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ON F_q-FUNCTIONS ASSOCIATED WITH RAMANUJAN'S SIXTH ORDER MOCK-THETA FUNCTIONS

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ABSTRACT

In present paper, we have considered the sixth order Mock theta functions defined by Andrews and Hickerson.We have defined generalized functions which can be reduced to Ramanujan's Mock theta functions of sixth order. We have shown that they are F_q-functions and have some properties. We have also given their integral representation.

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INTRODUCTION:

In his last letter to G.H. Hardy [10] written on January, 1920, Ramanujan wrote,"I discovered very interesting functions recently which I called 'Mock' θ-functions". In this letter S. Ramanujan provided a list of 17 mock theta functions of third order, fifth order and seventh order together with identities satisfied by them. Later on G.N.Watson [7, 8] introduced three new third order mock theta functions. There has been a phenomenal increase in number of papers published on Ramanujan's Mock theta functions in recent years specially the epoch-making discovery of Ramanujan's "Lost" Note book by G. E. Andrews. After this several mathematicians Andrews and Hickersons [6] B.Gordon and R.J. Mc Intosh [1] and Y.S. Choi [11] have wrote numerous papers on mock theta functions of sixth order, eight order and tenth order respectively.

Recently Bruce C Berndt and S.H.Chan [3] defined two new mock theta functions of order six. They also provided four transformation formulae relating the new mock theta function with Ramanujan's Mock theta functions of sixth order.Bhasker Srivastava [2] defined generalized functions which reduce to Ramanujan 's mock theta functions of order second and has shown that they are F_q-functions and deduced certain simple properties and integral representation.

In present paper, we will mainly follow the approach used by Bhasker Srivastava [2] for second order mock theta functions. The present paper will enhance the knowledge to know more about sixth order mock theta functions by placing them in the family of F_q- functions and representing them as q-integrals.

Trusdell [4] says the functions which satisfy the functional equation.

$$\frac{\partial}{\partial t}F(t,\alpha) = F(t,\alpha+1) \tag{1.1}$$

as F_q-functions.The q-analogue of (1.1) is the q-differential difference equation

$$D_{q,t}F(t,\alpha) = F(tq,\alpha) \tag{1.2}$$

where

$$tD_{q,t}F(t,\alpha)=F(t,\alpha)-F(tq,\alpha) \tag{1.3}$$

The functions which satisfy equation (1.2) are called F_q-functions.

In this paper, we have defined four generalized functions which reduce to Ramanujan's mock theta functions of order six and have proved that they are F_q-functions in section (4). In section (5) we have deduced some simple properties of generalized functions and in last section (6) we have given integral representation formulae for these functions.

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2. NOTATIONS:

Throughout this article, we assume that $\left|q^k\right|<1$ and use the following usual basic hyper geometric notations:

$$(x)_0=(x\,;q)_0=1$$

$$(x)_n=(x\,;q)_n=\prod_{k=0}^{n-1}(1-xq^k)$$

$$(x_1,x_2,\ldots\ldots\ldots,x_m)_n=(x_1\,;q)_n\ldots\ldots\ldots\ldots(x_m\,;q)_n$$

$$(x;q^k)_n=(1-x)(1-xq^k)\ldots\ldots\ldots(1-xq^{k(n-1)}) \quad , \quad n\geq 1$$

$${}_r\phi_s\left[x_1,x_2,\ldots\ldots\ldots,x_r\,;y_1,y_2,\ldots\ldots y_s\,;q,z\right]=$$

$$=\sum_{k=0}^{\infty}\frac{(x_1,x_2,\ldots\ldots,x_r\,;q)_k}{(q,y_1,y_2,\ldots\ldots,y_s\,;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k \quad ,|z|<1$$

where $\left(\frac{k}{2}\right)=\frac{k(k-1)}{2}$.

3. SIXTH ORDER MOCK THETA FUNCTIONS:

The sixth order mock theta functions, (Andrews and Hickerson [6])

$$\varphi(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(q\,;q^2)_n}{(-q;q)_{2n}} \tag{2.1}$$

$$\phi(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2}(q\,;q^2)_n}{(-q;q)_{2n+1}} \tag{2.2}$$

$$\rho(q)=\sum_{n=0}^{\infty}\frac{q^{\frac{n(n+1)}{2}}(-q\,;q)_n}{(q;q^2)_{n+1}} \tag{2.3}$$

$$\sigma(q)=\sum_{n=0}^{\infty}\frac{q^{\frac{(n+1)(n+2)}{2}}(-q\,;q)_n}{(q;q^2)_{n+1}} \tag{2.4}$$

