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OPEN MAPPING THEOREM ON INTUITIONISTIC 2-FUZZY2-NORMED LINEAR SPACE

[1]Thangaraj Beaula* & [2]Lilly Esthar Rani

1&2 Department of Mathematics, TBML College, Porayar, Tamil Nadu, India

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ABSTRACT

The concepts of fuzzy boundedness, fuzzy continuity and on intutionistic 2-fuzzy 2-normed linear space are introduced. Using these concepts some theorems are proved and as a result the famous Open Mapping Theorem is established in intuitionistic 2fuzzy 2-normed linear space.

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1. INTRODUCTION

The theory of fuzzy sets was introduced by L. Zadeh [9] in 1965. A satisfactory theory of 2-norm on a linear space has been introduced and developed by Gahler in 1964[2]. The concepts fuzzy norm and α-norm were introduced by Bag and Samanta in 2003[1]. Jialuzhang [3] has defined fuzzy linear space in a different way. The notion of 2-fuzzy 2-normed linear space of the set of all fuzzy sets of a set was introduced by R. M. Somasundaram and Thangaraj Beaula [6]. The concept of intuitionistic 2fuzzy 2-normed linear space of the set of all fuzzy sets of a set was introduced by Thangaraj Beaula and D. Lilly Esthar Rani [7].

We have introduced the concepts of fuzzy boundedness, fuzzy continuity on intutionistic 2-fuzzy 2-normed linear space. Using these concepts some theorems are proved and as a result the famous Open Mapping Theorem is established in intuitionistic 2fuzzy 2-normed linear space.

2. PRELIMINARIES

For the sake of completeness, we reproduce the following definitions due to Gahler [2], Bag and Samanta [1] and Jialuzhang [3].

Definition 2.1. [2] Let X be a real vector space of dimension greater than 1 and let $\|\bullet,\bullet\|$ be a real valued function on X x X satisfying the following conditions:

1. $\| x , y \| = 0$ if and only if x and y are linearly dependent,
2. $\| x , y \| = \| y, x \|$,
3. $\| \alpha x , y \| = | \alpha | \, \| x , y\|$, where α is real,
4. $\| x, y + z \| \le \| x , y \| + \| x , z \|$.

$\| \bullet,\bullet \|$ is called a 2-norm on X and the pair $(X, \|\bullet,\bullet\|)$ is called a linear 2-normed space.

Definition 2.2. [1] Let X be a linear space over K (field or real or complex numbers). A fuzzy subset N of X x R (R, the set of real numbers) is called a fuzzy norm on X if and only if for all x, u $\in$ X and c$\in$ K.

(N1) for all t$\in$R with $t \le 0$, N (x, t) = 0,

(N2) for all t$\in$R with t > 0, N (x, t) = 1, if and only if x = 0,

(N3) for all t$\in$R with t > 0, $N (cx, t) = N \left(x, \frac{t}{|c|}\right)$, if $c \neq 0$,

(N4) for all s, t $\in$ R, x, u$\in$X, $N (x+u, s + t) \ge \min \{N (x,s), N (u, t) \}$,

Corresponding author: [1]*Thangaraj Beaula**

1&2 Department of Mathematics, TBML College, Porayar, Tamil Nadu, India

(N5) N (x,•) is a non decreasing function of R and $\lim_{t \to \infty} N(x, t) = 1$

The pair (X, N) will be referred to as a fuzzy normed linear space.

Definition 2.3. [3] Let X be any non-empty set and F (X) be the set of all fuzzy sets on X. For U, V$\in$ F(X) and k $\in$ K the field of real numbers, define

U + V = { (x + y, $\lambda \wedge$ u) | (x,λ) $\in$ U, (y, μ) $\in$ V },

k U = { (kx,λ) | (x,λ) $\in$ U}

.

Definition 2.4.[3] A fuzzy linear space $\tilde{X}$ = X x (0, 1] over the number field K. where the addition and scalar multiplication operation on $\tilde{X}$ are defined by

$(x, \lambda) + (y, \mu) = (x + y, \lambda \wedge \mu)$,

k (x, λ) = (kx, λ) is a fuzzy normed space if to every (x,λ) $\in$ $\tilde{X}$ there is associated a non-negative real number, || (x, λ) ||, called the fuzzy norm of (x, λ), in such a way that

1. $\| (x,\lambda) \| = 0$ iff x = 0 the zero element of X, $\lambda \in (0,1]$
2. $\| k (x,\lambda) \| = |k| \ \| (x,\lambda) \|$ for all (x, λ) $\in \tilde{X}$ and all k $\in$ K
3. $\| (x, \lambda) + (y,\mu) \| \le \| (x,\lambda \wedge\mu) \| + \| (y, \lambda \wedge\mu) \|$ for all (x, λ) and (y, μ) $\in \tilde{X}$
4. $\| (x, \bigvee_t \lambda_t \| = \bigwedge_t \| (x, \lambda_t \|$ for $\lambda_t \in (0,1]$

