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NEW TRIPLE INTEGRAL RELATIONS ASSOCIATED WITH $\overline{\mathrm{H}}$-FUNCTION AND MULTIVARIABLE H-FUNCTION

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ABSTRACT

The main aim of the present paper is to derive some results related to triple integral relations on the product of two $\overline{\mathrm{H}}$-functions and the multivariable H-function. A large number of known and new results have also been obtained by proper choice of parameters.

***Key words**: $\overline{\mathrm{H}}$-function, Multivariable H-function, Feynman integrals, Wright hypergeometric function.*

***AMS Mathematical Classification**: 33C45, 33C60, 46T12.*

1. INTRODUCTION

The $\overline{\mathrm{H}}$-function, a generalization of Fox H-function introduced by Inayat-Hussain [4] and studied by Srivastava and Buschman [10] and others, is defined and represented in the following manner:

$$\overline{H}^{m,n}_{p,q}[z]=\overline{H}^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right]$$

$$=\frac{1}{2\pi i}\int_L \theta(s)\,z^s\,ds, \tag{1}$$

where

$$\theta(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\{\Gamma(1-a_j+\alpha_j s)\}^{A_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-b_j+\beta_j s)\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)} \tag{2}$$

and the contour L is the line from c – i ∞ to c + i ∞, suitably indented to keep poles of $\Gamma(b_j-\beta_j s),\ j=1,2,...,m$ to the right of the path and the singularities of $\{\Gamma(1-a_j+\alpha_j s)\}^{A_j}$, j = 1,2,…,n to the left of the path. For convergence conditions and other details of the $\overline{\mathrm{H}}$-function see Inayat Hussain [4], Srivastava and Buschman [10].

The series representation of $\overline{\mathrm{H}}$-function [3] is as follows

$$\overline{H}^{M,N}_{P,Q}[z]=\overline{H}^{M,N}_{P,Q}\left[z\,\middle|\,\begin{matrix}(e_j,E_j;\alpha'_j)_{1,N},(e_j,E_j)_{N+1,P}\\(f_j,F_j)_{1,M},(f_j,F_j;\beta'_j)_{M+1,Q}\end{matrix}\right]$$

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$$=\sum_{g'=1}^{M}\sum_{h=0}^{\infty}\frac{(-1)^h\,\phi(\eta_{g',h})}{h!\,F_{g'}}z^{\eta_{g',h}}, \tag{3}$$

where

$$\phi(\eta_{g',h})=\frac{\prod\limits_{\substack{j=1\\ j\neq g'}}^{M}\Gamma(f_j-F_j\eta_{g',h})\prod\limits_{j=1}^{N}\{\Gamma(1-e_j+E_j\eta_{g',h})\}^{\alpha'_j}}{\prod\limits_{j=M+1}^{Q}\{\Gamma(1-f_j+F_j\eta_{g',h})\}^{\beta'_j}\prod\limits_{j=N+1}^{P}\Gamma(e_j-E_j\eta_{g',h})} \tag{4}$$

and

$$\eta_{g',h}=\frac{f_{g'}+h}{F_{g'}}$$

The multivariable H-function due to Srivastava and Panda [7] is defined and represented as follows

$$H[z_1,\dots,z_r]=H^{0,\lambda:(U',V');\dots;(U^{(r)},V^{(r)})}_{A,C:[B',D'];\dots;[B^{(r)},D^{(r)}]}\left[\begin{array}{l}[(a):\theta',\dots,\theta^{(r)}]:[b':\phi'];\dots;[b^{(r)}:\phi^{(r)}];\\ [(c):\psi',\dots,\psi^{(r)}]:[d':\delta'];\dots;[d^{(r)}:\delta^{(r)}];\end{array} z_1,\dots,z_r\right]$$

$$=\frac{1}{(2\pi i)^r}\int_{L_1}\dots\int_{L_r}T(s_1,\dots,s_r)R_1(s_1)\dots R_r(s_r)z_1^{s_1}\dots z_r^{s_r}\,ds_1\dots ds_r,\ (i=\sqrt{-1}) \tag{5}$$

where

$$R_i(s_i)=\frac{\prod\limits_{j=1}^{U^{(i)}}\Gamma(d_j^{(i)}-\delta_j^{(i)}s_i)\prod\limits_{j=1}^{V^{(i)}}\Gamma(1-b_j^{(i)}+\phi_j^{(i)}s_i)}{\prod\limits_{j=U^{(i)}+1}^{D^{(i)}}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod\limits_{j=V^{(i)}+1}^{B^{(i)}}\Gamma(b_j^{(i)}-\phi_j^{(i)}s_i)},\ \forall i\in\{1,\dots,r\} \tag{6}$$

$$T(s_1,\dots,s_r)=\frac{\prod\limits_{j=1}^{\lambda}\Gamma\left(1-a_j+\sum\limits_{i=1}^{r}\theta_j^{(i)}s_i\right)}{\prod\limits_{j=1}^{C}\Gamma\left(1-c_j+\sum\limits_{i=1}^{r}\psi_j^{(i)}s_i\right)\prod\limits_{j=\lambda+1}^{A}\Gamma\left(a_j-\sum\limits_{i=1}^{r}\theta_j^{(i)}s_i\right)} \tag{7}$$

For the nature of contours, various sets of convergence conditions of the integral given by (5) and the other details about this function we may refer to [7].

