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PATCHES AND RESTRICTED HYPERGRAPHS

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ABSTRACT

A partial hypergraph (of a given simple hypergraph) has the property that each hyperedge is contained in the smaller vertex set on which this partial hypergraph is built. But this property does not hold for arbitrary subsets of the vertex set of the given simple hypergraph. This gives scope for a generalized notion of restricted hypergraphs, so that partial hypergraphs really turn out to be special cases of restricted hypergraphs.

Key words / phrases: *Simple hypergraph, Sperner, partial hypergraph.*

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1. INTRODUCTION

Let V be a nonempty finite set. The cardinality (or, size) of V is denoted by $|V|$. The set of all subsets (including the empty set φ) of V is denoted by 2^V which is called the *power set* [4] of V. The set of all nonempty subsets of V is denoted by 2^{V*}; that is, $2^{V*} = 2^V - \{\varphi\}$.

A *hypergraph* [1] on V is a pair H = (V, E) where E is a family of nonempty subsets of V with $\cup_{X \in E} X = V$. The set V is called the *vertex set* of H and each member of E is called a *hyperedge* of H. If the members of E are all distinct (that is, no two members are equal as subsets of V; or, $E \subseteq 2^{V*}$) then H is called *simple*. If no member of E is a subset (proper or otherwise) of another, then H is called a *Sperner* hypergraph. Some authors (instances: [1] and [2]) take Sperner hypergraphs to be simple and vice versa but there is distinction between the two: Sperner hypergraphs are necessarily simple but simple hypergraphs need not be Sperner [3]. If $\{y\} \in E$ for some vertex y, then {y} is called a *loop* at y.

All the hypergraphs in the coming discussion are assumed simple unless there is some unambiguous indication to the contrary. The motivation for this research work comes principally from the concept of partial hypergraphs [1].

2. PARTIAL HYPERGRAPHS AND PATCHES

Let H = (V, E), $F \in 2^{E*}$ and $W = \cup_{X \in F} X$. Then $W \in 2^{V*}$; and $H_W = (W, F)$ is a hypergraph on W, and is called the *partial hypergraph* (of H) on W.

2.1: Proposition. Let H = (V, E), $A \in 2^{V*}$, $e(A) = \{X \in E \mid X \subseteq A\}$ and $p(A) = A - \cup_{X \in e(A)} X$.

Then for A, $B \in 2^{V*}$, we have:
(i) $A \subseteq B \Longrightarrow e(A) \subseteq e(B)$;
(ii) $A \cap B = \varphi \Longrightarrow e(A) \cap e(B) = \varphi$;
(iii) $e(A) = \varphi \Leftrightarrow p(A) = A$;
(iv) $p(A) = \varphi \Leftrightarrow \cup_{X \in e(A)} X = A$;
(v) $A \in E \Longrightarrow p(A) = \varphi$; and
(vi) $p(A) = \varphi \Longrightarrow (A, e(A))$ is the partial hypergraph (of H) on A.

Proof.
(i) Assume $A \subseteq B$. Then $X \in e(A) \Longrightarrow X \subseteq A \Longrightarrow X \subseteq B \Longrightarrow X \in e(B)$.

(ii) Were $X \in e(A) \cap e(B)$, then $X \subseteq A$ and $X \subseteq B$, contradicting $A \cap B = \varphi$.

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(iii) e (A) = φ ⟹ p(A) = A is immediate. On the other hand, suppose e (A) ≠ φ. Then it is clear that $\cup_{X \in e(A)} X \neq \varphi$. And clearly $\cup_{X \in e(A)} X \subseteq A$. From these we have that p(A) ≠ A.

(iv), (v) and (vi) are obvious.

The set p (A) seen in 2.1 is called the *patch* on A in H (or, the patch on A in V under H). If p (A) ≠ φ then A is called a *patched set* in H (or, a patched set in V under H). If p(A) = A (equivalently, if e (A) = φ), then A is a *patch* in H.

2.2: Example. The converse of 2.1(i) is not true. Let H = (V, E), where V = {1, 2, 3, 4, 5, 6}, E = {X_1, X_2, X_3, X_4} with X_1 = {1}, X_2 = {2, 3}, X_3 = {3, 5} and X_4 = {4, 6}. Let A = {1, 3, 5, 6} and B={1, 2, 3, 4, 5}. Then e (A) = {X_1, X_3} and e (B) = {X_1, X_2, X_3}. Then e (A) ⊆ e (B) but A ⊈ B.