Definition 2.5.[6] Let X be a non-empty and F(X) be the set of all fuzzy sets in X. If f $\in$ F(X) then f = { (x,μ) | x $\in$X and $\mu \in$ (0,1]} Clearly f is a bounded function for $|f(x)| \le 1$. Let K be the space of real numbers, then F(X) is a linear space over the field K where the addition and scalar multiplication are defined by

f + g = { (x, μ) + (y, η) } = { (x +y, $\mu \wedge \eta$) / (x, μ) $\in$f and (y, η) $\in$g}

kf = { (kx, μ) / (x, μ) $\in$f} where k$\in$ K.

The linear space F(X) is said to be normed space if to every f$\in$F (X), there is associated a non- negative real number || f || called the norm of f in such a way that

1. ||f || = 0 if and only if f = 0.

For $\| f \| = 0 \Leftrightarrow \{\| (x, \mu)\| / (x,\mu) \in f\} = 0$

$\Leftrightarrow x = 0, \mu \in (0, 1]$

$\Leftrightarrow f = 0$

2. $\|kf \| = |k| \ \|f \|$, k $\in$ K.

For $\| kf \| = \{\|k (x, \mu)\| / (x, \mu) \in f, k \in K\}$

$= \{|k| \ \|x, \mu \|) | (x, \mu) \in f\}$

$= |k| \ \|f\|$.

3. $\|f + g \| \le \| f \| + \| g \|$ for every f, g$\in$F(X)

For,

$\|f + g \| = \{\| (x,\mu) + (y, \eta)\| / x, y \in X, \mu, \eta \in (0,1]\}$

$= \{\| (x+y), (\mu \wedge \eta)\| / x, y \in X, \mu, \eta \in (0,1]\}$

$\le \{\| x, \mu \wedge \eta\| + \|y, \mu \wedge \eta\| / (x,\mu) \in f \text{ and } (y, \eta) \in g\}$

$= \|f \| + \| g \|$.

and (F(X), ||•||) is a normed linear space.

Definition 2.6.[6] A 2-fuzzy set on X is a fuzzy set on F(X)

Definition 2.7.[6] Let F(X) be a linear space over the real field K. A fuzzy subset N of F(X) xF(X) x R. (R, the set of real numbers) is called a 2-fuzzy 2-norm on X (or fuzzy 2-norm on F(X)) if and only if,

(N1) for all $t\in R$ with $t \le 0$, $N(f_1, f_2, t) = 0$,
(N2) for all $t\in R$ with $t > 0$, $N(f_1, f_2\ t) = 1$, if and only if f_1 and f_2 are linearly dependent.
(N3) $N(f_1, f_2, t)$ is invariant under any permutation of f_1, f_2.
(N4) for all $t \in R$, with $t \ge 0$,

$$N(f_1, cf_2, t) = N\left(f_1, f_2, \frac{t}{|c|}\right) \text{ if } c \neq 0,\ c \in K \text{ (field)}$$

(N5) for all $s, t \in R$, $N(f_1, f_2 + f_3, s + t) \ge \min\{N(f_1, f_2, s), N(f_1, f_3, t)\}$
(N6) $N(f_1, f_2, \bullet): (0,\infty) \to [0, 1]$ is continuous.
(N7) $\lim_{t \to \infty} N(f_1, f_2, t) = 1$

Then F(X), N) is a fuzzy 2-normed linear space or (X, N) is a 2-fuzzy 2-normed linear space.

Definition 2.8. A binary operation $*: [0, 1] \times [0, 1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

1. $*$ is commutative and associative
2. $*$ is continuous
3. $a * 1 = a$, for all $a \in [0,1]$
4. $a * b \le c * d$ whenever $a \le c$ and $b \le d$ and $a, b, c, d \in [0,1]$

Definition 2.9. A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is a continuous t - conorm if it satisfies the following conditions:

1. $\Diamond$ is commutative and associative
2. $\Diamond$ is continuous
3. $a \Diamond 0 = a$, for all $a \in [0,1]$
4. $a \Diamond b \le c \Diamond d$ whenever $a \le c$ and $b \le d$ and $a, b, c, d \in [0,1]$

Note 2.10.
(1) For any $r_1, r_2 \in (0,1)$ with $r_1 > r_2$ there exist $r_3, r_4 \in (0,1)$ such that $r_1 * r_3 \ge r_2$ and $r_1 \ge r_4 \Diamond r_2$
(2) For any $r_5 \in (0,1)$, there exist $r_6, r_7 \in (0,1)$ such that $r_6 * r_6 \ge r_5$ and $r_7 \Diamond r_7 \ge r_5$