We define the following four generalized functions

$$\varphi(t,\alpha)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n^2+n\alpha-n}(q\,;q^2)_n}{(-q;q)_{2n}} \tag{2.5}$$

$$\phi(t,\alpha)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n^2+n\alpha+n+1}(q\,;q^2)_n}{(-q;q)_{2n+1}} \tag{2.6}$$

$$\rho(t,\alpha)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{\frac{n(n-1)}{2}+n\alpha}(-q\,;q)_n}{(q;q^2)_{n+1}} \tag{2.7}$$

$$\sigma(t,\alpha)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{\frac{n(n+1)}{2}+n\alpha+1}(-q;q)_n}{(q;q^2)_{n+1}} \tag{2.8}$$

4. GENERALIZED FUNCTIONS ARE F_Q-FUNCTIONS:

Proof: $$\sigma(t,\alpha)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{\frac{n(n+1)}{2}+n\alpha+1}(-q;q)_n}{(q;q^2)_{n+1}}$$

By definition

$$tD_{q,t}\,\sigma(t,\alpha)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{\frac{n(n+1)}{2}+n\alpha+1}(-q;q)_n}{(q;q^2)_{n+1}}-\frac{1}{(tq)_\infty}\sum_{n=0}^{\infty}\frac{(tq)_n q^{\frac{n(n+1)}{2}+n\alpha+1}(-q;q)_n}{(q;q^2)_{n+1}}$$

$$=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{\frac{n(n+1)}{2}+n\alpha+1}}{(q;q^2)_{n+1}}\left\{1-(1-tq^n)\right\}(-q;q)_n$$

$$=\frac{t}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{\frac{n(n+1)}{2}+n\alpha+1+n}(-q;q)_n}{(q;q^2)_{n+1}}$$

$$=\frac{t}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{\frac{n(n+1)}{2}+n(\alpha+1)+1}(-q;q)_n}{(q;q^2)_{n+1}}$$

$$=t\,\sigma(t,\alpha+1)$$

Hence, $\sigma(t,\alpha)$ is an F_q- function.

Similarly, it can be proved that $\phi(t,\alpha)$, $\varphi(t,\alpha)$, $\rho(t,\alpha)$ and are F_q-functions.

5. SIMPLE PROPERTIES:

(i) $D_{q,t}\rho(t,\alpha)=\frac{1}{q}\sigma(t,\alpha)$

(ii) $D_{q,t}\rho(t,\alpha)=\frac{1}{q}\sigma(q)$

(iii) $q\,D^2_{q,t}\left[\phi(t,\alpha)-\varphi(t,\alpha)\right]=\varphi(t,\alpha)$

(iv) $q\,D^2_{q,t}\left[\phi(t,\alpha)-\varphi(t,\alpha)\right]=\varphi(q)$

Proof of (i): $D_{q,t}\rho(t,\alpha)=\rho(t,\alpha+1)$

$$= \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n q^{\frac{n(n+1)}{2}+n(\alpha+1)} (-q;q)_n}{(q;q^2)_{n+1}}$$

$$= \frac{1}{q(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n q^{\frac{n(n+1)}{2}+n\alpha+1} (-q;q)_n}{(q;q^2)_{n+1}}$$

$$= \frac{1}{q} \sigma(t,\alpha)$$

This proves (i).

Proof of (ii): Taking $t = 0$, $\alpha = 1$ in (i), we find (ii).

(iii) $D_{q,t}^2 [\phi(t,\alpha) - \varphi(t,\alpha)] = \phi(t,\alpha+2) - \varphi(z,\alpha+2)$

So, $= \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n (-1)^n q^{n^2+n(\alpha+2)-n}}{(-q;q)_{2n}} (-q;q^2)_n - \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (t)_n q^{n^2+n(\alpha+2)+n+1}}{(q;q^2)_{2n+1}} (-q;q^2)_n$

$$= \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n (-1)^n q^{n^2+n\alpha+n}}{(-q;q)_{2n+1}} (1+q^{2n+1})(-q;q^2)_n - \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n (-1)^n q^{n^2+n\alpha+3n+1}}{(-q;q)_{2n+1}} (-q;q^2)_n$$

$$= \frac{q^{-1}}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n (-1)^n q^{n^2+n\alpha+n+1}}{(-q;q)_{2n+1}} (-q;q^2)_n$$

$$= \frac{1}{q} \varphi(t,\alpha).$$

$$q D_{q,t}^2 [\phi(t,\alpha) - \varphi(t,\alpha)] = \varphi(t,\alpha)$$

This proves (iii).

Proof of (iv): Put $t = 0$, $\alpha = 1$ in (iii), we find (iv).