For the sake of brevity, we used the following notations:

$$\Delta=\left(\frac{\pi}{2^{\sigma+\eta_{g',h}k'+2}}\right)\sum_{g'=1}^{M}\sum_{h=0}^{\infty}\frac{(-1)^h\,\phi(\eta_{g',h})}{h!\,F_{g'}}(x)^{\eta_{g',h}}\,\overline{H}^{m,n}_{p,q}\left[(\alpha')(2^{-k})\left|\begin{array}{l}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\ (b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{array}\right.\right] \tag{8}$$

$$\nabla=\left(\frac{\pi}{2^{\sigma+\eta_{g',h}k'+2}}\right)\sum_{g'=1}^{M}\sum_{h=0}^{\infty}\frac{(-1)^h\,\phi(\eta_{g',h})}{h!\,F_{g'}}(x)^{\eta_{g',h}}\frac{\ell_{d-1}\Gamma(\mu+1)\Gamma\left(\frac{1}{2}+\frac{\xi}{2}\right)}{(-1)^{\mu}2^{2+\mu}(\pi)^{\frac{1}{2}}\Gamma(\gamma)\Gamma\left(\gamma-\frac{\xi}{2}\right)}$$

$$.\ \overline{H}^{1,3}_{3,3}\left[(-\alpha')(2^{-k})\left|\begin{array}{l}(1-\gamma,1;1),(1-\gamma+\frac{\xi}{2},1;1),(1-\eta,1;1+\mu),(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\ (0,1),(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q},(-\frac{\xi}{2},1;1),(-\eta,1;1+\mu)\end{array}\right.\right] \tag{9}$$

2. RELATIONS

This section starts with the assumption of three new theorems on two $\overline{\mathrm{H}}$ -function associated with multivariable $\overline{\mathrm{H}}$ -function. These theorems can be used to establish known and various new results.

Theorem 1. Consider the $\overline{\mathrm{H}}$ -function and multivariable H-function defined in (1), (3) and (5) respectively, then

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{(tu)^\sigma}{(t^2+u^2+z^2)^\sigma} \cos\left(2\delta \tan^{-1}\frac{z}{t}\right)$$

$$.\overline{H}_{p,q}^{m,n}\left[\frac{(\alpha')(tu)^k}{(t^2+u^2+z^2)^k}\middle| \begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right]$$

$$.\ \overline{H}_{P,Q}^{M,N}\left[\frac{(x)\,(tu)^{k'}}{(t^2+u^2+z^2)^{k'}}\middle| \begin{matrix}(e_j,E_j;\alpha_j')_{1,N},(e_j,E_j)_{N+1,P}\\(f_j,F_j)_{1,M},(f_j,F_j;\beta_j')_{M+1,Q}\end{matrix}\right]$$

$$.\ H_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,\lambda:(U',V');\ldots;(U^{(r)},V^{(r)})}\left[\begin{matrix}y_1(tu)^{\sigma_1}(t^2+u^2+z^2)^{b_1-\sigma_1}\\ y_2(t^2+u^2+z^2)^{b_2}\\ \vdots\\ y_r(t^2+u^2+z^2)^{b_r}\end{matrix}\right] f(t^2+u^2+z^2)\,dt\,du\,dz$$

$$=\Delta\int_0^\infty H_{A,C:[B'+3,D'+3]*}^{0,\lambda:(U',V'+3)\,*}\left[\begin{matrix}y_1\rho^{2b_1}2^{-\sigma_1}\\ y_2\rho^{2b_2}\\ \vdots\\ y_r\,\rho^{2b_r}\end{matrix}\middle| \begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:\left[-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}\,k'}{2},\frac{\sigma_1}{2}\right],\\ [(c):\psi',\ldots,\psi^{(r)}]:\left[-\frac{1}{2}-\sigma-ks-\eta_{g',h}k',\sigma_1\right],\end{matrix}\right.$$

$$\left.\begin{matrix}\left[-\frac{1}{2}-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2},\frac{\sigma_1}{2}\right],\left[-\sigma-ks-\eta_{g',h}k',\sigma_1\right]*\\ \left[-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2}\pm\delta,\frac{\sigma_1}{2}\right]*\end{matrix}\right]\rho^2 f(\rho^2)\,d\rho \tag{10}$$

where

$$\left[\mathrm{Re}(\sigma)+k\min\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)+k'\min\mathrm{Re}\left(\frac{f_\ell}{F_\ell}\right)+\sigma_1\min\mathrm{Re}\left(\frac{d_{\ell'}'}{\delta_{\ell'}'}\right)+1\right]>0,$$

