x / R(x) \subset X\}$. Therefore lower approximation of a set consists of all elements that surely belong to the set.

Definition 2.3. [6] The upper approximation of X denoted by X_U is the union of all granules which have non-empty intersection with the set. That is $X_U=\{x / R(x) \cap X \neq \phi\}$. Therefore upper approximation of the set constitutes of all elements that possibly belong to the set.

Definition 2.4. [6] The boundary region of set is the difference between the upper and the lower approximation. Intuitively, the boundary region of the set consists of all elements that cannot be classified uniquely to the set or its complement, by employing available knowledge.

Definition 2.5. [6] A set is said to be a rough set, if it has a non-empty boundary region. If the boundary region is empty then the set is a crisp or exact set.

Result 2.6. [6] X is an exact set if $X_L=X_U$ and X is rough set if $X_L \neq X_U$.

Notation 2.7. [6] We denote a rough set X with lower approximation X_L and upper approximation X_U by $X=\left(X_L, X_U\right)$.

Result 2.8 [6] Let $A=(A_L, A_U)$ and $B= (B_L, B_U)$ be any two arbitrary rough sets. Then A is said to be a rough subset of B iff $A_L \subset B_L$ and $A_U \subset B_U$.

Definition 2.9 [6] Let $A= (A_L, A_U)$ and $B= (B_L, B_U)$ be any two arbitrary rough subsets of X. Then the union of rough sets A and B is defined as $A \cup B= (A_L \cup B_L, A_U \cup B_U)$ and intersection of A and B is defined as

$$A \cap B= (A_L \cap B_L, A_U \cap B_U).$$

Result 2.10. [6] For any two rough set A and B, $A \subset B$ iff $A \cap B=A$.

Definition 2.11. [12] A topological space is a pair (X, τ) consisting of a set X and a family τ of subsets of X satisfying the following three conditions.
(i) $\phi \in \tau$ and $X \in \tau$
(ii) τ is closed under arbitrary union
(iii) τ is closed under finite intersections.

Any family τ of subsets of X satisfying the above three conditions is known as a topology on X.

Definition 2.12. [12] A subset A of a topological space (X, τ) is said to be open iff $A \in \tau$, and A is said to be closed iff $A^C = X \setminus A \in \tau$.

Definition 2.13. [1] Let X be any non empty set. Define $\tau =\{X, \phi\}$. Then τ is the smallest topology on X and is known as indiscrete topology. Then (X, τ) is known as indiscrete topological space.

Result 2.14. [1] The only open sets of an indiscrete topological space (X, τ) are X and ϕ.

Definition 2.15. [1] Let X be any non empty set. Define $\tau =P(X)$, the power set of X. Then τ is the greatest topology on X and is known as discrete topology. Then (X, τ) is known as discrete topological space.

Result 2.16. [1] In a discrete topological space (X, τ) every subset of X is open.

Definition 2.17. [4] Suppose that τ^1 and τ^2 are two topologies on a given set X. If $\tau^1 \supset \tau^2$ we say that τ^1 is finer than τ^2 or τ^2 is coarser than τ^1. If τ^1 properly contains τ^2 we say that τ^1 is strictly finer than τ^2 or τ^2 is strictly coarser than τ^1.

Definition 2.18. [1] Let (X, τ) be a topological space. A subfamily *B* of τ is said to be a base for τ if every member of τ can be expressed as the union of some members of *B*.

Theorem 2.19. [12] Let (X, τ) be a topological space. A subfamily *B* of τ is a base for τ iff for every $x \in X$ and for every open set *U* containing *x*, $\exists B \in$ B such that $x \in B \subseteq U$.

Definition 2.20. [12] Let (X, τ) be a topological space and *A* be a subset of *X*. Then the closure of *A*, denoted by $\overline{A}$ is the intersection of all closed set containing *A*.

Theorem 2.21. [12] If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$

Definition 2.22. [4] A subset *A* of a topological space (X, τ) is dense in *X* if $\overline{A} = X$.

