

OPEN MAPPING THEOREM ON INTUITIONISTIC 2-FUZZY 2-NORMED LINEAR SPACE

¹Thangaraj Beula* & ²Lilly Esthar Rani

1&2 Department of Mathematics, TBML College, Porayar, Tamil Nadu, India

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ABSTRACT

The concepts of fuzzy boundedness, fuzzy continuity and on intuitionistic 2-fuzzy 2-normed linear space are introduced. Using these concepts some theorems are proved and as a result the famous Open Mapping Theorem is established in intuitionistic 2-fuzzy 2-normed linear space.

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1. INTRODUCTION

The theory of fuzzy sets was introduced by L. Zadeh [9] in 1965. A satisfactory theory of 2-norm on a linear space has been introduced and developed by Gahler in 1964[2]. The concepts fuzzy norm and α -norm were introduced by Bag and Samanta in 2003[1]. Jialuzhang [3] has defined fuzzy linear space in a different way. The notion of 2-fuzzy 2-normed linear space of the set of all fuzzy sets of a set was introduced by R. M. Somasundaram and Thangaraj Beula [6]. The concept of intuitionistic 2-fuzzy 2-normed linear space of the set of all fuzzy sets of a set was introduced by Thangaraj Beula and D. Lilly Esthar Rani [7].

We have introduced the concepts of fuzzy boundedness, fuzzy continuity on intuitionistic 2-fuzzy 2-normed linear space. Using these concepts some theorems are proved and as a result the famous Open Mapping Theorem is established in intuitionistic 2-fuzzy 2-normed linear space.

2. PRELIMINARIES

For the sake of completeness, we reproduce the following definitions due to Gahler [2], Bag and Samanta [1] and Jialuzhang [3].

Definition 2.1. [2] Let X be a real vector space of dimension greater than 1 and let $\|\bullet, \bullet\|$ be a real valued function on $X \times X$ satisfying the following conditions:

1. $\|x, y\| = 0$ if and only if x and y are linearly dependent,
2. $\|x, y\| = \|y, x\|$,
3. $\|\alpha x, y\| = |\alpha| \|x, y\|$, where α is real,
4. $\|x, y+z\| \leq \|x, y\| + \|x, z\|$.

$\|\bullet, \bullet\|$ is called a 2-norm on X and the pair $(X, \|\bullet, \bullet\|)$ is called a linear 2-normed space.

Definition 2.2. [1] Let X be a linear space over K (field or real or complex numbers). A fuzzy subset N of $X \times \mathbb{R}$ (\mathbb{R} , the set of real numbers) is called a fuzzy norm on X if and only if for all $x, u \in X$ and $c \in \mathbb{R}$.

(N1) for all $t \in \mathbb{R}$ with $t \leq 0$, $N(x, t) = 0$,

(N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(x, t) = 1$, if and only if $x = 0$,

(N3) for all $t \in \mathbb{R}$ with $t > 0$, $N(cx, t) = N(x, \frac{t}{|c|})$, if $c \neq 0$,

(N4) for all $s, t \in \mathbb{R}$, $x, u \in X$, $N(x+u, s+t) \geq \min \{N(x, s), N(u, t)\}$,

Corresponding author: ¹Thangaraj Beula*
1&2 Department of Mathematics, TBML College, Porayar, Tamil Nadu, India

(N5) $N(x, \bullet)$ is a non decreasing function of \mathbb{R} and $\lim_{t \rightarrow \infty} N(x, t) = 1$

The pair (X, N) will be referred to as a fuzzy normed linear space.

Definition 2.3. [3] Let X be any non-empty set and $F(X)$ be the set of all fuzzy sets on X . For $U, V \in F(X)$ and $k \in \mathbb{K}$ the field of real numbers, define

$$U + V = \{ (x + y, \lambda \wedge \mu) \mid (x, \lambda) \in U, (y, \mu) \in V \},$$

$$kU = \{ (kx, \lambda) \mid (x, \lambda) \in U \}$$

Definition 2.4.[3] A fuzzy linear space $\tilde{X} = X \times (0, 1]$ over the number field \mathbb{K} . where the addition and scalar multiplication operation on \tilde{X} are defined by

$$(x, \lambda) + (y, \mu) = (x + y, \lambda \wedge \mu),$$

$k(x, \lambda) = (kx, \lambda)$ is a fuzzy normed space if to every $(x, \lambda) \in \tilde{X}$ there is associated a non-negative real number, $\|(x, \lambda)\|$, called the fuzzy norm of (x, λ) , in such a way that

1. $\|(x, \lambda)\| = 0$ iff $x = 0$ the zero element of X , $\lambda \in (0, 1]$
2. $\|k(x, \lambda)\| = |k| \|(x, \lambda)\|$ for all $(x, \lambda) \in \tilde{X}$ and all $k \in \mathbb{K}$
3. $\|(x, \lambda) + (y, \mu)\| \leq \|(x, \lambda \wedge \mu)\| + \|(y, \lambda \wedge \mu)\|$ for all (x, λ) and $(y, \mu) \in \tilde{X}$
4. $\|(x, \bigvee_t \lambda_t)\| = \bigwedge_t \|(x, \lambda_t)\|$ for $\lambda_t \in (0, 1]$

