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SOME INEQUALITIES CONCERNING Q-GAMMA FUNCTIONS

W. T. Sulaiman*

Department of Computer Engineering, College of Engineering
University of Mosul, Iraq.
E-mail: waadsulaiman@hotmail.com

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ABSTRACT

Several new inequalities concerning q-gamma functions are proved.

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1. INTRODUCTION:

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{1.1}$$

The psi or digamma function, the logarithmic derivative of the gamma function is defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0. \tag{1.2}$$

The q-analogue of $\Gamma(x)$ is called q-gamma function, was introduced by Jackson in 1904 and defined for $x > 0$ and $0 < q < 1$ by

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}} \tag{1.3}$$

The q-gamma function satisfies the following

$$\lim_{q \to 1^{1-}} \Gamma_q(x) = \Gamma(x) \tag{1.4}$$

and

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q} \Gamma_q(x), \quad \Gamma_q(1) = 1. \tag{1.5}$$

The q-analogue of $\psi(x)$ is called the q-psi function defined by

$$\psi_q(x) = \frac{\Gamma_q'(x)}{\Gamma_q(x)} . \tag{1.6}$$

***Corresponding author: W. T. Sulaiman*, *E-mail: waadsulaiman@hotmail.com**
Department of Computer Engineering, College of Engineering University of Mosul, Iraq

From (1.3) and (1.6) it follows that

$$\psi_q(x) = -\log(1-q) + \log q \sum_{n=0}^{\infty} \frac{q^{n+x}}{1-q^{n+x}} = -\log(1-q) + \log q \sum_{n=0}^{\infty} \frac{q^{nx}}{1-q^{n}}. \quad (1.7)$$

It is well-known that ψ_q' is strictly completely monotonic on $(0,\infty)$ that is (see [1,page260])

$$(-1)^n \left(\psi_q'(x)\right)^{(n)} > 0 \quad \text{for} \quad x > 0,\ n \geq 0. \quad (1.8)$$

Concerning q-gamma functions, the following results were achieved

Theorem: 1.1[3]. *Let* $x \in [0,1]$, $q \in (0,1)$, $a \geq b > 0$, c, d *positive real numbers with* $bc > ad > 0$ *and* $\psi_q(b+ax) > 0.$ *Then*

$$\frac{\Gamma_q(a)^c}{\Gamma_q(b)^d} \leq \frac{\Gamma_q(a+bx)^c}{\Gamma_q(b+ax)^d} \leq \frac{\Gamma_q(a+b)^c}{\Gamma_q(a+b)^d}. \quad (1.9)$$

Theorem: 1.2[2]. *Let* $0 < q < 1$, $A \leq 0$, *and* $b \geq 0$. *Then the function*

$$f(x) = x^A \left[\Gamma(1+\tfrac{b}{x})\right]^x \quad (1.10)$$

decreases with respect to $x > 0$.

The object of the present paper is to give several new inequalities concerning the q-gamma functions.

2. RESULTS:

The following generalizes theorem 1.2.

Theorem: 2.1. *Let f be a non-negative real function such that* $f' < 0$ *and* $f'' \geq 0$. *Let* $0 < q < 1$, $\lim\limits_{x\to\infty} \frac{f'(x)}{f(x)} = 0$, $f(x)f''(x) \geq \left(f'(x)\right)^2$ *and* $b \geq 0$. *Then the function*

$$F(x) = f(x)\left[\Gamma_q(1+\tfrac{b}{x})\right]^x \quad (2.1)$$

decreases with respect to $x > 0$.

Proof: We have,

$$\log F(x) = \log f(x) + x \log \Gamma_q(1+\tfrac{b}{x}),$$

$$F'(x) = \left(\frac{f'(x)}{f(x)} - \frac{b}{x}\psi_q(1+\tfrac{b}{x}) + \log\Gamma_q(1+\tfrac{b}{x})\right)F(x) = g(\tfrac{1}{x})\,F(x).$$

By setting $x = 1/y$, we have

$$g(y) = \frac{f'(y^{-1})}{f(y^{-1})} - by\,\psi(1+by) + \log\Gamma_q(1+by).$$

Differentiating the above leads to

$$g'(y)=\frac{\left(f'(y^{-1})\right)^2-f(y^{-1})f''(y^{-1})}{y^2 f(y^{-1})}-by\psi'(1+by)<0.$$

Therefore g is non-increasing. As $g(0)=0$, then $g(y)$ and hence $g(x^{-1})<0$, which implies $F'(x)<0$. That is F is decreasing.