3. ROUGH TOPOLOGICAL SPACES

Definition 3.1. Let $X=\left(X_L, X_U\right)$ be a rough subset of the approximation space $\left(\Omega, R\right)$. Let τ_L and τ_U be any two topologies which contain only exact subsets of X_L and X_U respectively. Then the pair $\tau=\left(\tau_L, \tau_U\right)$ is said to be a Rough Topology on the rough set $X=\left(X_L, X_U\right)$ and the pair $\left(X, \tau\right)$ is known as a Rough Topological Space (*RTS*). Also in a rough topology $\tau=\left(\tau_L, \tau_U\right)$, τ_L is known as the lower rough topology and τ_U is known as the upper rough topology on *X*.

Definition 3.2. Let $A=\left(A_L, A_U\right)$ be any rough subset of a *RTS* $\left(X, \tau\right)$, where $X=\left(X_L, X_U\right)$ and $\tau=\left(\tau_L, \tau_U\right)$. Then *A* is said to be lower rough open if the lower approximation of *A* is in the lower rough topology. That is $A_L \in \tau_L$. Also *A* is said to be upper rough open if the upper approximation of *A* is in the upper rough topology. That is $A_U \in \tau_U$. *A* is said to be rough open iff *A* is lower rough open and upper rough open. That is $A=\left(A_L, A_U\right)$ is rough open with respect to the *RTS* $\left(X, \tau\right)$ iff $A_L \in \tau_L$ and $A_U \in \tau_U$.

Remark 3.3. We restrict the members of τ_L and τ_U to the set of all exact or definable subsets of X_L and X_U respectively, since they are only exactly defined in the approximation space $\left(\Omega, R\right)$ and we are totally unaware about the indefinable sets in the knowledge base. But when they are grouped to form the rough topology $\tau=\left(\tau_L, \tau_U\right)$, indefinable sets can also be rough open. Here the only condition is that, a subset of *X*, either exact or inexact is rough open iff its lower approximation is in the lower topology and its upper approximation is in the upper topology.

Theorem 3.4. Consider a *RTS* $\left(X, \tau\right)$, where $X=\left(X_L, X_U\right)$ and $\tau=\left(\tau_L, \tau_U\right)$. Let *T* be the collection of all rough open subsets of $\left(X, \tau\right)$. Then *T* is a topology on X.

Proof: Consider $X=\left(X_L, X_U\right)$ and $\tau=\left(\tau_L, \tau_U\right)$.

(i) We have $\phi \in \tau_L$ and $\phi \in \tau_U$. Therefore $\phi = (\phi, \phi)$ is rough open subset of *X*. Also $X_L \in \tau_L$ and $X_U \in \tau_U$ and hence $X=\left(X_L, X_U\right)$ is rough open subset of *X*. Thus $\phi = (\phi, \phi)$ and $X=\left(X_L, X_U\right) \in T$.

(ii) Let $A=(A_L, A_U)$ and $B=(B_L, B_U)$ be any two arbitrary members of *T*. Then *A* and *B* are rough open subsets of *X*. Therefore $A_L \in \tau_L$, $A_U \in \tau_U$, $B_L \in \tau_L$, and $B_U \in \tau_U$. Being topologies, τ_L and τ_U are closed under finite intersection and therefore $A_L \cap B_L \in \tau_L$ and $A_U \cap B_U \in \tau_U$. Hence $A \cap B = (A_L \cap B_L, A_U \cap B_U)$ is rough open subset of *X*. Which implies $A \cap B \in T$. Since *A* and *B* are arbitrary, *T* is closed under finite intersections.

(iii) Let $\left\{A_i = (A_{i_L}, A_{i_U}) / i \in I\right\}$ be any arbitrary family of rough open subsets of *X* and therefore a sub collection of *T*. $A_i = (A_{i_L}, A_{i_U}) \in T$ implies $A_{i_L} \in \tau_L$ and $A_{i_U} \in \tau_U$ for all $i \in I$. Being topologies, τ_L and τ_U are closed under arbitrary union and therefore we have $\bigcup_{i\in I} A_{i_L} \in \tau_L$ and $\bigcup_{i\in I} A_{i_U} \in \tau_U$. Which implies $\bigcup_{i\in I} A_i = \left(\bigcup_{i\in I} A_{i_L}, \bigcup_{i\in I} A_{i_U}\right)$ is rough open subset of *X*. Hence *T* is closed under arbitrary union.