Definition 2.5.[6] Let X be a non-empty and $F(X)$ be the set of all fuzzy sets in X . If $f \in F(X)$ then $f = \{ (x, \mu) \mid x \in X \text{ and } \mu \in (0, 1] \}$ Clearly f is a bounded function for $|f(x)| \leq 1$. Let \mathbb{K} be the space of real numbers, then $F(X)$ is a linear space over the field \mathbb{K} where the addition and scalar multiplication are defined by

$$f + g = \{ (x, \mu) + (y, \eta) \} = \{ (x + y, \mu \wedge \eta) \mid (x, \mu) \in f \text{ and } (y, \eta) \in g \}$$

$$kf = \{ (kx, \mu) \mid (x, \mu) \in f \} \text{ where } k \in \mathbb{K}.$$

The linear space $F(X)$ is said to be normed space if to every $f \in F(X)$, there is associated a non- negative real number $\|f\|$ called the norm of f in such a way that

1. $\|f\| = 0$ if and only if $f = 0$.

$$\text{For } \|f\| = 0 \Leftrightarrow \{ \|(x, \mu)\| \mid (x, \mu) \in f \} = 0$$

$$\Leftrightarrow x = 0, \mu \in (0, 1]$$

$$\Leftrightarrow f = 0$$

$$2. \|kf\| = |k| \|f\|, k \in \mathbb{K}.$$

$$\text{For } \|kf\| = \{ \|k(x, \mu)\| \mid (x, \mu) \in f, k \in \mathbb{K} \}$$

$$= \{ |k| \|(x, \mu)\| \mid (x, \mu) \in f \}$$

$$= |k| \|f\|.$$

$$3. \|f + g\| \leq \|f\| + \|g\| \text{ for every } f, g \in F(X)$$

For,

$$\|f + g\| = \{ \|(x, \mu) + (y, \eta)\| \mid x, y \in X, \mu, \eta \in (0, 1] \}$$

$$= \{ \|(x + y, (\mu \wedge \eta))\| \mid x, y \in X, \mu, \eta \in (0, 1] \}$$

$$\leq \{ \|(x, \mu \wedge \eta)\| + \|(y, \mu \wedge \eta)\| \mid (x, \mu) \in f \text{ and } (y, \eta) \in g \}$$

$$= \|f\| + \|g\|.$$

and $(F(X), \|\bullet\|)$ is a normed linear space.

Definition 2.6.[6] A 2-fuzzy set on X is a fuzzy set on $F(X)$

Definition 2.7.[6] Let $F(X)$ be a linear space over the real field K . A fuzzy subset N of $F(X) \times F(X) \times \mathbb{R}$, (\mathbb{R} , the set of real numbers) is called a 2-fuzzy 2-norm on X (or fuzzy 2-norm on $F(X)$) if and only if,

- (N1) for all $t \in \mathbb{R}$ with $t \leq 0$, $N(f_1, f_2, t) = 0$,
 (N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(f_1, f_2, t) = 1$, if and only if f_1 and f_2 are linearly dependent.
 (N3) $N(f_1, f_2, t)$ is invariant under any permutation of f_1, f_2 .
 (N4) for all $t \in \mathbb{R}$, with $t \geq 0$,

$$N(f_1, cf_2, t) = N\left(f_1, f_2, \frac{t}{|c|}\right) \text{ if } c \neq 0, c \in K \text{ (field)}$$

- (N5) for all $s, t \in \mathbb{R}$, $N(f_1, f_2 + f_3, s + t) \geq \min\{N(f_1, f_2, s), N(f_1, f_3, t)\}$
 (N6) $N(f_1, f_2, \bullet): (0, \infty) \rightarrow [0, 1]$ is continuous.
 (N7) $\lim_{t \rightarrow \infty} N(f_1, f_2, t) = 1$

Then $(F(X), N)$ is a fuzzy 2-normed linear space or (X, N) is a 2-fuzzy 2-normed linear space.

Definition 2.8. A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

1. $*$ is commutative and associative
2. $*$ is continuous
3. $a * 1 = a$, for all $a \in [0, 1]$
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0, 1]$

Definition 2.9. A binary operation \diamond : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-conorm if it satisfies the following conditions:

1. \diamond is commutative and associative
2. \diamond is continuous
3. $a \diamond 0 = a$, for all $a \in [0, 1]$
4. $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0, 1]$

Note 2.10.