Remark: 1 It may be mentioned that theorem 1.2 follows from theorem 2.1 by putting $f(x)=x^A,\ A\le 0.$ In a similar way as the Beta function, we define the q-beta function by

$$B_q(x,y)=\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)},\quad x,y>0. \tag{2.2}$$

Lemma: 2.2 Let $0<q<1,\ 0<s\le t,\ 0<t+m\le s+n,$ *then*

$$\frac{q^s}{1-q^m}\ge\frac{q^t}{1-q^n}.$$

Proof: We have

$$q^s+q^{t+m}\ge q^t+q^{s+n},$$

which implies

$$q^s-q^{s+n}\ge q^t-q^{t+m},$$

$$q^s\left(1-q^n\right)\ge q^t(1-q^m),$$

hence the result.

Theorem: 2.3 Let $x,\ a,\ b,\ \alpha>0.$ *Then the function* $B_q\left(\alpha a+bx,\alpha b+ax\right)$ *is non-increasing in* x.

Proof: Let

$$f(x)=B_q\left(\alpha a+bx,\alpha b+ax\right),$$

then, we have

$$\log f(x)=\log\Gamma_q(\alpha a+bx)+\log\Gamma(\alpha b+ax)-\log\Gamma\left((a+b)(\alpha+x)\right),$$

and

$$\begin{aligned}\frac{f'(x)}{f(x)}&=b\psi(\alpha a+bx)+a\psi(\alpha b+ax)-(a+b)\psi\left((a+b)(\alpha+x)\right)\\&=\log q\left(a\sum_{i=0}^{\infty}\frac{q^{\alpha a+bx+i}}{1-q^{\alpha a+bx+i}}+b\sum_{i=0}^{\infty}\frac{q^{\alpha b+ax+i}}{1-q^{\alpha b+ax+i}}-(a+b)\sum_{i=0}^{\infty}\frac{q^{(a+b)(\alpha+x)+i}}{1-q^{(a+b)(\alpha+x)+i}}\right)\\&=a\log q\left(\sum_{i=0}^{\infty}\left(\frac{q^{\alpha a+bx+i}}{1-q^{\alpha a+bx+i}}-\frac{q^{(a+b)(\alpha+x)+i}}{1-q^{(a+b)(\alpha+x)+i}}\right)q^i\right)\\&\quad+b\log q\left(\sum_{i=0}^{\infty}\left(\frac{q^{\alpha b+ax+i}}{1-q^{\alpha b+ax+i}}-\frac{q^{(a+b)(\alpha+x)+i}}{1-q^{(a+b)(\alpha+x)+i}}\right)q^i\right).\end{aligned}$$

Now, making use of lemma 2. With

$$s=\alpha a+bx,\quad t=(a+b)(\alpha+x),\quad m=\alpha a+bx+i\quad \textit{and}\quad n=(a+b)(\alpha+x)+i$$

and then exchang a and b, we get $f'(x)/f(x) \le 0,$ which implies $f'(x) \le 0$.

Therefore $f(x)$ is non-increasing. The proof is complete.

Theorem: 2.4 *Let* $x > 0,\ c,\ d,\ M,\ N > 0,\ Mb^2 < Nd^2,\ a > c,\ b > d,\ 0 < q < 1,\ f > 0$ *and* $f'' < 0$ *(that is* f' *is decreasing). Then the function*

$$L(x) = f(x)\frac{\left(\Gamma_q\left(1+\frac{a}{x}\right)\right)^{Mx}}{\left(\Gamma_q\left(1+\frac{b}{x}\right)\right)^{Nx}}$$

is decreasing for $x > 0.$

Proof: We have

$$\log L(x) = \log f(x) + Mx\log\left(\Gamma_q\left(1+\tfrac{a}{x}\right)\right) - Nx\log\left(\Gamma_q\left(1+b\right)\right),$$

$$\frac{L'(x)}{L(x)} = \frac{f'(x)}{f(x)} - b\frac{M}{x}\frac{\Gamma_q'\left(1+\frac{a}{x}\right)}{\Gamma_q\left(1+\frac{a}{x}\right)} + M\,\Gamma_q\left(1+\tfrac{a}{x}\right) + N\frac{b}{x}\frac{\Gamma_q'\left(1+\frac{b}{x}\right)}{\Gamma_q\left(1+\frac{b}{x}\right)} - N\Gamma_q\left(1+\tfrac{b}{x}\right)$$