$$(j=1,\ldots,m,\ \ell=1,\ldots,M,\ \ell'=1,\ldots,U')$$

$$\left[\mathrm{Re}(\sigma)+k\min\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)+k'\min\mathrm{Re}\left(\frac{f_\ell}{F_\ell}\right)+\sigma_1\min\mathrm{Re}\left(\frac{d_{\ell'}'}{\delta_{\ell'}'}\right)+2\right]>0,$$

$$\mathrm{Re}\left[\sum_{i=1}^{r} b_i\max\left(\frac{b_j^{(i)}-1}{\phi_j^{(i)}}\right)\right]<-\frac{3}{2};\qquad (j=1,\ldots,V^{(i)}),$$

$$|\arg(\alpha')| < T\pi/2,\ |\arg(x)| < T'\pi/2 \text{ and } |\arg(y_i)| < \frac{T_i\pi}{2};$$

$$T = \sum_{j=1}^{m}\beta_j + \sum_{j=1}^{n}|A_j\alpha_j| - \sum_{j=n+1}^{q}|B_j\beta_j| - \sum_{j=n+1}^{p}\alpha_j > 0,$$

$$T'' = \sum_{j=1}^{M}F_j + \sum_{j=1}^{N}|\alpha'_jE_j| - \sum_{j=M+1}^{Q}|\beta'_jF_j| - \sum_{j=N+1}^{P}E_j > 0,$$

$$T_i = -\sum_{j=\lambda+1}^{A}\theta_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{V^{(i)}}\phi_j^{(i)} - \sum_{j=V^{(i)}+1}^{B^{(i)}}\phi_j^{(i)} + \sum_{j=1}^{U^{(i)}}\delta_j^{(i)} - \sum_{j=U^{(i)}+1}^{D^{(i)}}\delta_j^{(i)} > 0,\ \forall i = \{1,\dots,r\}$$

where the asterisk * in (10) indicates that the parameters at these places are the same as the parameters of the H-function of several variables defined by (5).

Theorem 2. Consider the $\overline{\mathrm{H}}$ -function and multivariable H-function defined in (1), (3) and (5) respectively, then

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{(tu)^\sigma}{(t^2+u^2+z^2)^\sigma}\cos\left(2\delta\tan^{-1}\frac{z}{t}\right)$$

$$.\overline{H}_{p,q}^{m,n}\left[\frac{(\alpha')(tu)^k}{(t^2+u^2+z^2)^k}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]$$

$$.\ \overline{H}_{P,Q}^{M,N}\left[\frac{(x)(tu)^{k'}}{(t^2+u^2+z^2)^{k'}}\left|\begin{matrix}(e_j,E_j;\alpha'_j)_{1,N},(e_j,E_j)_{N+1,P}\\(f_j,F_j)_{1,M},(f_j,F_j;\beta'_j)_{M+1,Q}\end{matrix}\right.\right]$$

$$.\ H_{A,C:[B',D'];\dots;[B^{(r)},D^{(r)}]}^{0,\lambda:(U',V');\dots;(U^{(r)},V^{(r)})}\left[\begin{matrix}y_1(tu)^{-\sigma_1}(t^2+u^2+z^2)^{b_1+\sigma_1}\\y_2(t^2+u^2+z^2)^{b_2}\\\vdots\\y_r(t^2+u^2+z^2)^{b_r}\end{matrix}\right] f(t^2+u^2+z^2)\,dt\,du\,dz$$

$$= \Delta\int_0^\infty H_{A,C:[B'+3,D'+3]*}^{0,\lambda:(U'+3,V')*}\left[\begin{matrix}y_1\rho^{2b_1}2^{\sigma_1}\\y_2\rho^{2b_2}\\\vdots\\y_r\rho^{2b_r}\end{matrix}\left|\begin{matrix}[(a):\theta',\dots,\theta^{(r)}]:[b':\phi'];\left[\frac{3}{2}+\sigma+ks+\eta_{g',h}k',\sigma_1\right],\\ [(c):\psi',\dots,\psi^{(r)}]:\left[1+\frac{\sigma}{2}+\frac{ks}{2}+\frac{\eta_{g',h}k'}{2},\frac{\sigma_1}{2}\right],\\ \left[1+\frac{\sigma}{2}+\frac{ks}{2}+\frac{\eta_{g',h}k'}{2}\pm\delta,\sigma_{\frac{1}{2}}\right]*\\ \left[\frac{1}{2}+\frac{\sigma}{2}+\frac{ks}{2}+\frac{\eta_{g',h}k'}{2},\frac{\sigma_1}{2}\right],\left[1+\sigma+ks+\eta_{g',h}k',\sigma_1\right]*\end{matrix}\right.\right]\rho^2 f(\rho^2)\,d\rho \tag{11}$$

which is valid under the same conditions as given in (10).