Definition 2.11. An intuitionistic fuzzy 2- normed linear space (i.f-2-NLS) is of the form $A = \{F(X), N(f_1, f_2, t), M(f_1, f_2, t) / (f_1, f_2) \in F[(X)]^2\}$ where F(X) is a linear space over a field K, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm, N and M are fuzzy sets on $[F(X)]^2 \times (0,\infty)$ such that N denotes the degree of membership and M denotes the degree of non-membership of $(f_1, f_2, t) \in [F(X)]^2 \times (0, \infty)$ satisfying the following conditions:

(1) $N(f_1, f_2, t) + M(f_1, f_2, t) \le 1$
(2) $N(f_1, f_2, t) > 0$
(3) $N(f_1, f_2, t) = 1$ if and only if f_1, f_2 are linearly dependent
(4) $N(f_1, f_2, t)$ is invariant under any permutation of f_1, f_2
(5) $N(f_1, f_2, t) : (0, \infty) \to [0,1]$ is continuous in t.
(6) $N(f_1, cf_2, t) = N\left(f_1, f_2, \frac{t}{|c|}\right)$, if $c \neq 0$, $c \in K$
(7) $N(f_1, f_2, s) * N(f_1, f_3, t) \le N(f_1, f_2 + f_3, s + t)$
(8) $M(f_1, f_2, t) > 0$
(9) $M(f_1, f_2, t) = 0$ if and only if f_1, f_2 are linearly dependent
(10) $M(f_1, f_2, t)$ is invariant under any permutation of f_1, f_2
(11) $M(f_1, cf_2, t) = M\left(f_1, f_2, \frac{t}{|c|}\right)$ if $c \neq 0$, $c \in k$
(12) $M(f_1, f_2, s) \Diamond M(f_1, f_3, t) \ge M(f_1, f_2 + f_2, s + t)$
(13) $M(f_1, f_2, t) : (0, \infty) \to [0,1]$ is continuous in t.

3. FUZZY BOUDEDNESS AND FUZZY CONTINUITY ON INTUITIONISTIC 2- FUZZY NORMED LINEAR SPACE

Definition 3. 1. A sequence $\{f_n\}$ in an (IF 2-NLS) is said to converge to f if for given r > 0, t > 0, 0 < r < 1, there exists an integer $n_0 \in N$ such that

$N(f_n - f, g_1, t) > 1 - r$, $N(f_n - f, g_2, t) > 1 - r$

$M(f_n - f, g_1, t) < r$, $M(f_n - f, g_2, t) < r$ where g_1, g_2 are linearly independent (or) $N(f_n - f, g_i, t) \to 1$ as $n \to \infty$

for i = 1, 2 and
$M(f_n - f, gi, t) \to 0$ as $n \to \infty$ for i = 1, 2

Definition 3. 2. A sequence $\{f_n\}$ is a cauchy sequence if for given $\in > 0$,

$N(f_n - f_m, g_i, t) > 1 - \in$, $M(f_n - f_m, g_i, t) < \in$, $0 < \in < 1$, $t > 0$, g_i's are linearly independent, for i = 1, 2.

Definition 3. 3. Let $A = \{(F(X), N(f, f_2, t), M(f_1, f_2, t) / (f_1, f_2) \in [F(X)]^2)\}$ be an intuitionistic fuzzy 2-normed linear space then

$N((f_1, f_2), (f'_1, f'_2), t) = N((f_1 - f'_1), (f_2 - f'_2), t)$

$M(f_1, f_2), (f'_1, f'_2), t) = M((f_1 - f'_1), (f_2 - f'_2), t)$

are intuitionistic 2-fuzzy metrics defined on A and (A, N, M, *) is an intuitionistic 2-fuzzy metric space (i-2-f-m-s).

Definition 3. 4. Let (A, N, M, *) be an intuitionistic 2-fuzzy normed linear space. For t > 0, define the openball $B((f_1, f_2), r, t)$ with center $(f_1, f_2) \in A$ and radius 0 < r < 1 as

$B((f_1, f_2), r, t) = \{(g_1, g_2) \in A : N(f_1, g_1), (f_2, g_2), t) > 1 - r \; M(f_1 - g_1), (f_2 - g_2) < r\}$

Definition 3. 5. A subset $G \subset A$ is said to be open if for each $(f_1, f_2) \in G$, there exists t > 0 and 0 < r < 1 such that $B((f_1, f_2), r, t), r, t) \subset G$.

Definition 3. 6. Let $\mathcal{J}$ be the set of all open subsets of A, then it is called the the intuitionistic 2-fuzzy topology induced by the intuitionistic 2-fuzzy norm.