$$= \frac{f'(x)}{f(x)} - b\frac{M}{x}\psi_q\left(1+\tfrac{a}{x}\right) + M\,\Gamma_q\left(1+\tfrac{a}{x}\right) + N\frac{b}{x}\psi_q\left(1+\tfrac{b}{x}\right) - N\Gamma_q\left(1+\tfrac{b}{x}\right)$$

$$= \frac{G(x)}{f(x)}.$$

Now,

$$G'(x) = \frac{f''(x)}{f(x)} - \frac{\left(f'(x)\right)^2}{f^2(x)} + M\frac{a^2}{x^3}\psi_q'\left(1+\tfrac{a}{x}\right) - N\frac{b^2}{x^3}\psi_q'\left(1+\tfrac{b}{x}\right)$$

$$\le \frac{f''(x)}{f(x)} - \frac{\left(f'(x)\right)^2}{f^2(x)} + M\frac{a^2}{x^3}\left(\psi_q'\left(1+\tfrac{a}{x}\right) - \psi_q'\left(1+\tfrac{b}{x}\right)\right)$$

Since $\psi_q''(x) < 0,$ then ψ_q' is decreasing. Therefore $G'(x) \le 0$, which implies $L'(x) \le 0,$ and hence $L(x)$ is decreasing.

Theorem: 2.5 *Let* $x, y > 0,\ 0 < q < 1,\ c, d > 0,\ a \ge c,\ b \ge d, \Gamma_q(a) \ge 1,\ \psi_q(c+dx) \ge 0.$ *Then the function*

$$H(x) = \frac{\Gamma\left(a+\frac{b}{y}\right)^x}{\Gamma\left(c+\frac{d}{x}\right)^y}$$

is non- decreasing for $x > 0.$

Proof: Since $\psi_q' > 0,$ then ψ_q is increasing and therefore $\psi(a+bx) \ge \psi(c+dx)$ .Then we have

$$\log H(x) = x\log\Gamma_q\left(a+\tfrac{b}{y}\right) - y\log\Gamma_q\left(c+\tfrac{d}{x}\right),$$

$$\frac{H'(x)}{H(x)} = \log\Gamma\left(a+\tfrac{b}{y}\right) + \frac{yd}{x^2}\psi_q\left(c+\tfrac{d}{x}\right) = \frac{g(x)}{F(x)}.$$

Now,

$$g(z) = \log\Gamma_q(a+bz) + yz^2 d\psi_q(c+dz), \qquad x^{-1} = z.$$

$$g'(z) = b\psi_q(a+bz) + ydz^2\psi_q'(c+dz) + 2yz\psi_q(c+dz)$$

$$\geq (b+2yz)\psi_q(c+dz) + ydz^2\psi_q'(c+dz)$$
$$\geq 0.$$

Therefore g is non-decreasing. Since $g(0) \geq 0,$ then $g(x) \geq 0,$ and hence $H'(x) \geq 0.$

Theorem: 2.6 Let $f(x) \geq g(x),\ af'(x) \geq bg'(x) > 0,\ \psi(g(x)) > 0,\ a, b > 0,$ *then the function*

$$h(x) = \frac{\Gamma_q(f(x))^a}{\Gamma_q(g(x))^b}$$

is non-decreasing in x.

Proof: We have

$$\log h(x) = a\log\Gamma_q(f(x)) - b\log\Gamma_q(g(x)),$$
$$\frac{h'(x)}{h(x)} = a\,f'(x)\frac{\Gamma_q'(f(x))}{\Gamma_q(f(x))} - b\,g'(x)\frac{\Gamma_q'(g(x))}{\Gamma_q(g(x))}$$
$$= a\,f'(x)\psi_q(f(x)) - b\,g'(x)\psi_q(g(x)).$$

Making use of (1.7), we have

$$\psi(f(x)) - \psi(g(x)) = \log\sum_{n=0}^{\infty}\frac{\left(q^{f(x)} - q^{g(x)}\right)q^n}{\left(1-q^{n+f(x)}\right)\left(1-q^{n+g(x)}\right)} \geq 0,$$

which implies

$$\frac{h'(x)}{h(x)} \geq bg'(x)\left(\psi(f(x)) - \psi(g(x))\right) \geq 0.$$

Therefore $h'(x) \geq 0$ and h is non-decreasing.

Remark: 2 Theorem 1.1 follows from theorem 2.6 by putting

$f(x) = a + bx,\quad g(x) = b + ax,\ 0 \leq x \leq 1,$ and replacing a, b by c, d respectively.

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