Theorem 3. Consider the $\overline{\mathrm{H}}$ -function and multivariable H-function defined in (1), (3) and (5) respectively, then

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{(tu)^\sigma}{(t^2+u^2+z^2)^\sigma} \cos\left(2\delta \tan^{-1}\frac{z}{t}\right)$$

$$.\overline{H}^{m,n}_{p,q}\left[\frac{(\alpha')(tu)^k}{(t^2+u^2+z^2)^k}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right]$$

$$.\ \overline{H}^{M,N}_{P,Q}\left[\frac{(x)(tu)^{k'}}{(t^2+u^2+z^2)^{k'}}\middle|\begin{matrix}(e_j,E_j;\alpha_j')_{1,N},(e_j,E_j)_{N+1,P}\\(f_j,F_j)_{1,M},(f_j,F_j;\beta_j')_{M+1,Q}\end{matrix}\right]$$

$$.\ H^{0,\lambda:(U',V');...;(U^{(r)},V^{(r)})}_{A,C:[B',D'];...;[B^{(r)},D^{(r)}]}\left[\begin{matrix}y_1(tu)^{\sigma_1}(t^2+u^2+z^2)^{b_1-\sigma_1}\\ y_2(tu)^{\sigma_2}(t^2+u^2+z^2)^{b_2-\sigma_2}\\ \vdots\\ y_r(tu)^{\sigma_r}(t^2+u^2+z^2)^{b_r-\sigma_r}\end{matrix}\right] f(t^2+u^2+z^2)\,dt\,du\,dz$$

$$=\Delta\int_0^\infty H^{0,(\lambda+3)\,*}_{(A+3),(C+3)\,*}\left[\begin{matrix}y_1\rho^{2b_1}2^{-\sigma_1}\\ y_2\rho^{2b_2}2^{-\sigma_2}\\ \vdots\\ y_r\rho^{2b_r}2^{-\sigma_r}\end{matrix}\middle|\begin{matrix}\left[-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2};\frac{\sigma_1}{2},...,\frac{\sigma_r}{2}\right],\\ [(c):\psi',...,\psi^{(r)}],\left[-\frac{1}{2}-\sigma-ks-\eta_{g',h}k';\sigma_1,...,\sigma_r\right],\end{matrix}\right.$$

$$\left.\begin{matrix}\left[\frac{1}{2}-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}}{2}k';\frac{\sigma_1}{2},...,\frac{\sigma_r}{2}\right][-\sigma-ks-\eta_{g',h}k';\sigma_1,...,\sigma_r],[(a):\theta',...,\theta^{(r)}]\ *\\ \left[-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2}\pm\delta;\sigma_1,...,\sigma_r\right]*\end{matrix}\right]\rho^2 f(\rho^2)\,d\rho \qquad (12)$$

which is valid under the same conditions as given in (10).

Proof. To prove the theorem 1, first we change the left hand side of integral (10), from Cartesian to spherical polar form and then expressing the first $\overline{\mathrm{H}}$ -function in terms of Mellin-Barnes type of contour integral given by (1) and the second $\overline{\mathrm{H}}$ -function in series form given by (3), also the multivariable H-function in their contour form given by (5), interchanging the order of summations and integrations, we get the following form of integral say $(\in)$

$$\in=\sum_{g'=1}^{M}\sum_{h=0}^{\infty}\frac{(-1)^h\phi(\eta_{g',h})}{h!F_g}(x)^{\eta_{g',h}}\frac{1}{(2\pi i)^r}\int_{L_1}...\int_{L_r}T(S_1,...,S_r)R_1(S_1)...R_r(S_r)y_1^{S_1}...y_r^{S_r}$$

$$.\ \frac{1}{(2\pi i)}\int_L\theta(s)(\alpha')^s\left[\int_0^\infty\int_0^{\pi/2}\rho^{2+2b_1S_1+...+2b_rS_r}f(\rho^2)(\sin\theta)^{\sigma+ks+\eta_{g',h}k'+\sigma_1S_1+1}\right.$$

$$\left..(\cos\theta)^{\sigma+ks+\eta_{g',h}k'+\sigma_1S_1}\left\{\int_0^{\pi/2}(\cos\phi)^{\sigma+ks+\eta_{g',h}k'+\sigma_1S_1}\cos(2\delta\phi)d\phi\right\}d\theta\,d\rho\right]ds\,ds_1...ds_r. \qquad (13)$$

On evaluating the θ and ϕ integral occurring on the right hand side of (13) with the help of known result [5, Eq.5, p.16],see also [6, Eq. 3.2.7, p.62] and using the well known Beta function, we easily arrive at the desired result (10) after a little simplification. Theorems 2 and Theorem 3 can be proved on the similar way.