Definition 3.7. Let (A, N, M, *) be an i-2-f-m-s then a subset D of A is said to be intuitionistic2- fuzzy bounded if there exists t > 0 and 0 < r < 1 such that

$M((f_1, f_2), (g_1, g_2), t) > 1 - r$, $N(f_1, f_2), (g_1, g_2), t) < r$ for each $((f_1, f_2), (g_1, g_2)) \in [F(X)]^2$

Definition 3.8. Let $(A, N_1, M_1, *)$ $(B, N_2, M_2, *)$ be an intuitionistic 2-fuzzy normed linear space, a mapping $T: A \to B$ is said to be an intuitionistic fuzzy 2 - bounded if there exist constants $m_1, m_2 \in R^+$ such that for every $f \in A$ and for each t > 0,

$$N_2(Tf, Tg, t) > N_1\left(f, g, \frac{t}{m_1}\right)$$

$$M_2(Tf, Tg, t) > M_1\left(f, g, \frac{t}{m_2}\right)$$

Definition 3.9. Let $T : A \to B$ be a linear operator from IF 2-Banach Space A to IF 2 Banach space B. Then T is said to be an intuitionist 2 -fuzzy continuous if for each $\in$ with $0 < \in < 1$, there exists δ, $0 < \delta < 1$, such that

$N_1(f, g, t) \geq 1 - \delta$ and $M_1(f, g, t) \leq \delta$,

implies $N_2(Tf, Tg, t) \geq 1 - \in$ and $M_2(Tf, Tg, t) \leq \in$

Theorem3.10. A linear operator $T: (A, N_1, M_1, *) \to (B, N_2, M_2, *)$ is an intuitionistic 2- fuzzy bounded iff it is an intuitionistic 2- fuzzy continuous.

Proof. Assume $T: A \to B$ is an intuitionistic 2-fuzzy bounded. Then there exist constants $M_1, M_2 \in R^+$ such that for every $f \in A$ and for each $t > 0$,

$$N_2(Tf, Tg, t) \geq N_1\left(f, g, \frac{t}{M_1}\right) \text{ and}$$

$$M2(Tf, Tg, t) \leq M_1\left(f, g, \frac{t}{M_2}\right) \tag{1}$$

Suppose for $\in$, with $0 < \in < 1$, choose δ,with $0 < \delta < 1$, such that $N_1(f, g, t) \geq 1-\delta$ and $M_1(f, g, t) \leq \delta$ for any $t > 0$

and $$N_1\left(f, g, \frac{t}{M_1}\right) \geq 1 - \in$$

$$M_1\left(f, g, \frac{t}{M_2}\right) < \in \quad (\because M_1, M_2 > 0) \tag{2}$$

Using (2) in (1) we get

$$N_2(Tf, Tg, t) \geq 1 - \in \text{ and } M_2(Tf, Tg, t) \leq \in$$

Hence T is an intuitionistic 2- fuzzy continuous.

Conversely, Suppose T is an intuitionistic 2- fuzzy continuous.

For $\in$ with $0 < \in < 1$, there exists δ with $0 < \delta < 1$

such that $N_1(f, g,t) < 1 - \delta$, $M_1(f, g, t) < \delta$

implies $N_2(Tf, Tg, t) > 1 - \in$, $M_2(Tf, Tg, t) < \in$ (3)

Choose $M_1, M_2 \in R^+$ such that

$$N_1\left(f, g, \frac{t}{M_1}\right) \leq 1 - \in \text{ for given } N_1(f, g, t) > 1 - \delta \text{ and}$$

$$M_1\left(f, g, \frac{t}{M_2}\right) \geq \in \text{ for given } M_1(f, g, t) < \delta \tag{4}$$

Then applying (4) on (3) we get

$$N_2(Tf, Tg, t) > 1 - \in \geq N_1\left(f, g, \frac{t}{M_1}\right)$$

$$M_2(Tf, Tg, t) < \delta \leq M_1\left(f, g, \frac{t}{M_2}\right)$$

Therefore T is intuitionistic 2- fuzzy bounded.

Lemma 3.11. Let (F(X), N, M, *) be an intuitionistic 2-fuzzy normed linear space. Let T : F(X) $\to$ F(X) be an intuitionistic 2- fuzzy continuous. If $f_n \to f$ then $T(f_n) \to T(f)$ as $n \to \infty$.

Proof. Given $f_n \to f$ in (F(X), N, M, *)

Then for given $\in > 0$, $t > 0$, $0 < t < 1$ there exists an integer $n_0 \in N$ such that $N(f_n\text{-}f, gi, t) > 1 - \in$ and $M(f_n\text{-}f, g_i, t) < \in$ where g_i's are linearly independent for all $n \geq n_0$, i = 1, 2.