From (i), (ii) and (iii), the family *T* of all rough open subsets of *X* forms a topology on *X*.

Definition 3.5. In any rough set $X=\left(X_L, X_U\right)$, define τ_L = {$A \subseteq X_L$ / A is definable} and τ_U = {$B \subseteq X_U$ / B is definable}. Then τ_L and τ_U are topologies on X_L and X_U respectively and the rough topology $\tau = \left(\tau_L, \tau_U\right)$ is known as the Discrete rough topology on $X=\left(X_L, X_U\right)$ and the topological space $\left(X, \tau\right)$ is known as the Discrete Rough Topological Space on *X*.

Theorem 3.6. Every subset of a Discrete Rough Topological Space is rough open.

Proof: Consider any subset $A=\left(A_L, A_U\right)$ of the Discrete Rough Topological Space $\left(X, \tau\right)$. Where $X = \left(X_L, X_U\right)$ and $\tau = \left(\tau_L, \tau_U\right)$. Being the lower approximation of *A*, A_L is an exact subset of X_L and hence $A_L \in \tau_L$. Therefore *A* is lower rough open. Also being the upper approximation of *A*, A_U is exact subset of X_U and hence $A_U \in \tau_U$. Therefore *A* is upper rough open. That is $A=\left(A_L, A_U\right)$ is lower and upper rough open and therefore rough open subset of *X*. Since *A* is arbitrary, every subset of a Discrete Rough Topological Space is rough open.

Definition 3.7. In a rough set $X=\left(X_L, X_U\right)$, take τ_L ={ϕ, X_L} and τ_U ={ϕ, X_U}, then τ_L and τ_U are topologies on X_L and X_U respectively and the rough topology $\tau = \left(\tau_L, \tau_U\right)$ on *X* is known as the indiscrete rough topology on *X* and $\left(X, \tau\right)$ is known as the indiscrete rough topological space on *X*.

Remark 3.8. In indiscrete rough topology the only rough open subsets of *X* are not merely $X = \left(X_L, X_U\right)$ and ϕ=(ϕ, ϕ). But there can be several rough open subsets other than X and ϕ. For example construct a set *A* which consists exactly one and only one element from each equvalance class of *X*. Then *A* = (ϕ , *X*) is rough open. Thus the result 2.14 in general topological space is not valid in indiscrete rough topological space.

Example 3.9. Consider $X=\left(X_L, X_U\right)$. Let *A* and *B* be any two proper exact subsets of X_L and X_U respectively. Define τ_L as the family of all exact subsets of X_L that is contained in *A* together with X_L and τ_U as the family of all exact subsets of X_U that is contained in *B* together with X_U. Then τ_L and τ_U forms the lower and upper rough topologies on *X* and $\tau = \left(\tau_L, \tau_U\right)$ is a rough topology on *X*.

Definition 3.10 If $A = \left(A_L, A_U\right)$ is any sub rough set of a rough set $X=\left(X_L, X_U\right)$, then $A^{C_L} = X_L \backslash A_L$ is called the lower complement of *A* and $A^{C_U} = X_U \backslash A_U$, is called the upper complement of *A*.

Example 3.11. Consider $X = \left(X_L, X_U\right)$. Let *A* and *B* be any two proper exact subsets of X_L and X_U respectively. Define τ_L as the family of all exact subsets of X_L that is either a subset of *A* or a super set of A^{C_L} and define τ_U as the family of all exact subsets of X_U that is either a subset of B or a super set of B^{C_U} .Then τ_L and τ_U forms the lower and upper rough topologies on *X* and $\tau = \left(\tau_L, \tau_U\right)$ is a rough topology on *X*.

Definition 3.12. A subset *B*=(B_L, B_U) of the *RTS* $\left(X, \tau\right)$, where $X = \left(X_L, X_U\right)$ and $\tau = \left(\tau_L, \tau_U\right)$ is said to be lower rough closed set if $B^{C_L} = X_L \backslash B_L \in \tau_L$. Also *B* is said to be upper rough closed if $B^{C_U} = X_U \backslash B_U \in \tau_U$. *B* is said to be rough closed if it is lower rough closed and upper rough closed. That is A subset *B*= (B_L, B_U) of the *RTS* $\left(X, \tau\right)$ is rough closed subset iff its lower approximation is closed with respect to the lower topology and its upper approximation is closed with respect to the upper topology of $\left(X, \tau\right)$.