3. SPECIAL CASES

(I) If we take m = 1, n = 3 = p =q and replacing z by –z in the triple integral (10), (11) and (12) and using

$$g(\gamma,\eta,\xi,\mu;z)=\frac{\ell_{d-1}\Gamma(\mu+1)\Gamma\left(\frac{1}{2}+\frac{\xi}{2}\right)}{(-1)^{\mu}\,2^{2+\mu}\,(\pi)^{\frac{1}{2}}\,\Gamma(\gamma)\Gamma\left(\gamma-\frac{\xi}{2}\right)}\;\overline{H}^{1,3}_{3,3}\left[-z\left|\begin{matrix}(1-\gamma,1;1),(1-\gamma+\frac{\xi}{2},1;1),(1-\eta,1;1+\mu)\\(0,1),(-\frac{\xi}{2},1;1),(-\eta,1;1+\mu)\end{matrix}\right.\right], \tag{14}$$

where

$$\ell_d=\frac{2^{1-d}\,(\pi)^{-\frac{d}{2}}}{\Gamma(\frac{d}{2})}\qquad\text{([4], p.4121, Eqn.(5)).}$$

The above function is connected with a certain class of Feynman integrals, we get the following results

Theorem 4.

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{(tu)^{\sigma}}{(t^2+u^2+z^2)^{\sigma}}\cos\left(2\delta\tan^{-1}\frac{z}{t}\right)g\left(\gamma,\eta,\xi,\mu;\frac{(\alpha')(tu)^k}{(t^2+u^2+z^2)^k}\right)$$

$$\cdot\;\overline{H}^{M,N}_{P,Q}\left[\frac{(x)(tu)^{k'}}{(t^2+u^2+z^2)^{k'}}\left|\begin{matrix}(e_j,E_j;\alpha'_j)_{1,N},(e_j,E_j)_{N+1,P}\\(f_j,F_j)_{1,M},(f_j,F_j;\beta'_j)_{M+1,Q}\end{matrix}\right.\right]$$

$$\cdot\;H^{0,\lambda:(U',V');\ldots;(U^{(r)},V^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left[\begin{matrix}y_1(tu)^{\sigma_1}(t^2+u^2+z^2)^{b_1-\sigma_1}\\y_2(t^2+u^2+z^2)^{b_2}\\\vdots\\y_r(t^2+u^2+z^2)^{b_r}\end{matrix}\right]f(t^2+u^2+z^2)\,dt\,du\,dz$$

$$=\nabla\int_0^\infty H^{0,\lambda:(U',V'+3)\,*}_{A,C:[B'+3,D'+3]*}\left[\begin{matrix}y_1\rho^{2b_1}2^{-\sigma_1}\\y_2\rho^{2b_2}\\\vdots\\y_r\,\rho^{2b_r}\end{matrix}\left|\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:\left[-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}\,k'}{2},\frac{\sigma_1}{2}\right],\\ [(c):\psi',\ldots,\psi^{(r)}]:\left[-\frac{1}{2}-\sigma-ks-\eta_{g',h}k',\sigma_1\right],\end{matrix}\right.\right.$$

$$\left.\left.\begin{matrix}\left[-\frac{1}{2}-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2},\frac{\sigma_1}{2}\right],\left[-\sigma-ks-\eta_{g',h}k',\sigma_1\right]*\\\left[-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2}\pm\delta,\frac{\sigma_1}{2}\right]*\end{matrix}\right.\right]\rho^2 f(\rho^2)\,d\rho \tag{15}$$

which is valid under the same conditions as given in (10).

Theorem 5.

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{(tu)^{\sigma}}{(t^2+u^2+z^2)^{\sigma}}\cos\left(2\delta\tan^{-1}\frac{z}{t}\right)g\left(\gamma,\eta,\xi,\mu;\frac{(\alpha')(tu)^k}{(t^2+u^2+z^2)^k}\right)$$

$$. \; \overline{H}_{P,Q}^{M,N}\left[\frac{(x)(tu)^{k'}}{(t^2+u^2+z^2)^{k'}}\left|\begin{matrix}(e_j,E_j;\alpha_j')_{1,N},(e_j,E_j)_{N+1,P}\\(f_j,F_j)_{1,M},(f_j,F_j;\beta_j')_{M+1,Q}\end{matrix}\right.\right]$$

$$. \; H_{A,C:[B',D'];...;[B^{(r)},D^{(r)}]}^{0,\lambda:(U',V');...;(U^{(r)},V^{(r)})}\left[\begin{matrix}y_1(tu)^{-\sigma_1}(t^2+u^2+z^2)^{b_1+\sigma_1}\\ y_2(t^2+u^2+z^2)^{b_2}\\ \vdots \\ y_r(t^2+u^2+z^2)^{b_r}\end{matrix}\right] f(t^2+u^2+z^2)\,dt\,du\,dz$$