Since T is is intuitionistic 2- fuzzy continuous,

$(T(f_n - f), Tg_i, t) > 1 - \in$ and $M(T(f_n - f), Tg_i, t) < \in$

implies $N(Tf_n - Tf, g_i' t) > 1 - \in$ and $M(Tf_n - Tf, g_i', t) < \in$

Thus $Tf_n \to Tf$ as $n \to \infty$.

Lemma 3.12. Let (F(X), N, M, *) be an intuitionistic 2-fuzzy normed linear space then N and M are jointly continuous.

Proof. If $f_n \to f$ and $g_n \to g$ in (F(X), N, M, *) we have to prove that $N(f_n - f, g_n - g, t) > 1 - \in$ and $M(f_n - f, g_n - g, t) < \in$ as $n \to \infty$.

We know that $\lim\limits_{n\to\infty} N(f_n - f, f_i', t) = 1$ or $> 1 - \in$, $\lim\limits_{n\to\infty} N(g_n - g, f_i', t) = 1 > 1 - \in$ and $\lim\limits_{n\to\infty} M(f_n - f, f_i', t) = 0 < \in$, $\lim\limits_{n\to\infty} M(g_n - g, f_i', t) = 0 < \in$

$$
\begin{aligned}
N(f_n - f, g_n - g, t) &\geq N(f_n - f, f_i', t/2) * N(g_n - g, f_i', t/2) \\
&> (1 - \in) * (1 - \in) \\
&= 1 - \in
\end{aligned}
$$

$$
\begin{aligned}
\text{And, } M(f_n - f, g_n - g, t) &\leq M(f_n - f, f_i', t/2) \diamond M(g_n - g, f_i', t/2) \\
&< \in \diamond \in = \in
\end{aligned}
$$

Definition 3.13. Let (F(X), N, M, *) be an intuitionistic 2-fuzzy normed linear space. A subset A of F (X) is said to be is intuitionistic 2- fuzzy bounded if

$N(f, g, t) \geq 1 - M$ and $M(f, g, t) \leq M$ where M is a positive constant.

4. OPEN MAPPING THEOREM

If T is a continuous linear operator from the intuitionistic 2-fuzzy Banach space. (F(X), N_1, M_1, *) onto the intuitionistic 2-fuzzy Banach space (F(Y), N_2, M_2, *) then T is an open mapping.

Proof. Let us prove the theorem in various steps.

Step 1. Let A be an intuitionistic 2-fuzzy neighbourhood of $\overline{0} = (0,0)$ in $[F(X)]^2$. Let us show that $\overline{0} \in (T(A))°$.Let B be the the intuitionistic 2-fuzzy balanced neighbourhood of $\overline{0}$ such that $B + B \subset A$.

Since T (F(X)) = F(Y) and B is absorbing, it follows that $F(Y) = \bigcap\limits_{n} T(nB)$

There exists a positive integer n_0 such that T $\overline{(n_0 B)}$ has a non empty interior. Therefore $\overline{0} \in (\overline{T(B)})^0 - (\overline{T(B)})^0$

Also

$$T\left(\overline{B}\right)^0 - (T\left(\overline{B}\right))^0 \subset (T\left(\overline{B}\right)) - T\left(\overline{B}\right) \subset T\left(\overline{A}\right)$$

So the set $T(\overline{A})$ includes the the intuitionistic 2-fuzzy neighbourhood $(T(\overline{B}))^0 - (T(\overline{B}))^0$ of $\overline{0}$.

Step 2. Now it is shown that $\overline{0} \in (T(A))^0$ since $\overline{0} \in A$ and A is an open set there exists $0 < \alpha < 1$ and to $\in (0, \alpha)$ such that $B((0,0), \alpha, t_0) \subset A$. But for $0< \alpha < 1$ a sequence $\{\in_n\}$ can be found such that $\in_n \to 0$ and

$$1 - \alpha < \lim_n [(1- \in_1) * (1 - \in_2) * \ldots * (1 - \in_n)].$$

Again, $\overline{0} \in T(B(0, 0), \in_n, t_n'))$ where $t'_n = \frac{1}{2^n} t_0$, so by step 1 there exists $\delta_n \in (0,1)$ and $t_n > 0$ such that $B((0, 0), \delta_n, t_n) \subset T\,(\overline{B(0,0), \in_n, t_n'})$

Since the set $\left\{B(0,0), \frac{1}{n}, \frac{1}{n}\right\}$ is a countably locally base at zero and $t_n' \to 0$ as $n \to \infty$.

