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EXISTENCE THEORY FOR BOUNDARY VALUE PROBLEM OF RANDOM DIFFERENTIAL INCLUSION

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ABSTRACT

In this paper, I prove the existence of random solution for the boundary value problem of second order multi-valued differential inclusion with non-convex case using a nonlinear alternative of Leray-Schauder type and on a selection theorem for lower semi continuous maps..

Keywords: *Random differential inclusion, multi-valued random operator, random solution, boundary value problem.*

2000 Mathematics Subject Classification: *60H25, 47H10.*

1. INTRODUCTION

Consider the two point boundary value problem (BVP) of second order differential inclusions

$$x''(t) \in F\left(t, x(t), x'(t)\right) \quad \text{a.e.} \quad t \in J \tag{1.1}$$

satisfying the boundary conditions

$$\left.\begin{aligned} a_0 x(t_0) - a_1 x'(t_1) = c_0 \\ b_0 x(t_0) + b_1 x'(t_1) = c_1 \end{aligned}\right\} \tag{1.2}$$

where the functions involved in (1.1) and (1.2) satisfy the following properties :

(a) $F : J \times R \times R \to \mathcal{P}_f(R)$,

(b) $a_0, a_1, b_0, b_1 \in R^+$ satisfying $a_0 a_1 (t_1 - t_0) + a_0 b_1 + a_1 b_0 > 0$ and

(c) $c_0, c_1 \in R$.

Let $J = [t_0, t_1]$ be a closed and bounded interval in R for some real numbers $t_0,\ t_1 \in R$ with $t_0 < t_1$. let $P_f(R)$ denote the class of all non-empty subsets of R with a property *f*. By a solution of the BVP (1.1) – (1.2), I mean a function $x \in AC^1(J, R)$ whose second derivative exists and is a member of $L^1(J, R)$ in $F(t, x, x')$, i.e. there exists a $v \in L^1(J, R)$ such that $v(t) \in F(t, x(t), x'(t))$ for a. e. $t \in J$ and $x''(t) = v(t)$ for all $t \in J$ satisfying (1.2), where $AC^1(J, R)$ is the space of continuous real-valued functions whose first derivative exists and is absolutely continuous on J.

The special cases of the BVP (1.1)-(1.2) have been discussed in the literature for existence of the solutions. The special case of the form

$$x''(t) = f\left(t, x(t), x'(t)\right), \quad \text{a.e. } t \in J \tag{1.3}$$

satisfying the boundary conditions (1.2) where $f : J \times R \to R$, $a_0, a_1, b_0, b_1 \in R_+$, $c_0, c_1 \in R$ and $a_0 a_1 (t_1 - t_0) + a_0 b_1 + a_1 b_0 > 0$ has been discussed in Bernfeld and Lakshmikantham [2] for the existence of solutions . Again, when $c_0 = c_1, a_1 = 0 = b_1, a_0 = b_0$, and *F* not depending on x', the BVP (1.1) – (1.2) reduces

$$y'' \in F(t, y) \;\; a.e. \;\; t \in J, \;\; y(t_0) = y(t_1). \tag{1.4}$$

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where $y = x$. This is a BVP of second order differential inclusions considered in Benchohra and Ntouyas [3]. Finally, the special case of the BVP consisting of the equation

$$y''(t) \in F\left(t, y(t)\right), \quad a.e. \ \ t \in J \tag{1.5}$$

satisfying the boundary conditions (1.2) has been studied in Halidias and Papageorgiou [5] via the method of lower and upper solutions. Thus the BVP (1.1) - (1.2) is more general and so is its importance in the theory of differential inclusions. Here in the present paper, I discuss the BVP (1.1) - (1.2) via a nonlinear alternative of Leray-Schauder type and on a selection theorem for lower semi continuous maps. I prove the main existence results for the BVP (1.1) – (1.2) when the right hand side has nonconvex values.

2. AUXILIARY RESULTS

I apply the following nonlinear alternative in the sequel.