$$=\nabla\int_0^\infty H_{A,C:[B'+3,D'+3]*}^{0,\lambda:(U'+3,V')*}\left[\begin{matrix}y_1\rho^{2b_1}2^{\sigma_1}\\ y_2\rho^{2b_2}\\ \vdots\\ y_r\rho^{2b_r}\end{matrix}\left|\begin{matrix}[(a):\theta',...,\theta^{(r)}]:[b':\phi'];\left[\frac{3}{2}+\sigma+ks+\eta_{g',h}k',\sigma_1\right],\\ [(c):\psi',...,\psi^{(r)}]:\left[1+\frac{\sigma}{2}+\frac{ks}{2}+\frac{\eta_{g',h}k'}{2},\frac{\sigma_1}{2}\right],\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left[1+\frac{\sigma}{2}+\frac{ks}{2}+\frac{\eta_{g',h}k'}{2}\pm\delta,\sigma_{\frac{1}{2}}\right]*\\ \left[\frac{1}{2}+\frac{\sigma}{2}+\frac{ks}{2}+\frac{\eta_{g',h}k'}{2},\frac{\sigma_1}{2}\right],\left[1+\sigma+ks+\eta_{g',h}k',\sigma_1\right]*\end{matrix}\right]\rho^2 f(\rho^2)\,d\rho \tag{16}$$

which is valid under the same conditions as given in (10).

Theorem 6.

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{(tu)^\sigma}{(t^2+u^2+z^2)^\sigma}\cos\left(2\delta\tan^{-1}\frac{z}{t}\right)g\left(\gamma,\eta,\xi,\mu;\frac{(\alpha')(tu)^k}{(t^2+u^2+z^2)^k}\right)$$

$$. \; \overline{H}_{P,Q}^{M,N}\left[\frac{(x)(tu)^{k'}}{(t^2+u^2+z^2)^{k'}}\left|\begin{matrix}(e_j,E_j;\alpha_j')_{1,N},(e_j,E_j)_{N+1,P}\\(f_j,F_j)_{1,M},(f_j,F_j;\beta_j')_{M+1,Q}\end{matrix}\right.\right]$$

$$. \; H_{A,C:[B',D'];...;[B^{(r)},D^{(r)}]}^{0,\lambda:(U',V');...;(U^{(r)},V^{(r)})}\left[\begin{matrix}y_1(tu)^{\sigma_1}(t^2+u^2+z^2)^{b_1-\sigma_1}\\ y_2(t^2+u^2+z^2)^{b_2-\sigma_2}\\ \vdots \\ y_r(t^2+u^2+z^2)^{b_r-\sigma_r}\end{matrix}\right] f(t^2+u^2+z^2)\,dt\,du\,dz$$

$$=\nabla\int_0^\infty H_{(A+3),(C+3)*}^{0,(\lambda+3)*}\left[\begin{matrix}y_1\rho^{2b_1}2^{-\sigma_1}\\ y_2\rho^{2b_2}2^{-\sigma_2}\\ \vdots\\ y_r\rho^{2b_r}2^{-\sigma_r}\end{matrix}\left|\begin{matrix}\left[-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2},\frac{\sigma_1}{2},...,\frac{\sigma_r}{2}\right],\\ [(c):\psi',...,\psi^{(r)}],\left[-\frac{1}{2}-\sigma-ks-\eta_{g',h}k',\sigma_1,...,\sigma_r\right],\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left[\frac{1}{2}-\frac{\sigma}{2}-\frac{ks}{2}-\eta_{\frac{g',h}{2}}k';\frac{\sigma_1}{2},...,\frac{\sigma_r}{2}\right],[-\sigma-ks-\eta_{g',h}k';\sigma_1,...,\sigma_r],[(a):\theta',...,\theta^r]\ * \\ \left[-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2}\pm\delta;\sigma_{\frac{1}{2}},...,\sigma_{\frac{r}{2}}\right]*\end{matrix}\right]\rho^2 f(\rho^2)\,d\rho \qquad (17)$$

which is valid under the same conditions as given in (10).

(II) If we take $A_i = B_j = \alpha_k = \beta_\ell = 1(i = 1,...,n;\ j = m+1,...,q;\ k = 1,...,p;\ \ell = 1,...,q)$ in (10),

we arrive at the following result

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{(tu)^\sigma}{(t^2+u^2+z^2)^\sigma}\cos\left(2\delta\tan^{-1}\frac{z}{t}\right)G_{p,q}^{m,n}\left[\frac{(\alpha')(tu)^k}{(t^2+u^2+z^2)^k}\middle|\begin{matrix}(a_j,1)_{1,p}\\(b_j,1)_{1,q}\end{matrix}\right]$$

$$.\ \overline{H}_{P,Q}^{M,N}\left[\frac{(x)(tu)^{k'}}{(t^2+u^2+z^2)^{k'}}\middle|\begin{matrix}(e_j,E_j;\alpha_j')_{1,N},(e_j,E_j)_{N+1,P}\\(f_j,F_j)_{1,M},(f_j,F_j;\beta_j')_{M+1,Q}\end{matrix}\right]$$