It is to be shown that $B((0,0), \delta_1, t_1) \subset (T(A))^0$ .Suppose $(f_0, f'_0) \in B(0,0), \delta_1, t_1)$. Then $(f_0, f'_0) \in T\,(\overline{B(0,0), \in_2, t_2'})$ and so for $\delta_2 > 0$ and $\in_2 > 0$ the ball $B(f_0, f_0'), \delta_2, t_2)$ intersects $T(B(0,0), \delta_1, t')$. Therefore there exist $(f_1,f'_1) \in B(f_0, f_0'), \delta_2, t_2)$

(ie) $N_2[(f_0, f'_0) - T(f_1, f'_1)]\, g_i, t_2) >1 - \delta_2$ and $M_2((f_0, f_0') - T(f_1, f_1'), g_i, t_2) < \delta_2$ or equivalently,

$$((f_0, f_0) - T(f_1, f'_1) \in B(0,0), \delta_2, t_2) \subset T\,(\overline{B(0,0), \in_1, t'})$$

and by the similar argument there exists (f_2, f_2') in $B(0,0), \in_2, t')$, such that

$N_2 (f_0, f'_0) - T[(f_1, f'_1)+ (f_2, f'_2), g_i, t_3) = N_2 ((f_0, f_0') - T(f_1, f_1') - T(f_2, f_2'), g_i, t_3) > 1 - \delta_3$.

and $M_2 ((f_o, f_0') - T([f_1, f_1') + (f_2, f_2')], g_i, t_3) = M_2 ((f_0, f_0') - T(f_1, f'_1) -T(f_2, f_2'), g_i, t_3) > \delta_3$.

If this process is continued, it leads to a sequence $\{\{f_n, f_n')\}$ such that $(f_n, f_n') \in B((0,0), \in_n, t_n')$ and $N_2 ((f_0, f_0) - \sum_{j=1}^{n-1} T(f_j, f_j'), g_i, t_n) < 1 - \delta_n$,

$$M_2(f_0, f_0) - \sum_{j=1}^{n-1} T(f_j, f_j'), g_i, t_n) < \delta_n$$

Now if $n \in N$ and $\{p_n\}$ is a positive and increasing sequence, then

$$N_1\left(\sum_{j=1}^{n}(f_j, f_j') - \sum_{j=1}^{n+p_n}(f_j, f_j'), g_i, t\right) = N_1\left(\sum_{j=1}^{n+p_n}(f_j, f_j'), g_i, t\right)$$

$$\geq N_1 (f_{n+1}, f_{n+1}'), g_i, t_1) * N1 (f_{n+2}', f_{n+2}'), g_i, t_2) * \ldots * N1 (f_{n+p_n}, f_{n+p_n}'), g_i, t_{p_n})$$

where $t_1 + t_2 + \ldots + t_{p_n} = t$

$$M_1\left(\sum_{j=1}^{n}(f_j, f_j') - \sum_{j=1}^{n+p_n}(f_j, f_j'), g_i, t\right) = M_1\left(\left(\sum_{j=n+1}^{n+p_n}(f_j, f_j')\right), g_i, t\right)$$

$$\leq M_1 ((f_{n+1}, f_{n+1}'), g_i, t_1) \lozenge \ldots \lozenge M1 (f_n+p_n'\ f_{n+p_n}', g_i, t_n)$$

where $t_1 + t_2 + \ldots + t_{p_n} = t$.

By putting $t_0 = \min \{t_1, t_2 \ldots t_{pn}\}$ since $t_n' \to 0$ so there exists no such that $0 \leq t_n' \leq t_0$ for $n > n_0$.

Therefore,

$$N_1((f_{n+1}, f_{n+1}'), g_i, t_0) * \ldots * N_1(f_{n+pn}, f_{n+pn}'), g_i, t_0) \geq N_1((f_{n+1}, f_{n+1}'), t_{n+1}') * \ldots * N_1(f_{n+pn}, f_{n+pn}'), t_{n+pn}')$$

$$\geq (1 - \in_{n+1}) * \ldots * (1 - \in_{n+pn})$$

Also,

$$M_1((f_{n+1}, f_{n+1}'), t_0) \Diamond \ldots \Diamond M_1(f_{n+pn}, f_{n+pn}'), t_0) \leq M_1((f_{n+1}, f_{n+1}'), t_{n+1}') \Diamond \ldots \Diamond M_1(f_{n+pn}, f_{n+pn}'), t_{n+pn}')$$

$$\leq t_{n+1} \Diamond \ldots . \Diamond \in_{n+pn}$$

Hence $\lim\limits_{n\to\infty} N_1\left(\sum\limits_{j=n+1}^{n+p_n}(f_j, f_j'), t\right) \geq \lim\limits_{n\to\infty} ((1- \in_{n+1}) * \ldots .* (1- \in_{n+pn}) = 1$