2.3: Example. The converse of 2.1(ii) is not true. Let H = (V, E), where V = {1, 2, 3, 4} and E = {X_1, X_2, X_3, X_4}, with X_1 = {1, 2}, X_2 = {2, 3}, X_3 = {3, 4} and X_4 = {4, 1}. Let A = {1, 3, 4} and B = {2, 3}. Then e (A) = {X_3, X_4} and e (B) = {X_2}. Then e (A) ∩ e (B) = φ but A ∩ B ≠ φ.

2.4: Example. The converse of 2.1(v) is not true. Let V and E be as in 2.2, and A = {1, 2, 3}. Then we have e(A) = {X_1, X_2} and $\cup_{X \in e(A)} X = X_1 \cup X_2 = A$, from which p(A) = φ follows. Yet A ∉ E.

2.5: Proposition. Let H = (V, E) be Sperner and for A ∈ 2^{V*}, let e (A) be as defined in 2.1. Let A, B ∈ 2^{V*}. Then:
(i) e (A ∩ B) = e (A) ∩ e (B)
(ii) e (A) ∪ e (B) ⊆ e (A ∪ B)
(iii) e (A – B) ⊆ e (A) – e (B); so e (A^c) = e $(A)^c$, where A^c = V – A and e $(A)^c$ = E – e (A).

Proof.
(i) X ∈ e (A ∩ B) ⟹ X ∈ E and X ⊆ A ∩ B; so X ∈ e (A) and X ∈ e (B) follow. On the other hand if X ∈ e (A) ∩ e (B), then X ⊆ A ∩ B, and at once we have X ∈ e (A ∩ B).

(ii) X ∈ e (A) ∪ e (B) ⟹ X ∈ E and either X ⊆ A or X ⊆ B, whence X ∈ e (A ∪ B).

(iii) X ∈ e (A – B) ⟹ X ∈ E, X ⊆ A and X ∩ B = φ. Hence X ∈ e (A) and X ∉ e (B). Replacing A and B by V and A, respectively, we get the second part of (iii).

2.6: Example. Equality need not hold in 2.5(ii). Let H be as in 2.3. Let A = {1, 2, 4} and B = {1, 2, 3}. Then e (A) = {X_1, X_4}, e (B) = {X_1, X_2} and e (A ∪ B) = {X_1, X_2, X_3, X_4}.

2.7: Example. Equality need not hold in 2.5(iii). Let H be as in 2.3 and A, B be as in 2.6. Then A – B = {4}, e (A – B) = φ and e (A) – e (B) = {X_4}.

2.8: Example. In general, e (AΔ B) and e (A) Δ e (B), where Δ denotes symmetric difference [4] of sets, are not comparable by set inclusion.

If H = (V, E), let M (H) = {A ∈ 2^{V*} | p(A) = φ} and P (H) = {A ∈ 2^{V*} | p(A) ≠ φ}. Then M (H) ≠ φ because it contains every member of E. Also, M (H) ∩ P (H) = φ and M (H) ∪ P (H) = 2^{V*}.

2.9: Proposition. Let H = (V, E). If A, B ∈ M (H) then A ∪ B ∈ M (H) – i.e., M (H) is closed under set union.

Proof. By hypothesis, $\cup_{X \in e(A)} X = A$ and $\cup_{X \in e(B)} X = B$. Invoking 2.5(ii), we have at once that $\cup_{X \in e(C)} X = C$, where C = A ∪ B. Invoking 2.1(iv), we have p(A ∪ B) = φ, whence A ∪ B ∈ M (H).

2.10: Examples. M (H) is not closed under set intersection, set difference, symmetric difference or complementation. Let V = {1, 2, 3, 4, 5, 6, 7, 8}, X_1 = {1, 3}, X_2 = {2, 3, 4}, X_3 = {4, 5, 6}, X_4 = {7, 8} and E = {X_1, X_2, X_3, X_4} and H = (V, E).

(i) Let A = {1, 2, 3, 4} and B = {1, 3, 4, 5, 6, 7, 8}. Then e (A) = {X_1, X_2} and e (B) = {X_1, X_3, X_4}; and A, B ∈ M (H). Further, A ∩ B = {1, 3, 4}≠ φ and e (A ∩ B) = {X_1}. So p(A∩ B) ≠ φ, whence A∩ B ∉ M (H).

(ii) A = {1, 2, 3, 4} and B = {4, 5, 6, 7, 8}. It is easy to see that A, B ∈ M (H). Further, A − B = {1, 2, 3}, and e (A – B) = {X_1}, so A – B ∉ M (H).