$$.\ H_{A,C:[B',D'];...;[B^{(r)},D^{(r)}]}^{0,\lambda:(U',V');...;(U^{(r)},V^{(r)})}\begin{bmatrix}y_1(tu)^{\sigma_1}(t^2+u^2+z^2)^{b_1-\sigma_1}\\ y_2(t^2+u^2+z^2)^{b_2}\\ \vdots\\ y_r(t^2+u^2+z^2)^{b_r}\end{bmatrix}f(t^2+u^2+z^2)\,dt\,du\,dz$$

$$=\left(\frac{\pi}{2^{\sigma+\eta_{g',h}k'+2}}\right)\sum_{g'=1}^{M}\sum_{h=0}^{\infty}\frac{(-1)^h\phi(\eta_{g',h})}{h!F_g}(x)^{\eta_{g',h}}G_{p,q}^{m,n}\left[(\alpha')(2^{-k})\middle|\begin{matrix}(a_j,1)_{1,p}\\(b_j,\alpha)_{1,q}\end{matrix}\right]$$

$$.\int_0^\infty H_{A,C:[B'+3,D'+3)*}^{0,\lambda:(U',V+3)\ *}\left[\begin{matrix}[(a):\theta',...,\theta^{(r)}]:\left[-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2},\frac{\sigma_1}{2}\right],\\ [(c):\psi',...,\psi^{(r)}]:\left[-\frac{1}{2}-\sigma-ks-\eta_{g',h}k',\sigma_1\right],\end{matrix}\right.$$

$$\left.\begin{matrix}\left[-\frac{1}{2}-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2},\frac{\sigma_1}{2}\right],[-\sigma-ks-\eta_{g',h}k',\sigma_1]*\\ \left[-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2}\pm\delta;\frac{\sigma_1}{2}\right]*\end{matrix}\right]\rho^2 f(\rho^2)\,d\rho \qquad (18)$$

which is valid under the same conditions as given in (10), similarly we can find special case for (11) and (12).

(III) Replacing

$$\overline{H}_{p,q}^{m,n}\left[\frac{(\alpha')(tu)^k}{(t^2+u^2+z^2)^k}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right]$$

by

$$\overline{H}_{p,q+1}^{1,p}\left[\frac{(-\alpha')(tu)^k}{(t^2+u^2+z^2)^k}\middle|\begin{matrix}(1-a_j,\alpha_j;A_j)_{1,p}\\(0,1),(1-b_j,\beta_j;B_j)_{1,q}\end{matrix}\right]$$

In the left hand side of equation (10), we get

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{(tu)^\sigma}{(t^2+u^2+z^2)^\sigma}\cos\left(2\delta\tan^{-1}\frac{z}{t}\right)$$
$$.\ \overline{H}^{1,p}_{p,q+1}\left[\frac{(-\alpha')(tu)^k}{(t^2+u^2+z^2)^k}\middle|\begin{matrix}(1-a_j,\alpha_j;A_j)_{1,p}\\(0,1),(1-b_j,\beta_j;B_j)_{1,q}\end{matrix}\right]$$
$$.\ \overline{H}^{M,N}_{P,Q}\left[\frac{(x)(tu)^{k'}}{(t^2+u^2+z^2)^{k'}}\middle|\begin{matrix}(e_j,E_j;\alpha_j')_{1,N},(e_j,E_j)_{N+1,P}\\(f_j,F_j)_{1,M},(f_j,F_j;\beta_j')_{M+1,Q}\end{matrix}\right]$$
$$H^{0,\lambda:(U',V');\ldots;(U^{(r)},V^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left[\begin{matrix}y_1(tu)^{\sigma_1}(t^2+u^2+z^2)^{b_1-\sigma_1}\\y_2(t^2+u^2+z^2)^{b_2}\\\vdots\\y_r(t^2+u^2+z^2)^{b_r}\end{matrix}\right]f(t^2+u^2+z^2)\,dt\,du\,dz$$

$$=\frac{\pi}{2^{\sigma+\eta_{g',h}k'+2}}\sum_{g'=1}^{M}\sum_{h=0}^{\infty}\frac{(-1)^h\phi(\eta_{g',h})}{h!F_g}(x)^{\eta_{g',h}}\ {}_p\overline{\Psi}_q\left[\begin{matrix}(a_j,\alpha_j;A_j)_{1,p};\\(b_j,\beta_j;B_j)_{1,q};\end{matrix}(\alpha')\right]$$
$$.\ H^{0,\lambda:(U',V'+3)\,*}_{A,C:[B'+3,D'+3]*}\left[\begin{matrix}y_1\rho^{2b_1}2^{-\sigma_1}\\y_2\rho^{2b_2}\\\vdots\\y_r\rho^{2b_r}\end{matrix}\middle|\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:\left[-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2},\frac{\sigma_1}{2}\right],\\ [(c):\psi',\ldots,\psi^{(r)}]:\left[-\frac{1}{2}-\sigma-ks-\eta_{g',h}k',\sigma_1\right],\end{matrix}\right.$$
$$\left.\begin{matrix}\left[-\frac{1}{2}-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2},\frac{\sigma_1}{2}\right],\left[-\sigma-ks-\eta_{g',h}k',\sigma_1\right]*\\ \left[-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2}\pm\delta,\frac{\sigma_1}{2}\right]*\end{matrix}\right]\rho^2 f(\rho^2)\,d\rho \tag{19}$$