That is $N_1\left(\sum\limits_{j=n+1}^{n+p_n}(f_j, f_j'), t\right) \to 1$ for all $t>0$

and $\lim\limits_{n\to\infty} M_1\left(\sum\limits_{j=n+1}^{n+p_n}(f_j, f_j'), t\right) \leq \lim\limits_{n\to\infty} \in_{n+1} \Diamond \ldots . \Diamond \in_{n+pn}$

$=1$

That is $M_1\left(\sum\limits_{j=n+1}^{n+p_n}(f_j, f_j'), t\right) \to 1$ for all $t>0$

So the sequence $\{\sum\limits_{j=1}^{n} (f_j, f_j')\}$ is said to be a Cauchy sequence and consequently $\sum\limits_{j=1}^{\infty}(f_j, f_j')$ converges to some point $(f, f') \in F(X)$, because $F(X)$ is complete.

By fixing $t>0$, there exists n_0 such that $t>t_n$ for $n>n_0$, because $t_n \to 0$, it follows

$$N_2((f_0, f_0')\text{-}T(\sum_{j=1}^{n-1} T(f_j, f_j'), g_i, t) \geq N_2((f_0, f_0')\text{-}T(\sum_{j=1}^{n-1} T(f_j, f_j'), g_i, t_n)$$

$$\geq 1- \delta_n$$

Thus $N_2((f_0, f_0')\text{-}T(\sum\limits_{j=1}^{n-1} T(f_j, f_j'), g_i, t) \to 1$

Also $M_2((f_0, f_0')\text{-}T(\sum\limits_{j=1}^{n-1} T(f_j, f_j'), g_i, t) \leq M_2((f_0, f_0')\text{-}T(\sum\limits_{j=1}^{n-1} T(f_j, f_j'), g_i, t_n) \leq \delta_n$

Therefore $M_2((f_0, f_0')\text{-}T(\sum\limits_{j=1}^{n-1} T(f_j, f_j'), g_i, t) \to 1$ as $n \to\infty$

Hence $(f_0, f_0') = \lim\limits_{n\to\infty} T(\sum\limits_{j=1}^{n-1} T(f_j, f_j') = T(f, f')$

But $N_1((f,f'), g_i, t_0) \geq \limsup N_1(\sum\limits_{j=1}^{n}(f_j, f_j), g_i, t_0) \geq \limsup ((1- \in_1) * \ldots .* (1- \in_n)$

$= 1- \alpha$

and $M_1((f,f'), g_i, t_0) \leq \liminf M_1(\sum\limits_{j=1}^{n}(f_j, f_j), g_i, t_0)$

$\leq \liminf\limits_{n} (\delta_1 \Diamond \ldots . \Diamond \delta_n) = \alpha$

Hence $(f,f') \in B((0,0), \alpha, t_0)$

Step 3. Let G be an open subset of $[F(X)]^2$ and $(f,f') \in G$. Then

$T(G) = T(f,f')+T((-f,f')+G) \supset T(f,f')+(T(-(f,f')+G))^0$

Hence T(G) would be open because it includes a neighbourhood of each of its points.

Definition4.2. A fuzzy 2-linear functional F is a real valued function on A ×B where A and B are subspaces of (F(X), N) such that
(i) $F(f+h, g+h') = F(f, g) + F(f, h') + F(h, g) + F(h, h')$
(ii) $F(\alpha f, \beta g) = \alpha\beta\, F(f, g)$, $\alpha\ \beta \in [0,1]$

F is said to be bounded with respect to α-2-norm if there exists a constant $k\in [0,1]$ such that $|F(f, g)| \le k\, \|(f,g)\|_\alpha$
$\|F\| = g\ l\ b\ \{k : |F(f, g)| \le k\, \|(f, g)\|\alpha$ for every $(f,g) \in A \times B\}$

Theorem 4.3. Let T be an intuitionistic 2-fuzzy linear operator from intuitionsitic fuzzy 2-Banach space $(F(X), N_1, M_1, *)$ to intuitionistic fuzzy 2- Banach space $(F(Y), N_2, M_2,*)$. Suppose for every $\{f_n, f_n'\} \in (F(X), N_1,M_1)$ such that $(f_n, f_n') \to (f, f')$ and $(Tf_n, Tf_n') \to (g, g')$ for some $f, f' \in F(X)$, $g, g' \in F(Y)$ follows $T(f, f') = (g, g')$. Then T is continuous.

Proof. N and M on $(F(X), N_1, M_1, *)$ X $(F(Y), N_2, M_2, *)$ is given by

$$N((f_1, f_2), (g_1, g_2), t) = \min\{N_1(f_1, f_2, t), N_2(g_1, g_2, t)\}$$

$$= N_1(f_1, f_2, t) * N_2(g_1, g_2, t)$$

$$M((f_1, f_2), (g_1, g_2), t) = \max N_1(f_1, f_2, t), M(f_1, f_2, t)\}$$

where * is the usual continuous t-norm and <> is the usual continuous t-conorm. With this norm and conorm $(F(X), N_1, M_1, *)$ X (F(Y), N2, M2, *) is a complete intuitionistic 2-fuzzy normed linear space.