Theorem 2.1 (O'Regan [8]) Let U and $\overline{U}$ be the open and closed subsets in a normed linear space X such that $0 \in U$ and let $T : \overline{U} \to \mathcal{P}_{cp,cv}(X)$ be a completely continuous multi-valued operator. Then either

(i) the operator inclusion $x \in Tx$ has a solution, or

(ii) there is an element $u \in \partial U$ such that $\lambda u \in Tu$ for some $\lambda > 1$, where ∂U is the boundary of U.

Corollary 2.1 Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be the open and closed balls in a normed linear space X centered at origin 0 of radius r and let $T : \overline{\mathcal{B}_r(0)} \to \mathcal{P}_{cp,cv}(X)$ be a completely continuous multi-valued operator. Then either

(i) the operator inclusion $x \in Tx$ has a solution, or

(ii) there is an element $u \in X$ such that $\|u\| = r$ satisfying $\lambda u \in Tu$ for some $\lambda > 1$.

Corollary 2.2 Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be the open and closed balls in a normed linear space X centered at origin 0 of radius r and let $T : \overline{B_r(0)} \to X$ be a completely continuous multi-valued map. Then either (1) the operator inclusion $x = Tx$ has a solution, or (2) there is an element $u \in X$ such that $\|u\| = r$ and $u = \lambda Tu$ for some $\lambda < 1$.

Now, I state a selection theorem due to Bressan and Colombo [4].

Theorem 2.2 Let Y be a separable metric space and let $N : Y \to \mathcal{P}_f\left(L^1(J, \mathbb{R})\right)$ be a multi-valued operator which has property (BC). Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $g : Y \to L^1(J, R)$ such that $g(y) \in N(y)$ for every $y \in Y$.

3. EXISTENCE RESULTS

I have written a useful result from the theory of boundary value problems of ordinary differential equations.

Lemma 3.1 *If* $f \in L^1(J, R)$, then the BVP

$$x''(t) = f(t) \quad a.e. \quad t \in J \quad and \quad \begin{cases} a_0 x(t_0) - a_1 x'(t_1) = c_0 \\ b_0 x(t_0) + b_1 x'(t_1) = c_1 \end{cases} \tag{3.1}$$

has a unique solution x given by

$$x(t) = z(t) + \int_{t_0}^{t_1} G(t,s) f(s) ds, \qquad t \in J, \tag{3.2}$$

where z is a unique solution of the homogeneous differential equation

$$x''(t) = 0 \quad a.e. \quad t \in J \quad and \quad \begin{cases} a_0 x(t_0) - a_1 x'(t_1) = c_0 \\ b_0 x(t_0) + b_1 x'(t_1) = c_1 \end{cases} \tag{3.3}$$

given by $$z(t)=\frac{c_0a_1(t_1-t)+c_0b_1+c_1a_0(t-t_0)+c_1b_0}{a_0a_1(t_1-t_0)+a_0b_1+a_1b_0},\quad t\in J \tag{3.4}$$

and $G(t,s)$ is the Green's function associated to the differential equation

$$x''(t)=0\quad a.e.\quad t\in J\quad and\quad \begin{cases} a_0x(t_0)-a_1x'(t_1)=0\\ b_0x(t_0)+b_1x'(t_1)=0\end{cases} \tag{3.5}$$

given
$$G(t,s)=\begin{cases}\dfrac{(a_1(t_1-t)+b_1)(a_0(s-t_0)+b_0)}{a_0a_1(t_1-t_0)+a_0b_1+a_1b_0}, & t_0\le s\le t\le t_1,\\[2mm] \dfrac{(a_1(t_1-s)+b_1)(a_0(t-t_0)+b_0)}{a_0a_1(t_1-t_0)+a_0b_1+a_1b_0}, & t_0\le t\le s\le t_1,\end{cases} \tag{3.6}$$