where ${}_p\overline{\Psi}_q$ [3,p.271, Eqn.6] is the generalized Wright hypergeometric function because it gives ${}_p\Psi_q$ ([9], p.19, eq.(2.6.11)) for $A_j = 1$ (j = 1,…,p) and $B_j = 1$ (j = 1,…,q) in it. Similarly we can find special cases for (11) and (12).

(IV) Further on setting $\alpha_j = 1(j=1,\ldots,p)$ and $\beta_j = 1(j=1,\ldots,q)$ in (19), we arrive at

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{(tu)^\sigma}{(t^2+u^2+z^2)^\sigma}\cos\left(2\delta\tan^{-1}\frac{z}{t}\right)$$
$$.\ \overline{H}^{1,p}_{p,q+1}\left[\frac{(-\alpha')(tu)^k}{(t^2+u^2+z^2)^k}\middle|\begin{matrix}(1-a_j,1;A_j)_{1,p}\\(0,1),(1-b_j,1;B_j)_{1,q}\end{matrix}\right]$$
$$.\ \overline{H}^{M,N}_{P,Q}\left[\frac{(x)(tu)^{k'}}{(t^2+u^2+z^2)^{k'}}\middle|\begin{matrix}(e_j,E_j;\alpha_j')_{1,N},(e_j,E_j)_{N+1,P}\\(f_j,F_j)_{1,M},(f_j,F_j;\beta_j')_{M+1,Q}\end{matrix}\right]$$

$$H^{0,\lambda:(U',V');...;(U^{(r)},V^{(r)})}_{A,C:[B',D'];...;[B^{(r)},D^{(r)}]}\begin{bmatrix} y_1(tu)^{\sigma_1}(t^2+u^2+z^2)^{b_1-\sigma_1} \\ y_2(t^2+u^2+z^2)^{b_2} \\ \vdots \\ y_r(t^2+u^2+z^2)^{b_r} \end{bmatrix} f(t^2+u^2+z^2)\,dt\,du\,dz$$

$$=\left(\frac{\pi}{2^{\sigma+\eta_{g',h}k'+2}}\right)\sum_{g'=1}^{M}\sum_{h=0}^{\infty}\frac{(-1)^h\,\phi(\eta_{g',h})}{h!\,F_g}(x)^{\eta_{g',h}}\frac{\prod_{j=1}^{p}\{\Gamma(a_j)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(b_j)\}^{B_j}}\;{}_p\overline{F}_q\left[\begin{matrix}(a_j,1;A_j)_{1,p};\\(b_j,1;B_j)_{1,q};\end{matrix}(\alpha')\right]$$

$$.\,H^{0,\lambda:(U',V'+3)*}_{A,C:[B'+3,D'+3]*}\left[\begin{matrix} y_1\rho^{2b_1}2^{-\sigma_1} \\ y_2\rho^{2b_2} \\ \vdots \\ y_r\rho^{2b_r} \end{matrix}\right| \begin{matrix}[(a):\theta',...,\theta^{(r)}]:\left[-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2},\frac{\sigma_1}{2}\right], \\ [(c):\psi',...,\psi^{(r)}]:\left[-\frac{1}{2}-\sigma-ks-\eta_{g',h}k',\sigma_1\right],\end{matrix}$$

$$\left.\begin{matrix}\left[-\frac{1}{2}-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2},\frac{\sigma_1}{2}\right],\left[-\sigma-ks-\eta_{g',h}k',\sigma_1\right]* \\ \left[-\frac{\sigma}{2}-\frac{ks}{2}-\frac{\eta_{g',h}k'}{2}\pm\delta,\frac{\sigma_1}{2}\right]*\end{matrix}\right]\rho^2 f(\rho^2)\,d\rho \qquad (20)$$

where the function ${}_p\overline{F}_q$ ([3]) is a particular case of ${}_p\overline{\Psi}_q$ and also reduced to well known ${}_pF_q$ function for A_j = 1 (1,…,p) and B_j = 1 (j= 1,…,q) in it. Similarly we can find the special cases for (11) and (12).

(V) If we set $x \to 0$ in (10), (11), (12), (15), (16) and (17), we arrive at the results recently obtained by Chaurasia and Meghwal [2].

(VI) If we take $x \to 0$ and A_j = 1 (j = 1,…,n) and B_j = 1 (j = m+1,…,q) in (10), (11) and (12), we get the known results obtained by Chaurasia and Saxena [1].

(VII) A number of other special cases can also be obtained with the help of equation (10), (11) and (12) but we do not record them here due to lack of space.

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