For each $(f_1, f_2), (f_1', f_2') \in F(X)$ and $(g_1, g_2), (g_1', g_2') \in F(Y)$ and $t, s > 0$ it follows that

$$\begin{aligned} N(f_1, f_2), (g_1, g_2), t) * N(f_1', f_2'), (g_1', g_2'), s) &= [N_1(f_1, f_2, t) * N_2(g_1, g_2, t)] * [N_1(f_1', f_2', s) * N_2(g_1', g_2', s)] \\ &= [N_1(f_1, f_2, t) * N_1(f_1', f_2', s)] * [N_2(g_1, g_2, t) * N_2(g_1', g_2', s)] \\ &\le N_1(f_1 + f_1', f_2 + f_2', s + t) * N_2(g_1 + g_1', g_2 + g_2', t + s) \\ &= N((f_1 + f_1', f_2 + f_2'), (g_1 + g_1', g_2 + g_2'), s + t)) \end{aligned}$$

Again

$$\begin{aligned} M((f_1, f_2), (g_1, g_2), t) \lozenge M((f_1', f_2'), (g_1', g_2'), s) &= [M_1(f_1, f_2, t) \lozenge M_2(g_1, g_2, t)] \lozenge [M_1(f_1', f_2', s) \lozenge M_2(g_1', g_2', s)] \\ &= [M_1(f_1, f_2, t) \lozenge M_1(f_1', f_2', s)] \lozenge [M_2(g_1, g_2, t) \lozenge M_2(g_1', g_2', s)] \\ &\ge M_1(f_1 + f_1', f_2 + f_2', s+ t) \lozenge M_2(g_1 + g_1', g_2 + g_2', t + s) \\ &= M((f_1 + f_1', f_2 + f_2')(g_1 + g_1', g_2 + g_2')\, s + t) \end{aligned}$$

Now if $\{(f_n, f_n'), (g_n, g_n')\}$ is a cauchy sequence in

$(F(X) \times F(X) \times F(Y) \times F(Y), N, M, *)$ then there exists $n_0 \in N$ such that

$$N((f_n, f_n'), (g_n, g_n') - (f_m, f_m'), (g_m, g_m'), t) > 1 - \in$$

$M((f_n, f_n'), (g_n, g_n') - ((f_m, f_m'), (g_m, g_m'), t) < \in$ for every $\in > 0$ and $t > 0$

For $m, n > n_0$

$$\begin{aligned} N_1(f_n - f_m, f_n' - f_m', t) * N_2(g_n - g_m, g_n', g_m', t) &= N((f_n - f_m, f_n' - f_m'), (g_n - g_m, g_n' - g_m'), t) \\ &= N((f_n, f_n'), (g_n, g_n'), (f_m, f_m'), (g_m, g_m'), t) \\ &> 1 - \in \end{aligned}$$

$$\begin{aligned} M_1(f_n - f_m, f_n' - f_m', t) \lozenge M_2(g_n - g_m, g_n' - g_m', t) &= M((f_n - f_m, f_n' - f_m'), (g_n - g_m, g_n' - g_m'), t) \\ &= M((f_n, f_n'), (g_n, g_n'), (f_m, f_m'), (g_m, g_m'), t) \\ &< \in \end{aligned}$$

Therefore $\{(f_n, f_n')\}$ and $\{(g_n, g_n')\}$ are cauchy sequences in $(F(X), N_1, M_1, *)$ and $(F(Y), N_2, M_2, *)$ respectively and there exists $f \in F(X)$ and $g \in F(Y)$ such that $(f_n, f_n') \to (f, f')$ and $(g_n, g_n') \to (g, g')$ and consequently $\{(f_n, f_n'), (g_n, g_n')\}$converges in the intuitionistic 2-fuzzy normed linear space. Hence

$(F(X) \times F(X) \times F(Y) \times F(Y), N, M, *)$ is a complete intuitionistic 2-fuzzy normed linear space.

Here let $G = \{(f_n, f_n), T(f_n, f_n)\}$ for every $(f_n, f_n') \in F(X) \times F(X)$ be the graph of the fuzzy 2-linear operator T.

Suppose $(f_n, f_n') \to (f, f')$ and $T(f_n, f_n') \to (g, g')$.

Then from previous argument,

$\{(f_n, f_n'), (Tf_n, Tf_n')\}$ converges to $((f, f'), (g, g'))$ which belongs to G.

Therefore, $T(f, f') = (g, g')$. Thus T is continuous.

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