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RANDOM QUEUE LENGTH CONTROL CHART FOR (M / E_k / 1): (∞ / FCFS) QUEUEING MODEL

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ABSTRACT

Queueing theory involves the mathematical study of queues. Excessively long queues may result in loss of customers. Control charts constructed for random queue length N provide control limits for N so as to make effective use of time in order to achieve customer's satisfaction. Customer can have prior idea about expected waiting time, maximum waiting time and minimum waiting time respectively from central line, upper control limit and lower control limit of the chart so that, the performance of the system may be improved. Keeping this in view, in this paper the construction of control charts for random queue length is proposed for M /E_k /1 queueing model.

Keywords: *random queue length, control limits, phases of service.*

1. INTRODUCTION

World over competition is growing due to globalization, liberalization, privatization and automation. In order to meet this ever growing competition, quality is the most essential ingredient required in every sphere. Superior products and/or service excellence are the principal means of achieving competitive advantage. Shewhart developed control chart techniques to identify whether a production process is producing goods of set quality standards. Control charts are based on data of one or several quality related characteristics of the product or service. Montgomery (2010) proposed a number of applications of control charts in assuring quality in manufacturing industries.

In queueing system customer satisfaction can be increased by constructing control limits for N to make effective use of time. Shore (2000) developed control chart for random queue length, N of M /M /1 queueing model by considering the first three moments. Khaparde and Dhabe (2010) have developed control charts using weighted variance for random queue length of M /M /1 queueing model. In this paper, an attempt is made to find Shewhart control chart for random queue length of M /E_k /1 queueing model. This model finds applications in assembly and repairing machines, aircrafts etc. where the system undergoes many phases of service.

2. M /E_k /1 QUEUEING MODEL DESCRIPTION

Consider M /E_k /1 - a queueing system with Poisson arrival of rate λ and Erlang service of k phases with rate μ. Service time distribution follows Erlang with parameters (μ, k) having density function

$$f(t, \mu, k) = (k\mu)^k\, t^{k-1}\, e^{-k\mu t} /(k-1)!\ ,\ 0 \le t \le \infty,\quad k \ge 1$$

Let $t_1, t_2, \ldots, t_k$ be the respective service time for any customer in k phases, then the total service time t is given by $t = t_1 + t_2 + \ldots + t_k$. If the times $t_1, t_2, \ldots, t_k$ are independently and exponentially distributed with parameter kμ , then t has k-Erlangian distribution with parameter μ.

Let

$P_n(t)$	= Probability that there are n phases in the system (waiting and in service) at time t.
n	= Total number of phases in the system (waiting and in service)
k	= Number of phases in one unit in which phases are served one by one.

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Each arrival will increase the number of phases by k in the system and each time a phase is completed, the number of phases in the system is decreased by unity. Thus if there are m units waiting in the queue and one unit is already running under service at the s^{th} phase then n = mk + s. If μ is the number of units served per unit time, then μk will be the number of phases served per unit time and 1 / μk is the time taken by one service at each phase.

Steady state equations

$$P_0(t+\Delta t) = P_0(t)(1-\lambda\Delta t) + P_1(t)\,k\mu\,\Delta t + O(\Delta t)$$
$$P_n(t+\Delta t) = P_n(t)(1-(\lambda+k\mu)\Delta t) + P_{n-k}(t)\,\lambda\,\Delta t + P_{n+1}(t)\,k\mu\,\Delta t + O(\Delta t),\ n \geq 1 \quad (1)$$

Equation (1) gives

$$P_0'(t) = -\lambda P_0(t) + k\mu P_1(t)$$
$$P_n'(t) = -(\lambda + k\mu)P_n(t) + \lambda P_{n-k}(t) + k\mu P_{n+1}(t),\ n \geq 1 \quad (2)$$

The steady state equations corresponding to (2) are

$$0 = -\lambda P_0 + k\mu P_1$$
$$0 = -(\lambda + k\mu)P_n + \lambda P_{n-k} + k\mu P_{n+1},\ n \geq 1 \quad (3)$$

Let ρ = λ / μ be the traffic intensity. The equation (3) yields

$$kP_1 = \rho P_0,$$
$$(k+\rho)P_n = kP_{n+1} + P_{n-k},\ n \geq 1 \quad (4)$$

Equation (4) gives

$$P_0 = 1 - \rho \text{ and } P_n = (1-\rho)\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{m}(\rho / k)^m\,(-1)^i \binom{m}{i}\binom{m+j-1}{j}$$

where the summation is taken over all combinations of i, j and m satisfying n = m + ik + j for given k.