6. INTEGRAL REPRESENTATIONS:

The q-integral representation is introduced by Thomae [9] and Jackson [5] is defined as

$$\int_0^x f(z) d_q z = (1-q) \sum_{n=0}^{\infty} x f(xq^n) q^n \quad , \quad 0 < |q| < 1$$

Put $x = 1$

$$\int_0^1 f(z) d_q z = (1-q) \sum_{n=0}^{\infty} f(q^n) q^n$$

(i) $\phi(q^t,\alpha) = \frac{(1-q)^{-1}}{(q;q)_\infty} \int_0^1 z^{t-1} (zq;q)_\infty \phi(0,az) d_q z$

(ii) $\varphi(q^t,\alpha) = \frac{(1-q)^{-1}}{(q;q)_\infty} \int_0^1 z^{t-1} (zq;q)_\infty \varphi(0,az) d_q z$

(iii) $$\rho(q^t,\alpha)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 z^{t-1}(zq;q)_\infty\,\rho(0,az)\,d_q z$$

(iv) $$\sigma(q^t,\alpha)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 z^{t-1}(zq;q)_\infty\,\sigma(0,az)\,d_q z$$

where

$$\phi(0,az)=\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n}{(-q;q)_{2n}}(az)^n,\quad \varphi(0,az)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}}(az)^n$$

$$\rho(0,az)=\sum_{n=0}^{\infty}\frac{q^{\frac{n(n+1)}{2}}(-q;q)_n}{(-q;q^2)_{n+1}}(az)^n,\quad \sigma(0,az)=\sum_{n=0}^{\infty}\frac{q^{\frac{(n+1)(n+2)}{2}}(-q;q)_n}{(-q;q^2)_{n+1}}(az)^n$$

Proof: Limiting case of the q-beta integral is

$$\frac{1}{(q^x;q)_\infty}=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 z^{x-1}(zq;q)_\infty\,d_q z \tag{6.1}$$

Now, from (2.7), we have

$$\rho(t,\alpha)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{\frac{n(n-1)}{2}+n\alpha}(-q;q)_n}{(q;q^2)_{n+1}}$$

Writing q^t for t and $q^\alpha=a$, we have

$$\rho(t,\alpha)=\frac{1}{(q^t)_\infty}\sum_{n=0}^{\infty}\frac{(q^t)_\infty q^{\frac{n(n-1)}{2}+n\alpha}(-q;q)_n a^n}{(q;q^2)_{n+1}}$$

$$=\sum_{n=0}^{\infty}\frac{a^n q^{\frac{n(n-1)}{2}}(-q;q)_n}{(q;q^2)_{n+1}(q^{n+t},q)_\infty}$$

$$=\sum_{n=0}^{\infty}\frac{a^n q^{\frac{n(n-1)}{2}}(-q;q)_n}{(q;q^2)_{n+1}}\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 z^{n+t-1}(zq;q)_\infty\,d_q z$$

$$=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 z^{t-1}(zq;q)_\infty\sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}(-q;q)_n}{(q;q^2)_{n+1}}(az)^n\,d_q z \tag{6.2}$$

But

$$\rho(0,az) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}(-q;q)}{(q;q^2)_{n+1}}(az)^n \tag{6.3}$$

Using (6.3) ,(6.2) can be written as

$$\rho(q^t,\alpha) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 z^{t-1}(zq;q)_\infty \rho(0,az)\,d_q z\,.$$

Which is (iii).

Similarly we can prove (i), (ii) and (iv).

REFERENCES:

[1] B.Gordon and R.J.McIntosh, Som eight order mock theta functions, J.London, Math.Soc. 62(2) (2000), 321-335.
[2] B.Srivastava,A comprehencive study of second order mock theta functions,Bull.Korean Math., Soc., 42 (2000), 889-900.
[3] Bruce C.Bernadt and Song Heng Chen, Sixth order mock theta functions, Adv. In Math., 216 (2007), 771-789.
[4] C.Truesdell, An eassay toward a unified theory of special functions, Princeton University Press, Princenton, (1948).
[5] F.H.Jackson, On q-definite integral, J.Pure and Appl.Math. 41 (1910), 193-203.
[6] G.F.Andrews and D.Hickerson,Ramanujan's 'Lost'notebook VII:The sixth order mock theta functions, Adv. In Math., 89 (1991), 60-105.
[7] G.N.Watson, The final problem: An account of the mock theta functions, J.London Math Soc.11 (1936), 55-80.
[8] G.N.Watson, The mock theta functions (2), Proc.London Math Soc.42 (1937), 274-304.
[9] J.Thomac, Beitragezur Theoreider durchdie heinesche reihe, J.Reine Angew.Math, 70 (1869), 258-281.
[10] S.Ramanujan, Collected paper, Cambridge Univ.Press London, 1927, reprinted Chelsea, New York, (1962).
[11] Y.S.Choi, Tenth order mock theta functions in Ramanujan's 'Lost' Notebook, Invent. Math., 136 (1999), 497-569.