Theorem 3.13. Consider the *RTS* (X,τ), where $X=(X_L,X_U)$ and $\tau=(\tau_L,\tau_U)$. Let *F* be the collection of all rough closed subsets of (X,τ). Then *F* has the following properties.

(i) $\phi=(\phi,\phi)$ and $X=(X_L,X_U)\in F$

(ii) *F* is closed under finite union

(iii) *F* is closed under arbitrary intersections.

Proof: Consider the *RTS* (X,τ), where $X=(X_L,X_U)$ and $\tau=(\tau_L,\tau_U)$ and let *F* be the collection of all rough closed subsets of (X,τ).

(i) We have $\phi^{C_L}=X_L\setminus\phi=X_L\in\tau_L$ and $\phi^{C_U}=X_U\setminus\phi=X_U\in\tau_U$. Therefore $\phi=(\phi,\phi)$ is rough closed. Now $X^{C_L}=X_L\setminus X_L=\phi\in\tau_L$ and $X^{C_U}=X_U\setminus X_U=\phi\in\tau_U$. Therefore $X=(X_L,X_U)$ is rough closed. So $\phi=(\phi,\phi)$ and $X=(X_L,X_U)\in F$.

(ii) Let $A=(A_L,A_U)$ and $B=(B_L,B_U)$ be any to arbitrary members of *F*. Then *A* and *B* are rough closed subsets of *X*.

Therefore $A^{C_L}=X_L\setminus A_L\in\tau_L$, $A^{C_U}=X_U\setminus A_U\in\tau_U$, $B^{C_L}=X_L\setminus B_L\in\tau_L$ and $B^{C_U}=X_U\setminus B_U\in\tau_U$. Then we have $(A\cup B)^{C_L}=X_L\setminus(A_L\cup B_L)=(X_L\setminus A_L)\cap(X_L\setminus B_L)=A^{C_L}\cap B^{C_L}\in\tau_L$, since τ_L is closed under finite intersections. Also $(A\cup B)^{C_U}=X_U\setminus(A_U\cup B_U)=(X_U\setminus A_U)\cap(X_U\setminus B_U)=A^{C_U}\cap B^{C_U}\in\tau_U$, since τ_U is closed under finite intersections. That is $(A\cup B)^{C_L}\in\tau_L$ and $(A\cup B)^{C_U}\in\tau_U$. Therefore $A\cup B=(A_L\cup B_L, A_U\cup B_U)$ is rough closed. Since *A* and *B* are arbitrary, *F* is closed under finite union.

(iii) Let $\{A_i=(A_{i_L},A_{i_U})/i\in I\}$ be any arbitrary family of rough closed subsets of *X*. Which implies $A_i=(A_{i_L},A_{i_U})\in F$ for all $i\in I$. Therefore $A_i^{C_L}=X_L\setminus A_{i_L}\in\tau_L$ and $A_i^{C_U}=X_U\setminus A_{i_U}\in\tau_U$. Now $\left(\bigcap_{i\in I}A_i\right)^{C_L}=X_L\setminus\bigcap_{i\in I}A_{i_L}=\bigcup_{i\in I}(X_L\setminus A_{i_L})\in\tau_L$, since τ_L is closed under arbitrary union. Similarly $\left(\bigcap_{i\in I}A_i\right)^{C_U}=X_U\setminus\bigcap_{i\in I}A_{i_U}=\bigcup_{i\in I}(X_U\setminus A_{i_U})\in\tau_U$, since τ_U is closed under arbitrary union. That is $\left(\bigcap_{i\in I}A_{i_L}\right)^{C_L}\in\tau_L$ and $\left(\bigcap_{i\in I}A_{i_U}\right)^{C_U}\in\tau_U$. Hence $\bigcap_{i\in I}A_i=\left(\bigcap_{i\in I}A_{i_L},\bigcap_{i\in I}A_{i_U}\right)$ is rough closed subset of *X*. Therefore $\bigcap_{i\in I}A_i\in F$.