Performance measures

(i) $E(L_s)$ = Average number of units in the system

$$= \sum_{n=0}^{\infty} nP_n$$
$$= \frac{(k+1)}{2k}\,\frac{\rho 2}{1-\rho} + \rho \quad (5)$$

(ii) $E(L_q)$ = Average number of units in the queue

$$= \frac{(k+1)}{2k}\,\frac{\rho 2}{1-\rho} \quad (6)$$

(iii) $E(W_s)$ = Average time a unit spends in the system

$$= \frac{(k+1)}{2k}\,\frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu} \quad (7)$$

(iv) $E(W_q)$ = Average time a unit spends in the queue

$$= \frac{(k+1)}{2k}\,\frac{\lambda}{\mu(\mu-\lambda)} \quad (8)$$

Let L_s = N = Number of units in the system.

From (5)

$$E(N) = \frac{(k+1)}{2k}\,\frac{\rho 2}{1-\rho} + \rho \quad (9)$$

$$E(N^2) = \sum_{n=0}^{\infty} n^2 P_n$$

$$= \rho(1 + \rho k)(6k + 3\rho - 5\rho k + 2\rho k^3 - 2\rho^2 k^3 + 2\rho^2 k) / (6k(1-\rho)^2) \quad (10)$$

V(N) = Variance of number of units in the system

$= E(N^2) - (E(N))^2$

$$= \rho(1+\rho k)(6k+3\rho-5\rho k+2\rho k^3-2\rho^2k^3+2\rho^2k)/(6k(1-\rho)^2) - \left(\frac{(k+1)}{2k}\frac{\rho 2}{1-\rho}+\rho\right)^2 \qquad (11)$$

3. CONSTRUCTION OF CONTROL CHARTS FOR RANDOM QUEUE LENGTH, N

Shewhart type control charts are constructed by approximating the statistic under consideration by a normal distribution. The parameters of the control chart are

UCL $= E(N) + 3\sqrt{V(N)}$
CL $= E(N)$
LCL $= E(N) - 3\sqrt{V(N)}$

E(N) and V(N) can be obtained from equations (9) and (11).

4. NUMERICAL ANALYSIS

Assessment of random queue length by means of control chart is carried out with numerical illustrations for various values of k and various traffic intensities.

The parameters of the control chart when k = 1 are

UCL $= (\rho + 3\sqrt{\rho}) / (1-\rho)$
CL $= \rho / (1-\rho)$
LCL $= (\rho - 3\sqrt{\rho}) / (1-\rho)$

and the parameters of the control chart when k = 2 are

UCL $= (\rho(4-\rho) + 3(\rho(16+28\rho+16\rho^2-33\rho^3))^{1/2}) / (4(1-\rho))$
CL $= \rho(4-\rho) / (4(1-\rho))$
LCL $= (\rho(4-\rho) - 3(\rho(16+28\rho+16\rho^2-33\rho^3))^{1/2}) / (4(1-\rho))$

Table.1 gives the numerical values of parameters of the control charts corresponding to k = 1 and k = 2 for certain values of ρ.