Remark 3.1 It is known that the function *z* belongs to the class $C^1(J,R)$. Therefore it is bounded on *J* and there is a constant $C_1>0$ with

$$C_1=\max\left\{\frac{c_0a_1(t_1-t_0)+c_0b_1+c_1a_0(t_1-t_0)+c_1b_0}{a_0a_1(t_1-t_0)+a_0b_1+a_1b_0},\frac{c_0b_1-c_0a_1+c_1a_0+c_1b_0}{a_0a_1(t_1-t_0)+a_0b_1+a_1b_0}\right\}$$

such that $$\|z\|=\max\left\{\sup_{t\in J}|z(t)|,\sup_{t\in J}|z'(t)|\right\}\le C_1.$$

Remark 3.2 It is easy to see that the Green's function $G(t,s)$ of Lemma 3.1 is continuous in $J\times J$ and $G_t(t,s)$ is continuous in $(a,b)\times(a,b)\setminus\{(t,t)\mid t\in J\}$ and satisfy the inequalities

$$|G(t,s)|=G(t,s)\le\frac{(a_1(t_1-t_0)+b_1)(a_0(t_1-t_0)+b_0)}{a_0a_1(t_1-t_0)+a_0b_1+a_1b_0}=K_1, \tag{3.7}$$

$$|G_t(t,s)|=\begin{cases}\dfrac{|-a_1|(a_0(s-t_0)+b_0)}{a_0a_1(t_1-t_0)+a_0b_1+a_1b_0}, t_0<s<t<t_1,\\[2mm] \dfrac{(a_1(t_1-s)+b_1)a_0}{a_0a_1(t_1-t_0)+a_0b_1+a_1b_0}, t_0<t<s<t_1.\end{cases}$$

and
$$=\max\left\{\frac{a_1(a_0(t_1-t_0)+b_0)}{a_0a_1(t_1-t_0)+a_0b_1+a_1b_0},\frac{(a_1(t_1-t_0)+b_1)a_0}{a_0a_1(t_1-t_0)+a_0b_1+a_1b_0}\right\} \tag{3.8}$$
$$=K_2.$$

Now, I study the case, when *F* is not necessarily convex valued. I give result, based on the Leray-Schauder Alternative for single valued maps combined with a selection theorem due to Bressan and Colombo [4] for lower semicontinuous multi-valued operators with decomposable values.

The following assumptions will be needed for proving main existence resul

(H_1) There exists a function $\phi\in L^1(J,R)$ with $\phi(t)>0$ for a.e. $t\in J$ and there is a nondecreasing function $\psi:R^+\to(0,\infty)$ such that
$\|F(t,x,y)\|_{\mathcal{P}}=\sup\{|u|:u\in F(t,x,y)\}\le\phi(t)\psi(\max\{|x|,|y|\})$ for a.e. $t\in J$ and for all $x,y\in R$.

(H_2) The multi-valued function $t \mapsto F(t,x,y)$ is measurable and integrably bounded for all $x, y \in R$.

(H_3) The multi-fraction $F : J \times R \times R \to \mathcal{P}_{cl}(R)$ satisfies

$$H\left(F(t,x_1,y_1),F(t,x_2,y_2)\right) \le \ell_1(t)|x_1 - y_1| + \ell_2(t)|x_2 - y_2| \quad a.e. \ t \in J$$

for all $x_1, x_2, y_1, y_2 \in R$.

(H_4) The multi-function $F : J \times R \times R \to \mathcal{P}_{cp}(R)$ satisfies

a) $(t,x,y) \mapsto F(t,x,y)$ is $(\mathcal{L} \otimes \mathcal{B} \otimes \mathcal{B})$ - measurable, and
b) $(x,y) \mapsto F(t,x,y)$ is lower semi-continuous for almost every $t \in J$.

