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SEMIRINGS SATISFYING THE IDENTITIES

T. Vasanthi* & N. Sulochana

Department of Applied Mathematics, Yogi Vemana University, Kadapa – 516003(A.P), India

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ABSTRACT

In this paper, we study the properties of semirings satisfying the identity a + ab = b for all a, b in S. We establish that a + b = ab = b for all a, b in S if (S, ·) is band.

Keywords: *Non-negatively ordered; Non-positively ordered; PRD; IMP; Mono semiring.*

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1. INTRODUCTION:

A triple (S, +, ·) is called a semiring if (S, +) is a semigroup; (S, ·) is semigroup; a(b + c) = ab + ac and (b + c)a = ba + ca for every a, b, c in S. A semiring (S, +, ·) is said to be a totally ordered semiring if the additive semigroup (S, +) and multiplicative semigroup (S, ·) are totally ordered semigroups under the same total order relation. An element x in a totally ordered semigroup (S, ·) is non-negative (non-positive) if $x^2 \geq x$($x^2 \leq x$). A totally ordered semigroup (S, ·) is said to be non-negatively (non-positively) ordered if every one of its elements is non-negative (non-positive). (S, ·) is positively (negatively) ordered in strict sense if $xy \geq x$ and $xy \geq y$ ($xy \leq x$ and $xy \leq y$) for every x and y in S. A semigroup (S, +) is said to be a band if a + a = a for all a in S. A semiring (S, +, ·) is said to satisfy Integral Multiple Property (IMP) if $a^2 = na$ for all a in S where the positive integer n depends on the element a. A semiring (S, +, ·) with additive identity zero which is multiplicative zero is said to be zero square ring if $x^2 = 0$ for all $x \in S$. Zeroid of a semiring (S, +, ·) is the set of all x in S such that x + y = y or y + x = y for some y in S. We may also term this as the zeroid of (S, +, ·). A semiring (S, +, ·) is said to be a Positive Rational Domain (PRD) if and only if (S, ·) is an abelian group. A semiring (S, +, ·) with additive identity zero is said to be zerosumfree semiring if x + x = 0 for all $x \in S$. A semiring (S, +, ·) said to satisfy a mono semiring if a + b = ab for every a, b in S.

2. Semirings satisfying the identity a + ab = b for all a, b in S:

Theorem 2.1: Let (S, +, ·) be a zero square semiring with additive identity 0. If S satisfying the identity a + ab = b for all a, b in S then $S^2 = \{0\}$.

Proof: consider a + ab = b for all a, b in S

$\Rightarrow (a + ab)\, b = b^2$

$\Rightarrow ab + ab^2 = b^2$

$\Rightarrow ab + a.0 = 0$ (since (S, +, ·) is a zero square semiring)

$\Rightarrow ab + 0 = 0$

$\Rightarrow ab = 0$

Also a + ab = b

$\Rightarrow b(a + ab) = b^2$

$\Rightarrow ba + b(ab) = b^2$

Corresponding author: *T. Vasanthi**
Department of Applied Mathematics, Yogi Vemana University, Kadapa – 516003(A.P), India

$\Rightarrow ba + b.0 = 0$

$\Rightarrow ba + 0 = 0$

$\Rightarrow ba = 0$

$\therefore ab = ba = 0$

Hence $S^2 = \{0\}$

Theorem 2.2: Let (S, +, ·) be a semiring and satisfying the identity a + ab = b for all a, b in S. If (S, ·) is a band, then
(i) a + ab = a + b = ab = b and a(a + b) = a(ab) = b for all a, b in S.
(ii) (S, +) is band

Proof: (i) consider a + ab = b for all a, b in S

$\Rightarrow a^2 + a^2b = b$ (since (S, ·) is a band)

$\Rightarrow a(a + ab) = b$

$\Rightarrow ab = b$

Also a + ab = b

$\Rightarrow a + b = b$

Therefore a + ab = a + b = ab = b

And also $a(a + b) = a^2 + ab = a + ab = b$

$a(ab) = a^2b = b$

Therefore a(a + b) = a(ab) = b

(ii) Suppose a + a.a = a, for all a in S

$\Rightarrow a + a^2 = a$

$\Rightarrow a + a = a$ (since (S, ·) is a band)

This is evident from the following example:

Example:

+	a	b	c
a	a	b	c
b	a	b	c
	a	b	c

•	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

Theorem 2.3: Let (S, +, ·) be a semiring satisfying the identity a + ab = b for all a, b in S and let S contain multiplicative identity 1. Assume that either a or b can be the multiplicative identity but not both. Then the following are true.
(i) 1 + b = b and ab = b for all a, b in S
(ii) S is a mono semiring
(iii) (S, +) is a commutative
(iv) (S, ·) is a band
(v) $a^n + b^n = a + b$ for all $n \geq 1$

Proof: (i). Given that (S, +, ·) be a semiring

Let 1 is the multiplicative identity of S

Let S satisfy the condition a + ab = b for all a, b in S

Since $1 \in S$, $1 + 1.b = b$, for all $b \in S$

$1 + b = b$, for all $b \in S$

Consider a + ab = b

$a(1 + b)= b$

$ab = b$ ----------> (I)

$\therefore$ a + ab = ab

(ii) Consider a + ab = b for all a, b in S

$\Rightarrow a + a + ab = a + b$

$\Rightarrow a + a(1 + b) = a + b$

$\Rightarrow a +ab = a + b$

$\Rightarrow ab = a + b$

Hence S is a mono semiring.