- (1) For any $r_1, r_2 \in (0, 1)$ with $r_1 > r_2$ there exist $r_3, r_4 \in (0, 1)$ such that $r_1 * r_3 \geq r_2$ and $r_1 \geq r_4 \diamond r_2$
 (2) For any $r_5 \in (0, 1)$, there exist $r_6, r_7 \in (0, 1)$ such that $r_6 * r_6 \geq r_5$ and $r_7 \diamond r_7 \geq r_5$

Definition 2.11. An intuitionistic fuzzy 2-normed linear space (i.f-2-NLS) is of the form $A = \{F(X), N(f_1, f_2, t), M(f_1, f_2, t) / (f_1, f_2) \in [F(X)]^2\}$ where $F(X)$ is a linear space over a field K , $*$ is a continuous t-norm, \diamond is a continuous t-conorm, N and M are fuzzy sets on $[F(X)]^2 \times (0, \infty)$ such that N denotes the degree of membership and M denotes the degree of non-membership of $(f_1, f_2, t) \in [F(X)]^2 \times (0, \infty)$ satisfying the following conditions:

- (1) $N(f_1, f_2, t) + M(f_1, f_2, t) \leq 1$
- (2) $N(f_1, f_2, t) > 0$
- (3) $N(f_1, f_2, t) = 1$ if and only if f_1, f_2 are linearly dependent
- (4) $N(f_1, f_2, t)$ is invariant under any permutation of f_1, f_2
- (5) $N(f_1, f_2, t): (0, \infty) \rightarrow [0, 1]$ is continuous in t .

$$(6) \quad N(f_1, cf_2, t) = N\left(f_1, f_2, \frac{t}{|c|}\right), \text{ if } c \neq 0, c \in K$$

- (7) $N(f_1, f_2, s) * N(f_1, f_3, t) \leq N(f_1, f_2 + f_3, s + t)$
- (8) $M(f_1, f_2, t) > 0$
- (9) $M(f_1, f_2, t) = 0$ if and only if f_1, f_2 are linearly dependent
- (10) $M(f_1, f_2, t)$ is invariant under any permutation of f_1, f_2

$$(11) \quad M(f_1, cf_2, t) = M\left(f_1, f_2, \frac{t}{|c|}\right) \text{ if } c \neq 0, c \in k$$

- (12) $M(f_1, f_2, s) \diamond M(f_1, f_3, t) \geq M(f_1, f_2 + f_3, s + t)$
- (13) $M(f_1, f_2, t): (0, \infty) \rightarrow [0, 1]$ is continuous in t .

3. FUZZY BOUEDNESS AND FUZZY CONTINUITY ON INTUITIONISTIC 2- FUZZY NORMED LINEAR SPACE

Definition 3. 1. A sequence $\{f_n\}$ in an (IF 2-NLS) is said to converge to f if for given $r > 0, t > 0, 0 < r < 1$, there exists an integer $n_0 \in \mathbb{N}$ such that

$$N(f_n - f, g_1, t) > 1 - r, N(f_n - f, g_2, t) > 1 - r$$

$$M(f_n - f, g_1, t) < r, M(f_n - f, g_2, t) < r \text{ where } g_1, g_2 \text{ are linearly independent (or) } N(f_n - f, g_i, t) \rightarrow 1 \text{ as } n \rightarrow \infty$$

for $i = 1, 2$ and

$$M(f_n - f, g_i, t) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for } i = 1, 2$$

Definition 3. 2. A sequence $\{f_n\}$ is a cauchy sequence if for given $\epsilon > 0$,

$$N(f_n - f_m, g_i, t) > 1 - \epsilon, M(f_n - f_m, g_i, t) < \epsilon, 0 < \epsilon < 1, t > 0, g_i \text{'s are linearly independent, for } i = 1, 2.$$

Definition 3. 3. Let $A = \{(F(X), N(f, f_2, t), M(f_1, f_2, t) / (f_1, f_2) \in [F(X)]^2)\}$ be an intuitionistic fuzzy 2-normed linear space then

$$N((f_1, f_2), (f'_1, f'_2), t) = N((f_1 - f'_1), (f_2 - f'_2), t)$$

$$M(f_1, f_2), (f'_1, f'_2), t) = M((f_1 - f'_1), (f_2 - f'_2), t)$$

are intuitionistic 2-fuzzy metrics defined on A and $(A, N, M, *)$ is an intuitionistic 2-fuzzy metric space (i-2-f-m-s).

Definition 3. 4. Let $(A, N, M, *)$ be an intuitionistic 2-fuzzy normed linear space. For $t > 0$, define the openball $B((f_1, f_2), r, t)$ with center $(f_1, f_2) \in A$ and radius $0 < r < 1$ as

$$B((f_1, f_2), r, t) = \{(g_1, g_2) \in A : N(f_1, g_1), (f_2, g_2), t) > 1 - r, M(f_1 - g_1), (f_2 - g_2) < r\}$$

Definition 3. 5. A subset $G \subset A$ is said to be open if for each $(f_1, f_2) \in G$, there exists $t > 0$ and $0 < r < 1$ such that $B((f_1, f_2), r, t), r, t) \subset G$.

Definition 3. 6. Let \mathcal{J} be the set of all open subsets of A , then it is called the the intuitionistic 2-fuzzy topology induced by the intuitionistic 2-fuzzy norm.