Lemma 3.2 Let $F : J \times R \times R \to \mathcal{P}_{cp}(R)$ *be an integrably bounded multi-valued function satisfying* (H_4). *Then F is of lower semi-continuous type.*

MAIN RESULT

Theorem 3.3 Assume the hypotheses (H_1)-(H_4) hold and there exists a real number r > 0 satisfying

$$r > C_1 + \max\{K_1, K_2\} \|\phi\|_{L^1} \psi(r), \tag{3.9}$$

where C_1, K_1 *and* K_2 are the constants defined in Remark 3.2. Then the BVP (1.1) – (1.2) has at least one solution on J.

Proof: First, I transform the BVP (1.1) – (1.2) into a fixed-point problem in a suitable normed linear space. The problem of existence of a solution of BVP (1.1) – (1.2) reduces to finding a solution of the integral equation

$$x(t) = z(t) + \int_{t_0}^{t_1} k(t,s) f\left(x(s)\right) ds, \quad t \in J, \tag{3.10}$$

where $f(x(\cdot)) \in L^1$ with $f\left(x(t)\right) \in F\left(t, x(t), x'(t)\right) \ \ a.e. \ \ t \in J$. I study the integral equation (6.3.13) in the space $AC^1(J,R)$. Let $X = AC^1(J,R)$ and define an open ball $\mathcal{B}_r(0)$ in *X* centered at origin 0 of radius *r*, where the real number r > 0 satisfies the inequality (3.9). Define the operator *T* on $\overline{\mathcal{B}_r(0)}$ by

$$Tx(t) = z(t) + \int_{t_0}^{t_1} k(t,s) f\left(x(s)\right) ds. \tag{3.11}$$

Then the integral equation (3.11) is equivalent to the operator equation

$$x(t) = Tx(t), \quad t \in J. \tag{3.12}$$

I will show that the multi-valued operator *T* satisfies all the conditions of Corollary 2.2.

First, I show that *T* is continuous on $\overline{\mathcal{B}_r(0)}$. Since (H_1) holds, then

$$\left| f\left(x(t)\right) \right| \le \phi(t) \psi\left(\max\left\{|x(t)|, |x'(t)|\right\}\right) \quad a.e. \ \ t \in J$$

for all $x \in AC^1(J,R)$. Let $\{x_n\}$ be a sequence in $\overline{\mathcal{B}_r(0)}$ converging to a point $x \in \overline{\mathcal{B}_r(0)}$.

Then,
$$\left| f\left(x_n(t)\right) \right| \le \phi(t) \psi(r) \quad a.e. \ \ t \in J.$$

Hence, by the dominated convergence theorem and continuity of *f*, I have

$$\lim_{n\to\infty} Tx_n(t) = z(t) + \int_{t_0}^{t_1} G(t,s) f\left(x_n(s)\right) ds$$

$$= Tx(t)$$

and

$$\lim_{n\to\infty} \left(Tx_n\right)'(t) = z'(t) + \int_{t_0}^{t_1} G_t(t,s) f\left(x_n(s)\right) ds$$

$$= \left(Tx\right)'(t)$$

for all $t \in J$. As a result, *T* is continuous on $\overline{\mathcal{B}_r(0)}$. Next, using theorem as "Assume that (1) F is Carath'eodory and (2) $\left(H_3\right)$ hold. Suppose that there is a real number r > 0 such that $r > C_1 + \max\left\{K_1, K_2\right\} \|\phi\|_{L^1} \psi(r)$. Then the BVP (1.1)-(1.2 has at least one solution u such that $\|u\| \le r$ .Following the arguments as in above mentioned theorem , it is shown that *T* is a compact operator on $\overline{\mathcal{B}_r(0)}$. Now an application of Corollary 2.2 yields that either (i) the operator equation $x = Tx$ has a solution $\overline{\mathcal{B}_r(0)}$, or (ii) there is an element $u \in X$ such that $\|u\| = r$ and $u = \lambda Tu$ for some $\lambda \in (0,1)$. If the assertion (ii) holds, then we obtain a contradiction to multivalued operator definition. Hence assertion (i) holds and the BVP (1.1 – (1.2) has a solution $u \in AC^1(J, R)$ such that $\|u\| \le r$. This completes the proof.

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