(iii) To show that (S, +) is commutative

Since a + ab = b for all a, b in S

$\Rightarrow a + ab + a = b + a$

$\Rightarrow a + b + a = b + a$

$\Rightarrow ab + a = b + a$ (since S is mono semiring)

$\Rightarrow a(b + 1) = b + a$

$\Rightarrow ab = b + a$ (since S is mono semiring, b.1 = b + 1, for all 1, b in S,)

$\Rightarrow a + b = b + a$

(iv) Consider $a + a^2 = a(1 + a)$

$= a.a$

$= a^2$

Taking a = b in a + ab = b, for all a, b in S

$\Rightarrow a + a.a= a$, for all a in S

$\Rightarrow a + a^2 = a$

$\therefore a^2 = a + a^2 = a \Rightarrow a= a^2$

Hence $(S, \cdot)$ is a band

(v) $a^2 + b^2 = a + b$

$\Rightarrow a^3 + b^3 = a^2.a + b^2.b = a + b$

Similarly $a^n + b^n = a + b$, for all $n \geq 1$

Note: - If both a and b is equal to 1 then S reduces to a singleton set

Theorem 2.4: Let $(S, +, \cdot)$ be a semiring and let $a + ab = b$ for all a, b in S. If $(S, +)$ is right cancellative then
(i) $(S, +)$ is a band
(ii) $(S, \cdot)$ is a band if S satisfies IMP

Proof: (i) Consider $a + ab = b$ for all a, b in S
$\Rightarrow a^2 + a^2b = ab$

But $a^2 + a^2b = b$ for all a^2, b in S

$\therefore$ $ab = b$ for all a, b in S

$\Rightarrow a + ab = a + b$

$\Rightarrow b = a + b$

$\Rightarrow a + b = a + a + b$

By using $(S, +)$ is right cancellative

$\Rightarrow a = a + a \longrightarrow$ (I)

$\therefore$ $(S, +)$ is a band

(ii) $a = a + a = 2a$ ($\because$ From (I))

$a + a = 2a + a = 3a$

Continuing like this
$\Rightarrow na = a \longrightarrow$ (II)

Implies S satisfies IMP, i.e, $a^2 = na \longrightarrow$ (III)

$\therefore$ From (II) and (III), $a^2 = a$ for all a in S

Hence $(S, \cdot)$ is a band

Theorem 2.5: Let $(S, +, \cdot)$ be a zerosumfree semiring with additive identity zero. Then S satisfies the identity $a + ab = b$ for all a, b in S if and only if S is a mono semiring.

Proof: Assume $a + ab = b$ for all a, b in S

$\Rightarrow a + a + ab = a + b$

$\Rightarrow 0 + ab = a + b$ (since S is a zerosumfree semiring)

$\Rightarrow ab = a + b$

$\therefore$ S is a mono semiring
Conversely
Assume S is a mono semiring

Suppose $a + a = 0$

$\Rightarrow a + a + b = 0 + b$

$\Rightarrow a + a + b = b$

$\Rightarrow a + ab = b$ for all a, b in S

Theorem 2.6: Let (S, +, ·) be a zerosumfree semiring satisfying the identity a + ab = b for all a, b in S. Then S is a zero square semiring.

Proof: consider a + a.a = a for all a in S

$\Rightarrow a + a + a^2 = a + a$

$\Rightarrow 0 + a^2 = 0$

$\Rightarrow a^2 = 0$

$\therefore$ S is a zero square semiring

Theorem 2.7: Let (S, +, ·) be a PRD satisfying the identity a + ab = b for all a, b in S. Then the following are true

(a) $(ab^{-1})^{-1} = ab^{-1} + a + b$

(b) $a + a = a^{-1}$ for all a in S. In particular $a = a^{-1}$ if (S, +) is a band

(c) $(1 + a^{-1}) = (1 + a)\,(b^{-1} + 1)$ for all , b in S

Proof: **(a)** Suppose a + ab = b for all a, b in S

$\Rightarrow a^{-1}(a + ab) = a^{-1}b$

$\Rightarrow a^{-1}a + a^{-1}ab = a^{-1}b$

$\Rightarrow 1 + b = a^{-1}b \longrightarrow$ (I)

Consider a + ab = b for all a, b in S

$\Rightarrow (a + ab)b^{-1} = bb^{-1}$

$\Rightarrow ab^{-1} + a = 1$

$\Rightarrow ab^{-1} + a + b = 1 + b$

$\Rightarrow ab^{-1} + a + b = a^{-1}b$ ($\because$ from (I))