Thus *F* is closed under arbitrary intersections.

Hence the theorem.

Theorem 3.14. Let $X=(X_L,X_U)$ be any rough set and *F* be any collection of subsets of *X*, which satisfies the three properties in theorem 3.13. Then *F* defines a unique rough topology *T* on *X* such that *F* coincides with the family of all rough closed subsets of the *RTS* (X,T).

Proof: Consider a rough set $X=(X_L,X_U)$ and F be a family of rough subsets of X having the following three properties.

(i) $\phi=(\phi,\phi)$ and $X=(X_L,X_U)\in F$.

(ii) F is closed under finite unions

(iii) F is closed under arbitrary intersections.

Then we have to prove that there exists a unique rough topology, $T=(T_L,T_U)$ such that F coincides with the family of all rough closed subsets of (X,T).

Define $T_L=\{B^{C_L}/B\in F\}$ and $T_U=\{B^{C_U}/B\in F\}$. It is given that $\phi=(\phi,\phi)\in F$. Therefore $\phi^{C_L}=X_L\setminus\phi=X_L\in T_L$. Also we have $X=(X_L,X_U)\in F$. Therefore $X^{C_L}=X_L\setminus X_L=\phi\in T_L$.

Thus $\phi \in T_L$ and $X_L \in T_L$ (1)

Let $A, B \in T_L$. Then $A = D^{C_L}$, and $B = E^{C_L}$, for some $D, E \in F$. Then

$$A \cap B = D^{C_L} \cap E^{C_L} = (X_L \setminus D_L) \cap (X_L \setminus E_L) = (X_L \setminus (D_L \cup E_L)) = (D \cup E)^{C_L} \in T_L \quad (2)$$

Since $D, E \in F$ and F is closed under finite unions implies $D \cup E \in F$. Thus T_L is closed under finite intersections.

Let $\{A_i / i \in I\}$ be any arbitrary family of members of T_L. Then by the construction of T_L, $A_i = D_i^{C_L}$ for some $D_i \in F$, for all $i \in I$. Then $\bigcup_{i \in I} A_i = \bigcup_{i \in I} D_i^{C_L} = \bigcup_{i \in I} (X_L \setminus D_{iL}) = X_L \setminus \bigcap_{i \in I} D_{iL} = \left(\bigcap_{i \in I} D_i\right)^{C_L} \in T_L$. Since $D_i \in F$ for all $i \in I$ and F is closed under arbitrary intersections implies

$$\bigcap_{i \in I} D_i \in F. \quad (3)$$

Thus T_L is closed under arbitrary unions.

Hence from (1), (2) and (3) T_L is a topology on X_L.

Similarly by the same arguments on X_U, T_U is a topology on X_U.

Thus $T = (T_L, T_U)$ is a rough topology on $X = (X_L, X_U)$.

Also from the construction of T_L and T_U, clearly they are unique and so is T.

Let $B = (B_L, B_U)$ be any arbitrary member of F. Then $B^{C_L} \in T_L$ and $B^{C_U} \in T_U$ by the construction of T_L and T_U. Therefore $(B^{C_L}, B^{C_U}) \in T$

Hence F coincides with the family of all rough closed subsets of (X, T).

Hence the theorem.

Definition 3.15. Let $\tau^1 = (\tau^1{}_L, \tau^1{}_U)$ and $\tau^2 = (\tau^2{}_L, \tau^2{}_U)$ are any two rough topologies on $X = (X_L, X_U)$. If $\tau^1{}_L \subseteq \tau^2{}_L$ and $\tau^1{}_U \subseteq \tau^2{}_U$ then we say that τ^2 is finer than τ^1 or τ^1 is coarser than τ^2. If $\tau^1{}_L$ is a proper sub collection of $\tau^2{}_L$ and $\tau^1{}_U$ is a proper sub collection of $\tau^2{}_U$ then we say that τ^2 is strictly finer than τ^1 or τ^1 is strictly coarser than τ^2.

Theorem 3.16. For any two rough topologies on a rough set $X = (X_L, X_U)$ one is finer or coarser than the other iff their lower and upper rough topologies are so.