Definition 3.7. Let $(A, N, M, *)$ be an i-2-f-m-s then a subset D of A is said to be intuitionistic2- fuzzy bounded if there exists $t > 0$ and $0 < r < 1$ such that

$$M((f_1, f_2), (g_1, g_2), t) > 1 - r, N(f_1, f_2), (g_1, g_2), t) < r \text{ for each } ((f_1, f_2), (g_1, g_2)) \in [F(X)]^2$$

Definition 3.8. Let $(A, N_1, M_1, *) (B, N_2, M_2, *)$ be an intuitionistic 2-fuzzy normed linear space, a mapping $T: A \rightarrow B$ is said to be an intuitionistic fuzzy 2 - bounded if there exist constants $m_1, m_2 \in \mathbb{R}^+$ such that for every $f \in A$ and for each $t > 0$,

$$N_2(Tf, Tg, t) > N_1\left(f, g, \frac{t}{m_1}\right)$$

$$M_2(Tf, Tg, t) > M_1\left(f, g, \frac{t}{m_2}\right)$$

Definition 3.9. Let $T: A \rightarrow B$ be a linear operator from IF 2-Banach Space A to IF 2 Banach space B . Then T is said to be an intuitionist 2 -fuzzy continuous if for each ϵ with $0 < \epsilon < 1$, there exists $\delta, 0 < \delta < 1$, such that

$$N_1(f, g, t) \geq 1 - \delta \text{ and } M_1(f, g, t) \leq \delta,$$

implies $N_2(Tf, Tg, t) \geq 1 - \epsilon$ and $M_2(Tf, Tg, t) \leq \epsilon$

Theorem3.10. A linear operator $T: (A, N_1, M_1, *) \rightarrow (B, N_2, M_2, *)$ is an intuitionistic 2- fuzzy bounded iff it is an intuitionistic 2- fuzzy continuous.

Proof. Assume $T: A \rightarrow B$ is an intuitionistic 2-fuzzy bounded. Then there exist constants $M_1, M_2 \in \mathbb{R}^+$ such that for every $f \in A$ and for each $t > 0$,

$$\begin{aligned} N_2(Tf, Tg, t) &\geq N_1\left(f, g, \frac{t}{M_1}\right) \text{ and} \\ M_2(Tf, Tg, t) &\leq M_1\left(f, g, \frac{t}{M_2}\right) \end{aligned} \quad (1)$$

Suppose for ϵ , with $0 < \epsilon < 1$, choose δ , with $0 < \delta < 1$, such that $N_1(f, g, t) \geq 1 - \delta$ and $M_1(f, g, t) \leq \delta$ for any $t > 0$

$$\begin{aligned} \text{and } N_1\left(f, g, \frac{t}{M_1}\right) &\geq 1 - \epsilon \\ M_1\left(f, g, \frac{t}{M_2}\right) &< \epsilon \quad (\because M_1, M_2 > 0) \end{aligned} \quad (2)$$

Using (2) in (1) we get

$$N_2(Tf, Tg, t) \geq 1 - \epsilon \text{ and } M_2(Tf, Tg, t) \leq \epsilon$$

Hence T is an intuitionistic 2- fuzzy continuous.

Conversely, Suppose T is an intuitionistic 2- fuzzy continuous.

For ϵ with $0 < \epsilon < 1$, there exists δ with $0 < \delta < 1$

such that $N_1(f, g, t) < 1 - \delta$, $M_1(f, g, t) < \delta$

$$\text{implies } N_2(Tf, Tg, t) > 1 - \epsilon, M_2(Tf, Tg, t) < \epsilon \quad (3)$$

Choose $M_1, M_2 \in \mathbb{R}^+$ such that

$$\begin{aligned} N_1\left(f, g, \frac{t}{M_1}\right) &\leq 1 - \epsilon \text{ for given } N_1(f, g, t) > 1 - \delta \text{ and} \\ M_1\left(f, g, \frac{t}{M_2}\right) &\geq \epsilon \text{ for given } M_1(f, g, t) < \delta \end{aligned} \quad (4)$$

Then applying (4) on (3) we get

$$\begin{aligned} N_2(Tf, Tg, t) &> 1 - \epsilon \geq N_1\left(f, g, \frac{t}{M_1}\right) \\ M_2(Tf, Tg, t) &< \delta \leq M_1\left(f, g, \frac{t}{M_2}\right) \end{aligned}$$

Therefore T is intuitionistic 2- fuzzy bounded.

Lemma 3.11. Let $(F(X), N, M, *)$ be an intuitionistic 2-fuzzy normed linear space. Let $T : F(X) \rightarrow F(X)$ be an intuitionistic 2- fuzzy continuous. If $f_n \rightarrow f$ then $T(f_n) \rightarrow T(f)$ as $n \rightarrow \infty$.