$\Rightarrow ab^{-1} + a + b = (b^{-1}a)^{-1}$

$\Rightarrow ab^{-1} + a + b = (ab^{-1})^{-1}$ (since S is PRD)

Hence $(ab^{-1})^{-1} = ab^{-1} + a + b$ for all a, b in S

(b) Suppose a + ab = b for all a, b in S

$\Rightarrow a^{-1}b^{-1}(a + ab) = (a^{-1}b^{-1})b$

$\Rightarrow b^{-1} + 1 = a^{-1} \longrightarrow$ (II) (since (S, ·) is an abelian group)

Also a + ab = b

$\Rightarrow b^{-1}(a + ab) = b^{-1}b$

$\Rightarrow b^{-1}a + a = 1 \longrightarrow$ (III)

$\Rightarrow b^{-1} + b^{-1}a + a = b^{-1} + 1$

$\Rightarrow b^{-1} + b^{-1}a + a = a^{-1}$ ($\because$ from (II))

$\Rightarrow a + a = a^{-1}$ for all a in S (since $b^{-1} + b^{-1}a = a$ for all b^{-1}, a in S)

In particular, if (S, +) is band

$\Rightarrow a + a = a$ and $a + a = a^{-1}$

$\therefore\ a = a^{-1}$ for all in S

(c) Adding (II) and (III) we have

$$1 + a^{-1} = b^{-1}a + a + b^{-1} + 1$$

$$= ab^{-1} + a.1 + 1.b^{-1} + 1.1$$

$$= a(b^{-1} + 1) + 1(b^{-1} + 1)$$

$$= (a + 1)(b^{-1} + 1) \text{ for all a, b in S}$$

3. Ordering on a + ab = b for all a, b in S:

Theorem 3.1: If (S, +, ·) be a totally ordered semiring satisfying the identity a + ab = b for all a, b in S and (S, +) is commutative. If (S, +) is non-negatively ordered (non-positively ordered), then (S, +) is p.t.o (n.t.o).

Proof: Assume (S, +) is non-negatively ordered

$$\Rightarrow a + a \geq a$$

$$\Rightarrow a + a + ab \geq a + ab$$

$$\Rightarrow a + b \geq b \longrightarrow \text{(I)}$$

Suppose $a + b < a$

$$\Rightarrow a + b + ab \leq a + ab$$

$$\Rightarrow a + ab + b \leq b \qquad \text{(since (S, +) is commutative)}$$

$$\Rightarrow b + b \leq b$$

Which is a contradiction to (S, +) is non-negatively ordered

Therefore $a + b \geq a \longrightarrow$ (II)

From (I) and (II) we have, $a + b \geq a$ and $a + b \geq b$

Therefore (S, +) is p.t.o

Similarly, we can prove that (S, +) is n.t.o if (S, +) is non-positively ordered

Theorem 3.2: If (S, +, ·) be a totally ordered semiring satisfying the identity a + ab = b for all a, b in S and (S, +) is commutative. If (S, ·) is non-negatively ordered (non-positively ordered), then (S, ·) is p.t.o (n.t.o).

Proof: Suppose $ab < a$

$$\Rightarrow ab^2 \leq ab < a$$

$$\Rightarrow ab^2 \leq a$$

$$\Rightarrow ab^2 + ab \leq a + ab$$

$$\Rightarrow (ab + a)b \leq b$$

$$\Rightarrow b^2 \leq b \qquad \text{(since (S, +) is commutative)}$$

Which is contradiction to (S, ·) non-negatively ordered
Therefore $ab \geq a \longrightarrow$ (I)

Also $b = a + ab$

Implies $ab = a^2 + a^2b = b$ for all a^2, b in S

Therefore $ab = b$

Obviously $ab \geq b \longrightarrow$ (II)

From (I) and (II) we have, (S, ·) is p.t.o

Similarly, we can prove that (S, ·) is n.t.o if (S, ·) is non-positively ordered

Theorem 3.3: If (S, +, ·) be a t.o semiring satisfying the identity $a + ab = b$ for all a, b in S and (S, +) is band. If (S, ·) is p.t.o(n.t.o), then (S, +) is p.t.o(n.t.o).

Proof: Suppose (S, +) is band

Consider $a + ab = b$ for all a, b in S

$\Rightarrow a + (a + a)\, b = b$

$\Rightarrow a + ab + ab = b$

$\Rightarrow b + ab = b \longrightarrow$ (I)

Assume (S, ·) is p.t.o

Which implies $ab \geq a$

$\Rightarrow a + ab \geq a + a$

$\Rightarrow b \geq a + a$

$\Rightarrow a + b \geq a + (a + a)$

$\Rightarrow a + b \geq a + a$ (since (S, +) is band)

$\Rightarrow a + b \geq a \longrightarrow$ (II)

Now $a + b \geq a$

$\Rightarrow a + b + ab \geq a + ab$

$\Rightarrow a + b \geq b \longrightarrow$ (III) ($\because$ from (I))

From (II) and (III) we have (S, +) is p.t.o

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