(iii) A, B as in (ii). Here A Δ B = {1, 2, 3, 5, 6, 7, 8} ∉ M (H) because e (A Δ B) = {X_1, X_4}.

(iv) A = {1, 2, 3, 4} ∈ M (H). But A^C = {5, 6, 7, 8} ∉ M (H) because e (A^C) = {X_4}.

2.11: Proposition. Let H = (V, E). Then P (H) = φ ⟺ {x} ∈ E for each x ∈ V.

Proof. (⟹) Let x ∈ V be given and A = {x}. By hypothesis, A ∈ M (H). So $\cup_{X \in e(A)} X = A$.

But then e (A) = e ({x}) = {X ∈ E | X ⊆ {x}}, whence {x} ∈ E.

(⟸) Suppose {x} ∈ E for each x ∈ V. Let B ∈ 2^{V^*} be given and let S = {{x} | x ∈ B}. Clearly S ⊆ E and $\cup_{X \in S} X = B$, whence B ∈ M (H). Consequently 2^{V^*} ⊆ M (H), from which P (H) = φ.

The set P (H) is not closed under set union, intersection, difference or symmetric difference.

3. RESTRICTED HYPERGRAPHS

3.1: Proposition. Suppose H = (V, E). For A ∈ 2^{V^*}, let
(i) e (A) and p(A) be as in 2.1;

(ii) C (A) = e (A) if p(A) = φ, and C (A) = e (A) ∪ {p(A)} if p(A) ≠ φ.

Then (A, C (A)) is a hypergraph on A.

Proof. Clearly C (A) is a nonempty subfamily of 2^{V^*}. Note that A = $(\cup_{X \in e(A)} X) \cup p(A)$, so that given x ∈ A we have either x ∈ X for some X ∈ e (A) or x ∈ p(A). Also, $\cup_{X \in e(A)} X$ and p(A) make a partition of A, whence (A, C (A)) is a hypergraph on A.

The hypergraph (A, C (A)) seen in 3.1 is called the *restricted hypergraph* on the subset A of V, and is denoted by $H|_A$. We also call this hypergraph the *restriction* of H to A. If A = V, then e (A) = E so that $H|_V$ = H.

3.2: Proposition. Given H = (V, E). Then a unique $H|_A$ exists for each A ∈ 2^{V^*}.

Proof. The existence follows from 3.1. For A ∈ 2^{V^*}, the sets e (A) and p(A) are unique, and so is C (A).

3.3: Proposition. If $H|_A$ is non-trivial, then e (A) ≠ φ. (A hypergraph is *nontrivial* if some hyperedge X does not equal the vertex set V.)

Proof. If e (A) = φ, then p(A) = A, so that C (A) = {A}, whence $H|_A$ is trivial.

3.4: Example. The converse of 4.4 is not true. Let H = (V, E) be Sperner, and A ∈ E. Then p(A) = φ, so e (A) = {A}. But $H|_A$ is trivial because C (A) = {A}.

3.5: Proposition. $H|_A$ is simple for each A ∈ 2^{V^*}.

Proof. Let A ∈ 2^{V^*} be given and $H|_A$ = (A, C (A)). If e (A) = φ, then $H|_A$ is clearly simple. In the case e (A)≠ φ, then write e (A) = {Y_1, …, Y_t} for some positive integer t. It is clear that p(A) is disjoint with every member of e (A). Were $H|_A$ not simple, then for some distinct positive integers j and k (both ≤ t) we would have $Y_j = Y_k$ in e (A). But then H would not be simple.

3.6: Proposition. If p (A) = φ, then $H|_A$ is the partial hypergraph on A generated by e (A).

The proof follows from the preceding discussions on e (A), p (A) and $H|_A$.

4. SUMMING UP

(i) Given simple hypergraph H, patches in a subset of vertices are aggregates of vertices in the subset whose union cannot contain any hyperedge.

(ii) A partial hypergraph is a special case of a restricted hypergraph (3.6).

(iii) The only class of hypergraphs that cannot have any patches is that of the ones in which there is a loop at every vertex (consequence of 2.11). This class is very rarely dealt with in applications.

So, a majority of simple hypergraphs do have patches. This could have applications in image processing.

(iv) In a restricted hypergraph, at most one hyperedge is a patch. This patch could be of negligible size, and this could be of interest in image processing applications involving thresholds. The authors are studying such application possibilities.

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