Table.1: Parameters of control chart

	k=1			k=2		
ρ	CL	UCL	LCL	CL	UCL	LCL
0.05	0.0526	0.7588	-0.6535	0.0520	0.7891	-0.6851
0.1	0.1111	1.1652	-0.9430	0.1083	1.2548	-1.0381
0.15	0.1764	1.5434	-1.1905	0.1699	1.7152	-1.3755
0.2	0.2500	1.9271	-1.4271	0.2375	2.2029	-1.7279
0.25	0.3333	2.3333	-1.6667	0.3125	2.7355	-2.1105
0.3	0.4285	2.7759	-1.9188	0.3964	3.3277	-2.5348
0.35	0.5384	3.2689	-2.1920	0.4913	3.9951	-3.0124
0.4	0.6667	3.8289	-2.4956	0.6000	4.7569	-3.5569
0.45	0.8182	4.4772	-2.8408	0.7261	5.6380	-4.1858
0.5	1.0000	5.2426	-3.2426	0.8750	6.6724	-4.9224
0.55	1.2222	6.1664	-3.7219	1.0542	7.9082	-5.7999
0.6	1.5000	7.3095	-4.3095	1.2750	9.4175	-6.8676
0.65	1.8571	8.7676	-5.0534	1.5554	11.3126	-8.2019
0.7	2.3333	10.6999	-6.0333	1.9250	13.7795	-9.9295
0.75	3.0000	13.3923	-7.3923	2.4375	17.1524	-12.277
0.8	4.0000	17.4164	-9.4164	3.2000	22.0976	-15.6976
0.85	5.6667	24.1058	-12.772	4.4625	30.1675	-21.2425
0.9	9.0000	37.4605	-19.4605	6.9750	46.0130	-32.0630
0.95	19.0000	77.4808	-39.4808	14.4875	92.8749	-63.8999
0.99	99.0000	397.4962	-199.4960	74.4975	464.9870	-315.9920

The parameters of the control chart when the number of phases in service of the customer corresponding to k = 3 and k = 4 are respectively

$$UCL = (\rho (3 - \rho) + 3 (\rho (9 + 39\rho + 45\rho^2 - 73 \rho^3))^{1/2}) / (3 (1 - \rho))$$
$$CL = \rho (3 - \rho) / (3(1 - \rho))$$
$$LCL = (\rho (3 - \rho) - 3 (\rho (9 + 39\rho + 45\rho^2 - 73\rho^3))^{1/2}) / (3 (1 - \rho))$$

and

$$UCL = (\rho (8 - 3\rho) + 3 (\rho (64 + 488 \rho + 912 \rho^2 - 1289 \rho^3))^{1/2}) / (8(1 - \rho))$$
$$CL = \rho (8 - 3\rho) / (8(1 - \rho))$$
$$LCL = (\rho (8 - 3\rho) - 3 (\rho (64 + 488 \rho + 912 \rho^2 - 1289 \rho^3))^{1/2}) / (8(1 - \rho))$$

Table.2 gives the parameters of control charts for N for selected traffic intensities, when k = 3 and k = 4.

Table.2: Parameters of control chart

	k=3			**k=4**		
ρ	CL	UCL	LCL	CL	UCL	LCL
0.05	0.0518	0.8343	-0.7308	0.0516	0.8914	-0.7881
0.1	0.1074	1.3877	-1.1729	0.1069	1.5541	-1.3402
0.15	0.1676	1.9682	-1.6329	0.1665	2.2826	-1.9495
0.2	0.2333	2.6061	-2.1394	0.2312	3.1038	-2.6413
0.25	0.3056	3.3183	-2.7072	0.3021	4.0342	-3.4300
0.3	0.3857	4.1202	-3.3488	0.3804	5.0904	-4.3298
0.35	0.4756	5.029	-4.0782	0.4678	6.2927	-5.3571
0.4	0.5778	6.0680	-4.9124	0.5667	7.6667	-6.5333
0.45	0.6955	7.2648	-5.8739	0.6801	9.2469	-7.8867
0.5	0.8333	8.6590	-6.9929	0.8125	11.0806	-9.4556
0.55	0.9981	10.3088	-8.3125	0.9701	13.2347	-11.2944
0.6	1.2000	12.2960	-9.8960	1.1625	15.8073	-13.4822
0.65	1.4548	14.7504	-11.8408	1.4045	18.9492	-16.1403
0.7	1.7889	17.8850	-14.3072	1.7208	22.9063	-19.4646
0.75	2.2500	22.0790	-17.5790	2.1563	28.1111	-23.7986
0.8	2.9333	28.0825	-22.2158	2.8000	35.4104	-29.8104
0.85	4.0611	37.6314	-29.5092	3.8604	46.7408	-39.0199
0.9	6.3000	55.8981	-43.2981	5.9625	67.8108	-55.8858
0.95	12.9830	108.6294	-82.6627	12.2312	126.7746	-102.3120
0.99	66.3300	521.0150	-388.3550	62.2463	576.9032	-452.4110