Proof: Proof is immediate from definition 3.15.

Corollary 3.17. The discrete rough topology is finer than and the indiscrete rough topology is coarser than any other rough topologies on any rough set. Thus the discrete rough topology is the biggest rough topology and the indiscrete rough topology is the smallest rough topology on any arbitrary rough set.

Definition 3.18. In a $RTS(X, \tau)$, where $X = (X_L, X_U)$ and $\tau = (\tau_L, \tau_U)$, a sub collection β_L of τ_L is said to be a base for τ_L if every member of τ_L can be expressed as the union of some members of β_L. Another sub collection β_U of τ_U is said to be a base for τ_U if every member of τ_U can be expressed as the union of some members of β_U. If β_L is a base for τ_L and β_U is a base for τ_U then the pair $\beta = (\beta_L, \beta_U)$ is known as a rough base for the rough topology $\tau = (\tau_L, \tau_U)$ on $X = (X_L, X_U)$. Then β_L is known as the lower base for $\tau = (\tau_L, \tau_U)$ and β_U is known as the upper base for $\tau = (\tau_L, \tau_U)$.

Theorem 3.19. Consider the $RTS(X,\tau)$, where $X=(X_L, X_U)$ and $\tau=(\tau_L,\tau_U)$. Let β_L and β_U are families of subsets of X_L and X_U respectively then $\beta=(\beta_L,\beta_U)$ is a rough base for τ iff for any rough open set $A=(A_L, A_U)$ of (X,τ), $x\in A_L$ and $y\in A_U$ then there exist $B_L\in\beta_L$ and $B_U\in\beta_U$ such that $x\in B_L\subseteq A_L$ and $y\in B_U\subseteq A_U$.

Proof: Consider the $RTS(X,\tau)$, where $X=(X_L,X_U)$ and $\tau=(\tau_L,\tau_U)$. Let β_L and β_U are families of subsets of X_L and X_U respectively such that $\beta=(\beta_L,\beta_U)$ is a rough base for τ. Also consider any rough open set $A=(A_L, A_U)$ of (X,τ) and let $x\in A_L$ and $y\in A_U$ be arbitrary points.

Now $x\in A_L$, $A_L\in\tau_L$ and β_L is a base for τ_L implies A_L can be expressed as the union of some members of β_L. Hence $\exists B_i\in\beta_L$ such that $x\in B_i$ and $B_i\subseteq A_L$. Choose such a B_i as B_L. Therefore $x\in B_L\subseteq A_L$.

Similarly by the same argument, $\exists B_U\in\beta_U$, such that, $y\in B_U\subseteq A_U$.

Conversely suppose β_L and β_U are families of subsets of X_L and X_U respectively such that for any rough open set $A=(A_L, A_U)$ of (X,τ), $x\in A_L$ and $y\in A_U$ then there exist $B_L\in\beta_L$ and $B_U\in\beta_U$ such that $x\in B_L\subseteq A_L$ and $y\in B_U\subseteq A_U$. To prove $\beta=(\beta_L,\beta_U)$ is a rough base for τ. Let $C=(C_L, C_U)$ be any arbitrary rough open subset of the $RTS(X,\tau)$. For each $x\in C_L$, then there exist $B_{x_L}\in\beta_L$ such that $x\in B_{x_L}\subseteq C_L$, by our assumption. Therefore $C_L=\bigcup_{x\in C_L} B_{x_L}$. Which implies C_L can be expressed as the union of some members of β_L. Hence β_L is a lower base for the rough topology τ, since $C=(C_L, C_U)$ is arbitrary.

Similarly by the same argument C_U can be expressed as the union of some members of β_U and there for β_U is an upper base for the rough topology τ. Thus $\beta=(\beta_L,\beta_U)$ is a rough base for τ. Hence the theorem.