Proof. Given $f_n \rightarrow f$ in $(F(X), N, M, *)$

Then for given $\epsilon > 0, t > 0, 0 < t < 1$ there exists an integer $n_0 \in \mathbb{N}$ such that $N(f_n - f, g_i, t) > 1 - \epsilon$ and $M(f_n - f, g_i, t) < \epsilon$ where g_i 's are linearly independent for all $n \geq n_0, i = 1, 2$.

Since T is intuitionistic 2- fuzzy continuous,

$$(T(f_n - f), Tg_i, t) > 1 - \epsilon \text{ and } M(T(f_n - f), Tg_i, t) < \epsilon$$

$$\text{implies } N(Tf_n - Tf, g_i', t) > 1 - \epsilon \text{ and } M(Tf_n - Tf, g_i', t) < \epsilon$$

Thus $Tf_n \rightarrow Tf$ as $n \rightarrow \infty$.

Lemma 3.12. Let $(F(X), N, M, *)$ be an intuitionistic 2-fuzzy normed linear space then N and M are jointly continuous.

Proof. If $f_n \rightarrow f$ and $g_n \rightarrow g$ in $(F(X), N, M, *)$ we have to prove that $N(f_n - f, g_n - g, t) > 1 - \epsilon$ and $M(f_n - f, g_n - g, t) < \epsilon$ as $n \rightarrow \infty$.

We know that $\lim_{n \rightarrow \infty} N(f_n - f, f_i', t) = 1$ or $> 1 - \epsilon$, $\lim_{n \rightarrow \infty} N(g_n - g, f_i', t) = 1 > 1 - \epsilon$ and $\lim_{n \rightarrow \infty} M(f_n - f, f_i', t) = 0 < \epsilon$,

$$\lim_{n \rightarrow \infty} M(g_n - g, f_i', t) = 0 < \epsilon$$

$$N(f_n - f, g_n - g, t) \geq N(f_n - f, f_i', t/2) * N(g_n - g, f_i', t/2)$$

$$> (1 - \epsilon) * (1 - \epsilon)$$

$$= 1 - \epsilon$$

$$\text{And, } M(f_n - f, g_n - g, t) \leq M(f_n - f, f_i', t/2) \diamond M(g_n - g, f_i', t/2)$$

$$< \epsilon \diamond \epsilon = \epsilon$$

Definition 3.13. Let $(F(X), N, M, *)$ be an intuitionistic 2-fuzzy normed linear space. A subset A of $F(X)$ is said to be intuitionistic 2- fuzzy bounded if

$$N(f, g, t) \geq 1 - M \text{ and } M(f, g, t) \leq M \text{ where } M \text{ is a positive constant.}$$

4. OPEN MAPPING THEOREM

If T is a continuous linear operator from the intuitionistic 2-fuzzy Banach space $(F(X), N_1, M_1, *)$ onto the intuitionistic 2-fuzzy Banach space $(F(Y), N_2, M_2, *)$ then T is an open mapping.

Proof. Let us prove the theorem in various steps.

Step 1. Let A be an intuitionistic 2-fuzzy neighbourhood of $\bar{0} = (0,0)$ in $[F(X)]^2$. Let us show that $\bar{0} \in (T(A))^0$. Let B be the intuitionistic 2-fuzzy balanced neighbourhood of $\bar{0}$ such that $B + B \subset A$.

Since $T(F(X)) = F(Y)$ and B is absorbing, it follows that $F(Y) = \bigcap_n T(nB)$

There exists a positive integer n_0 such that $T(\overline{n_0 B})$ has a non empty interior. Therefore $\bar{0} \in (\overline{T(B)})^0 - (\overline{T(B)})^0$

Also

$$T(\bar{B})^0 - (T(\bar{B}))^0 \subset (T(\bar{B})) - T(\bar{B}) \subset T(\bar{A})$$

So the set $T(\overline{A})$ includes the the intuitionistic 2-fuzzy neighbourhood $(T(\overline{B}))^0 - (T(\overline{B}))^0$ of $\overline{0}$.

Step 2. Now it is shown that $\overline{0} \in (T(A))^0$ since $\overline{0} \in A$ and A is an open set there exists $0 < \alpha < 1$ and $\epsilon \in (0, \alpha)$ such that $B((0,0), \alpha, t_0) \subset A$. But for $0 < \alpha < 1$ a sequence $\{\epsilon_n\}$ can be found such that $\epsilon_n \rightarrow 0$ and

$$1 - \alpha < \lim_n [(1 - \epsilon_1) * (1 - \epsilon_2) * \dots * (1 - \epsilon_n)].$$

Again, $\overline{0} \in T(B(0, 0), \epsilon_n, t_n')$ where $t_n' = \frac{1}{2^n} t_0$, so by step 1 there exists $\delta_n \in (0,1)$ and $t_n > 0$ such that $B((0, 0), \delta_n, t_n) \subset T(\overline{B(0,0)}, \epsilon_n, t_n')$

Since the set $\left\{ B(0,0), \frac{1}{n}, \frac{1}{n} \right\}$ is a countably locally base at zero and $t_n' \rightarrow 0$ as $n \rightarrow \infty$.