Similarly the parameters of control charts for k = 8 and k=10 are respectively

$$UCL = (\rho (16 - 7 \rho) + 3 (\rho (256 + 7056 \rho + 36960 \rho^2 - 43057 \rho^3))^{1/2}) / (16 (1 - \rho))$$
$$CL = (\rho (16 - 7 \rho)) / 16 (1 - \rho)$$
$$LCL = (\rho (16 - 7 \rho) - 3 (\rho (256 + 7056 \rho + 36960 \rho^2 - 43057 \rho^3))^{1/2}) / (16 (1 - \rho))$$

and

$$UCL = (\rho (20 - 9 \rho) + 3 (\rho (400 + 16620 \rho + 117360 \rho^2 - 132081 \rho^3))^{1/2}) / (20 (1 - \rho))$$
$$CL = (\rho (20 - 9 \rho)) / 20 (1 - \rho)$$
$$LCL = (\rho (20 - 9 \rho) - 3 (\rho (400 + 16620 \rho + 117360 \rho^2 - 132081 \rho^3))^{1/2}) / (20 (1 - \rho))$$

The numerical analysis corresponding to k = 8 and k=10 are presented in Table.3.

Table.3: Parameters of control chart

ρ	k=8			k=10		
	CL	UCL	LCL	CL	UCL	LCL
0.05	0.0515	1.2156	-1.1127	0.0515	1.4224	-1.3196
0.1	0.1063	2.4708	-2.2583	0.1061	3.0422	-2.8300
0.15	0.1649	3.9862	-3.6565	0.1646	5.0397	-4.7106
0.2	0.2281	5.7756	-5.3193	0.2275	7.4221	-6.9671
0.25	0.2969	7.8563	-7.2625	0.2958	10.2090	-9.6169
0.3	0.3723	10.2540	-9.5094	0.3707	13.4310	-12.6896
0.35	0.4560	13.0049	-12.0930	0.4537	17.1358	-16.2285
0.4	0.5500	16.1579	-15.0579	0.5467	21.3868	-20.2934
0.45	0.6571	19.7797	-18.4655	0.6525	26.2712	-24.9662
0.5	0.7813	23.9614	-22.3989	0.7750	31.9085	-30.3585
0.55	0.9281	28.8307	-26.9745	0.9197	38.4654	-36.6259
0.6	1.1063	34.5716	-32.3591	1.0950	46.1813	-43.9913
0.65	1.3290	41.4603	-38.8022	1.3139	55.4136	-52.7857
0.7	1.6188	49.9339	-46.6964	1.5983	66.7244	-63.5278
0.75	2.0156	60.7363	-56.7051	1.9875	81.0618	-77.0868
0.8	2.6000	75.2669	-70.0669	2.5600	100.1890	-95.0690
0.85	3.5594	96.5782	-89.4594	3.4992	127.8959	-120.8980
0.9	5.4563	133.1753	-122.2630	5.3550	174.5248	-163.8150
0.95	11.1031	224.2148	-202.0090	10.8775	286.1638	-264.4090
0.99	56.1206	822.5564	-710.3150	54.8955	965.2202	-855.4290

5. CONCLUSION

The performance of the queueing system M /E_k /1 for the random queue length N is studied through control charts for various values of k. From the numerical values we see that if the number of phases k increases, CL and UCL values increase and LCL values increase negatively. Increase in k makes the band width wider between the UCL and LCL of the chart.

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