Definition 3.20. Let (X,τ) be any RTS, where $X=(X_L,X_U)$ and $\tau=(\tau_L,\tau_U)$. Let $A=(A_L,A_U)$ be any rough subset of X. Then the lower closure of A is the closure of A_L in (X_L,τ_L) and is defined as the intersection of all closed subsets of (X_L,τ_L) containing A_L and it is denoted by $\overline{A_L}$. Also the upper closure of A is the closure of A_U in (X_U,τ_U) and is defined as the intersection of all closed subsets of (X_U,τ_U) containing A_U and it is denoted by $\overline{A_U}$. Then the rough closure of $A=(A_L, AU)$ is denoted by $\overline{A}$ and is defined as $\overline{A}=(\overline{A_L},\overline{A_U})$.

Theorem 3.21. $\overline{A}=(\overline{A_L},\overline{A_U})$ is a rough closed subset of (X,τ). Moreover $\overline{A}$ is the smallest rough closed subset of (X,τ) containing A.

Proof: Let $A=(A_L, A_U)$ be any rough subset of a RTS (X,τ) and $\overline{A}=(\overline{A_L},\overline{A_U})$. Which implies $\overline{A_L}=\cap\{B\subseteq X_L / A_L\subseteq B, X_L-B\in\tau_L\}$ and $\overline{A_U}=\cap\{B\subseteq X_U / A_U\subseteq B, X_U-B\in\tau_U\}$. Now being the arbitrary intersection of closed subsets, $\overline{A_L}$ is a closed subset of (X_L,τ_L) and $\overline{A_U}$ is a closed subset of (X_U,τ_U). Hence $\overline{A}=(\overline{A_L},\overline{A_U})$ is a rough closed subset of (X,τ). Also by being the intersection of all closed sets containing A_L, $\overline{A_L}$ is the smallest closed subset of (X_L,τ_L) containing A_L. Similarly $\overline{A_U}$ is the smallest closed subset of (X_U,τ_U), containing A_U. Therefore $\overline{A}=(\overline{A_L},\overline{A_U})$ is the smallest rough closed subset of (X,τ) containing A.

Theorem 3.22. $A=(A_L, A_U)$ is rough closed $\Leftrightarrow$ $\overline{A}=A$

Proof: If A is rough closed implies A is the smallest rough closed subset containing A. Therefore $\overline{A}=A$ by theorem 3.21.

Conversely suppose $\overline{A} = A$. Then $\left(\overline{A_L}, \overline{A_U}\right)$ = (A_L, A_U) . Which implies $\overline{A_L} = A_L$ and $\overline{A_U} = A_U$. Therefore A_L and A_U are closed subsets of (X_L, τ_L) and (X_U, τ_U) respectively. Thus $A = (A_L, A_U)$ is rough closed.

Corollary 3.23. $\overline{X} = X$ and $\overline{\phi} = \phi$

Proof: Proof follows from theorem 3.22

Theorem 3.24. Let $A = (A_L, A_U)$ and $B = (B_L, B_U)$ be any rough subset of the *RTS*, (X, τ). If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$

Proof: Suppose $A \subseteq B$. That is (A_L, A_U) $\subseteq$ (B_L, B_U). Which implies $A_L \subseteq B_L$ and $A_U \subseteq B_U$. Then $\overline{A_L} \subseteq \overline{B_L}$ and $\overline{A_U} \subseteq \overline{B_U}$, by theorem 2.21.
Hence $\left(\overline{A_L}, \overline{A_U}\right) \subseteq \left(\overline{B_L} \subseteq \overline{B_U}\right)$
That is $\overline{A} \subseteq \overline{B}$.

Theorem 3.25. For any rough subset $A = (A_L, A_U)$ of the *RTS* (X, τ), $\overline{\overline{A}} = \overline{A}$.

Proof: Let $A = (A_L, A_U)$, then $\overline{A} = \left(\overline{A_L}, \overline{A_U}\right)$. Which implies $\overline{A}$ is rough closed subset of the *RTS* (X, τ) by theorem 3.21. Hence $\overline{\overline{A}} = \overline{A}$ by theorem 3.22.