It is to be shown that $B((0,0), \delta_1, t_1) \subset (T(A))^0$. Suppose $(f_0, f'_0) \in B(0,0), \delta_1, t_1$. Then $(f_0, f'_0) \in T(\overline{B(0,0)}, \epsilon_2, t_2')$ and so for $\delta_2 > 0$ and $\epsilon_2 > 0$ the ball $B(f_0, f'_0), \delta_2, t_2$ intersects $T(B(0,0), \delta_1, t_1)$. Therefore there exist $(f_1, f'_1) \in B(f_0, f'_0), \delta_2, t_2$

(ie) $N_2[(f_0, f'_0) - T(f_1, f'_1)](g_i, t_2) > 1 - \delta_2$ and $M_2((f_0, f'_0) - T(f_1, f'_1), g_i, t_2) < \delta_2$ or equivalently,

$$((f_0, f'_0) - T(f_1, f'_1)) \in B(0,0), \delta_2, t_2 \subset T(\overline{B(0,0)}, \epsilon_1, t_1')$$

and by the similar argument there exists (f_2, f'_2) in $B(0,0), \epsilon_2, t_2'$, such that

$$N_2(f_0, f'_0) - T[(f_1, f'_1) + (f_2, f'_2), g_i, t_3] = N_2((f_0, f'_0) - T(f_1, f'_1) - T(f_2, f'_2), g_i, t_3) > 1 - \delta_3.$$

$$\text{and } M_2((f_0, f'_0) - T[(f_1, f'_1) + (f_2, f'_2)], g_i, t_3) = M_2((f_0, f'_0) - T(f_1, f'_1) - T(f_2, f'_2), g_i, t_3) > \delta_3.$$

If this process is continued, it leads to a sequence $\{(f_n, f'_n)\}$ such that $(f_n, f'_n) \in B((0,0), \epsilon_n, t_n')$ and $N_2((f_0, f'_0) -$

$$\sum_{j=1}^{n-1} T(f_j, f'_j), g_i, t_n) < 1 - \delta_n,$$

$$M_2(f_0, f'_0) - \sum_{j=1}^{n-1} T(f_j, f'_j), g_i, t_n) < \delta_n$$

Now if $n \in \mathbb{N}$ and $\{p_n\}$ is a positive and increasing sequence, then

$$N_1 \left(\sum_{j=1}^n (f_j, f'_j) - \sum_{j=1}^{n+p_n} (f_j, f'_j), g_i, t \right) = N_1 \left(\sum_{j=1}^{n+p_n} (f_j, f'_j), g_i, t \right) \\ \geq N_1(f_{n+1}, f'_{n+1}), g_i, t_1) * N_1(f_{n+2}, f'_{n+2}), g_i, t_2) * \dots * N_1(f_{n+p_n}, f'_{n+p_n}), g_i, t_{p_n})$$

where $t_1 + t_2 + \dots + t_{p_n} = t$

$$M_1 \left(\sum_{j=1}^n (f_j, f'_j) - \sum_{j=1}^{n+p_n} (f_j, f'_j), g_i, t \right) = M_1 \left(\left(\sum_{j=n+1}^{n+p_n} (f_j, f'_j) \right), g_i, t \right) \\ \leq M_1((f_{n+1}, f'_{n+1}), g_i, t_1) \diamond \dots \diamond M_1(f_{n+p_n}, f'_{n+p_n}), g_i, t_{p_n})$$

where $t_1 + t_2 + \dots + t_{p_n} = t$.

By putting $t_0 = \min \{t_1, t_2, \dots, t_{p_n}\}$ since $t_n' \rightarrow 0$ so there exists no such that $0 \leq t_n' \leq t_0$ for $n > n_0$.

Therefore,

$$N_1((f_{n+1}, f_{n+1}'), g_i, t_0) * \dots * N_1(f_{n+pn}, f_{n+pn}'), g_i, t_0) \geq N_1((f_{n+1}, f_{n+1}'), t_{n+1}') * \dots * N_1(f_{n+pn}, f_{n+pn}'), t_{n+pn}') \\ \geq (1 - \epsilon_{n+1}) * \dots * (1 - \epsilon_{n+pn})$$

Also,

$$M_1((f_{n+1}, f_{n+1}'), t_0) \diamond \dots \diamond M_1(f_{n+pn}, f_{n+pn}'), t_0) \leq M_1((f_{n+1}, f_{n+1}'), t_{n+1}') \diamond \dots \diamond M_1(f_{n+pn}, f_{n+pn}'), t_{n+pn}') \\ \leq t_{n+1} \diamond \dots \diamond \epsilon_{n+pn}$$

$$\text{Hence } \lim_{n \rightarrow \infty} N_1 \left(\sum_{j=n+1}^{n+p_n} (f_j, f_j'), t \right) \geq \lim_{n \rightarrow \infty} ((1 - \epsilon_{n+1}) * \dots * (1 - \epsilon_{n+pn})) = 1$$

$$\text{That is } N_1 \left(\sum_{j=n+1}^{n+p_n} (f_j, f_j'), t \right) \rightarrow 1 \text{ for all } t > 0$$

$$\text{and } \lim_{n \rightarrow \infty} M_1 \left(\sum_{j=n+1}^{n+p_n} (f_j, f_j'), t \right) \leq \lim_{n \rightarrow \infty} \epsilon_{n+1} \diamond \dots \diamond \epsilon_{n+pn} \\ = 1$$

$$\text{That is } M_1 \left(\sum_{j=n+1}^{n+p_n} (f_j, f_j'), t \right) \rightarrow 1 \text{ for all } t > 0$$