Theorem 3.26. If $A = (A_L, A_U)$ and $B = (B_L, B_U)$ be any rough subset of the *RTS* (X, τ). Then $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof: Let $A = (A_L, A_U)$ and $B = (B_L, B_U)$ be any rough subset of the *RTS* (X, τ). Then $\overline{A} = \left(\overline{A_L}, \overline{A_U}\right)$ and $\overline{B} = \left(\overline{B_L}, \overline{B_U}\right)$. We have $A_L \subseteq \overline{A_L}$ and $B_L \subseteq \overline{B_L}$. Therefore $A_L \cup B_L \subseteq \overline{A_L} \cup \overline{B_L}$. Then $\overline{A_L \cup B_L} \subseteq \overline{A_L} \cup \overline{B_L}$, Since $\overline{A_L} \cup \overline{B_L}$ is closed set containing $A_L \cup B_L$ and $\overline{A_L \cup B_L}$ is the smallest closed set containing $A_L \cup B_L$.

Similarly $\overline{A_U \cup B_U} \subseteq \overline{A_U} \cup \overline{B_U}$. Hence $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

For the other way inclusion we have $A_L \subseteq A_L \cup B_L$ and $B_L \subseteq A_L \cup B_L$. Hence $\overline{A_L} \subseteq \overline{A_L \cup B_L}$ and $\overline{B_L} \subseteq \overline{A_L \cup B_L}$, by theorem 2.21. Therefore $\overline{A_L} \cup \overline{B_L} \subseteq \overline{A_L \cup B_L}$

Similarly $\overline{A_U} \cup \overline{B_U} \subseteq \overline{A_U \cup B_U}$. Therefore $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Hence ($\overline{A_L \cup B_L}$, $\overline{A_U \cup B_U}$)=($\overline{A_L} \cup \overline{B_L}$, $\overline{A_U} \cup \overline{B_U}$). Which implies $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Definition 3.27. A rough subset *A* of the *RTS* (X, τ) is said to be dense in *X* if $\overline{A} = X$. That is a rough subset $A = (A_L, A_U)$ is dense in *X* if $\overline{A_L} = X_L$ and $\overline{A_U} = X_U$.

Theorem 3.28. A rough subset $A = (A_L, A_U)$of a RTS (X, τ) is dense in *X* iff for every non empty rough open sets $B = \left(B_L, B_U\right)$ of (X, τ), $A_L \cap B_L \neq \phi$ and $A_U \cap B_U \neq \phi$

Proof: Suppose $A = (A_L, A_U)$ is dense in *X*. Then $\overline{A} = \left(\overline{A_L}, \overline{A_U}\right) = \left(X_L, X_U\right) = X$

Therefore $\overline{A_L} = X_L$ and $\overline{A_U} = X_U$. Now let $B = \left(B_L, B_U\right)$be any non empty rough open set in (X, τ). Then $A \cap B = \left(A_L \cap B_L, A_U \cap B_U\right)$.

Suppose $A_L \cap B_L = \phi$. Then $A_L \subseteq (X_L \setminus B_L)$. Which implies $\overline{A_L} \subseteq (X_L \setminus B_L)$, since $B_L \in \tau_L$ and therefore $(X_L \setminus B_L)$ is closed. But $(X_L \setminus B_L)$ is a proper subset of X_L, which contradicts $\overline{A_L} = X_L$. Hence $A_L \cap B_L \neq \phi$.

Similarly $A_U \cap B_U \neq \phi$

Conversely suppose $A = (A_L, A_U)$ be a rough subset of X such that for every non empty rough open sets B = (B_L, B_U) of (X, τ), $A_L \cap B_L \neq \phi$ and $A_U \cap B_U \neq \phi$. Then the only closed set containing A_L in (X_L, τ_L) is X_L and the only closed set containing A_U in (X_U, τ_U) is X_U. Therefore $\overline{A_L} = X_L$ and $\overline{A_U} = X_U$. Hence $\overline{A} = (\overline{A_L}, \overline{A_U}) = (X_L, X_U) = X$. Therefore A is dense in X. Hence the theorem.

4. CONCLUSIONS AND FUTURE WORK

Rough set theory is a mathematical tool to deal with vagueness or imperfect knowledge using a boundary region approach. Rough set theory is not an alternative to classical set theory but it is embedded in it. In this paper we defined general topology on an arbitrary rough set. Then some topological properties of the resulting Rough Topological Space, are studied. This is just a beginning of a new area and with the help of the ideas presented in this paper several topological properties of the *RTS*, the rough topological spaces can be studied. So there are a lot of research scopes in this area.

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