So the sequence $\left\{ \sum_{j=1}^n (f_j, f_j') \right\}$ is said to be a Cauchy sequence and consequently $\sum_{j=1}^{\infty} (f_j, f_j')$ converges to some point $(f, f') \in F(X)$, because $F(X)$ is complete.

By fixing $t > 0$, there exists n_0 such that $t > t_n$ for $n > n_0$, because $t_n \rightarrow 0$, it follows

$$N_2((f_0, f_0') - T \left(\sum_{j=1}^{n-1} T(f_j, f_j'), g_i, t \right) \geq N_2((f_0, f_0') - T \left(\sum_{j=1}^{n-1} T(f_j, f_j'), g_i, t_n \right) \\ \geq 1 - \delta_n$$

$$\text{Thus } N_2((f_0, f_0') - T \left(\sum_{j=1}^{n-1} T(f_j, f_j'), g_i, t \right) \rightarrow 1$$

$$\text{Also } M_2((f_0, f_0') - T \left(\sum_{j=1}^{n-1} T(f_j, f_j'), g_i, t \right) \leq M_2((f_0, f_0') - T \left(\sum_{j=1}^{n-1} T(f_j, f_j'), g_i, t_n \right) \leq \delta_n$$

$$\text{Therefore } M_2((f_0, f_0') - T \left(\sum_{j=1}^{n-1} T(f_j, f_j'), g_i, t \right) \rightarrow 1 \text{ as } n \rightarrow \infty$$

$$\text{Hence } (f_0, f_0') = \lim_{n \rightarrow \infty} T \left(\sum_{j=1}^{n-1} T(f_j, f_j') \right) = T(f, f')$$

$$\text{But } N_1((f, f'), g_i, t_0) \geq \limsup_{j=1}^n N_1 \left(\sum_{j=1}^n (f_j, f_j'), g_i, t_0 \right) \geq \limsup ((1 - \epsilon_1) * \dots * (1 - \epsilon_n)) \\ = 1 - \alpha$$

$$\text{and } M_1((f, f'), g_i, t_0) \leq \liminf_{j=1}^n M_1 \left(\sum_{j=1}^n (f_j, f_j'), g_i, t_0 \right)$$

$$\leq \liminf_{n = \alpha} (\delta_1 \diamond \dots \diamond \delta_n)$$

$$\text{Hence } (f, f') \in B((0,0), \alpha, t_0)$$

Step 3. Let G be an open subset of $[F(X)]^2$ and $(f, f') \in G$. Then

$$T(G) = T(f, f') + T((-f, f') + G) \supset T(f, f') + (T((-f, f') + G))^0$$

Hence $T(G)$ would be open because it includes a neighbourhood of each of its points.

Definition4.2. A fuzzy 2-linear functional F is a real valued function on $A \times B$ where A and B are subspaces of $(F(X), N)$ such that

- (i) $F(f+h, g+h') = F(f, g) + F(f, h') + F(h, g) + F(h, h')$
(ii) $F(\alpha f, \beta g) = \alpha\beta F(f, g), \alpha \beta \in [0,1]$

F is said to be bounded with respect to α -2-norm if there exists a constant $k \in [0,1]$ such that $|F(f, g)| \leq k \|(f,g)\|_\alpha$
 $\|F\| = \sup \{k : |F(f, g)| \leq k \|(f, g)\|_\alpha \text{ for every } (f, g) \in A \times B\}$

Theorem 4.3. Let T be an intuitionistic 2-fuzzy linear operator from intuitionistic fuzzy 2-Banach space $(F(X), N_1, M_1, *)$ to intuitionistic fuzzy 2- Banach space $(F(Y), N_2, M_2, *)$. Suppose for every $\{f_n, f_n'\} \in (F(X), N_1, M_1)$ such that $(f_n, f_n') \rightarrow (f, f')$ and $(Tf_n, Tf_n') \rightarrow (g, g')$ for some $f, f' \in F(X), g, g' \in F(Y)$ follows $T(f, f') = (g, g')$. Then T is continuous.

Proof. N and M on $(F(X), N_1, M_1, *) \times (F(Y), N_2, M_2, *)$ is given by

$$N((f_1, f_2), (g_1, g_2), t) = \min \{N_1(f_1, f_2, t), N_2(g_1, g_2, t)\}$$

$$= N_1(f_1, f_2, t) * N_2(g_1, g_2, t)$$

$$M((f_1, f_2), (g_1, g_2), t) = \max \{M_1(f_1, f_2, t), M_2(g_1, g_2, t)\}$$

where * is the usual continuous t-norm and \diamond is the usual continuous t-conorm. With this norm and conorm $(F(X), N_1, M_1, *) \times (F(Y), N_2, M_2, *)$ is a complete intuitionistic 2-fuzzy normed linear space.

For each $(f_1, f_2), (f_1', f_2') \in F(X)$ and $(g_1, g_2), (g_1', g_2') \in F(Y)$ and $t, s > 0$ it follows that

$$\begin{aligned} N(f_1, f_2), (g_1, g_2), t) * N(f_1', f_2'), (g_1', g_2'), s) &= [N_1(f_1, f_2, t) * N_2(g_1, g_2, t)] * [N_1(f_1', f_2', s) * N_2(g_1', g_2', s)] \\ &= [N_1(f_1, f_2, t) * N_1(f_1', f_2', s)] * [N_2(g_1, g_2, t) * N_2(g_1', g_2', s)] \\ &\leq N_1(f_1 + f_1', f_2 + f_2', s + t) * N_2(g_1 + g_1', g_2 + g_2', t + s) \\ &= N((f_1 + f_1', f_2 + f_2'), (g_1 + g_1', g_2 + g_2'), s + t) \end{aligned}$$

Again

$$\begin{aligned} M((f_1, f_2), (g_1, g_2), t) \diamond M((f_1', f_2'), (g_1', g_2'), s) &= [M_1(f_1, f_2, t) \diamond M_2(g_1, g_2, t)] \diamond [M_1(f_1', f_2', s) \diamond M_2(g_1', g_2', s)] \\ &= [M_1(f_1, f_2, t) \diamond M_1(f_1', f_2', s)] \diamond [M_2(g_1, g_2, t) \diamond M_2(g_1', g_2', s)] \\ &\geq M_1(f_1 + f_1', f_2 + f_2', s + t) \diamond M_2(g_1 + g_1', g_2 + g_2', t + s) \\ &= M((f_1 + f_1', f_2 + f_2'), (g_1 + g_1', g_2 + g_2'), s + t) \end{aligned}$$

Now if $\{(f_n, f_n'), (g_n, g_n')\}$ is a cauchy sequence in

$(F(X) \times F(X) \times F(Y) \times F(Y), N, M, *)$ then there exists $n_0 \in \mathbb{N}$ such that

$$N((f_n, f_n'), (g_n, g_n') - (f_m, f_m'), (g_m, g_m'), t) > 1 - \epsilon$$

$$M((f_n, f_n'), (g_n, g_n') - (f_m, f_m'), (g_m, g_m'), t) < \epsilon \text{ for every } \epsilon > 0 \text{ and } t > 0$$

For $m, n > n_0$

$$\begin{aligned} N_1(f_n - f_m, f_n' - f_m'), t) * N_2(g_n - g_m, g_n' - g_m'), t) &= N((f_n - f_m, f_n' - f_m'), (g_n - g_m, g_n' - g_m'), t) \\ &= N((f_n, f_n'), (g_n, g_n'), (f_m, f_m'), (g_m, g_m'), t) \\ &> 1 - \epsilon \end{aligned}$$

$$\begin{aligned} M_1(f_n - f_m, f_n' - f_m'), t) \diamond M_2(g_n - g_m, g_n' - g_m'), t) &= M((f_n - f_m, f_n' - f_m'), (g_n - g_m, g_n' - g_m'), t) \\ &= M((f_n, f_n'), (g_n, g_n'), (f_m, f_m'), (g_m, g_m'), t) \\ &< \epsilon \end{aligned}$$

Therefore $\{(f_n, f_n')\}$ and $\{(g_n, g_n')\}$ are Cauchy sequences in $(F(X), N_1, M_1, *)$ and $(F(Y), N_2, M_2, *)$ respectively and there exists $f \in F(X)$ and $g \in F(Y)$ such that $(f_n, f_n') \rightarrow (f, f')$ and $(g_n, g_n') \rightarrow (g, g')$ and consequently $\{(f_n, f_n'), (g_n, g_n')\}$ converges in the intuitionistic 2-fuzzy normed linear space. Hence

$(F(X) \times F(X) \times F(Y) \times F(Y), N, M, *)$ is a complete intuitionistic 2-fuzzy normed linear space.

Here let $G = \{(f_n, f_n'), T(f_n, f_n')\}$ for every $(f_n, f_n') \in F(X) \times F(X)$ be the graph of the fuzzy 2-linear operator T .

Suppose $(f_n, f_n') \rightarrow (f, f')$ and $T(f_n, f_n') \rightarrow (g, g')$.

Then from previous argument,

$\{(f_n, f_n'), (Tf_n, Tf_n')\}$ converges to $((f, f'), (g, g'))$ which belongs to G .

Therefore, $T(f, f') = (g, g')$. Thus T is continuous.

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