GENERALIZED CONTRACTION MAPPING IN FUZZY METRIC SPACES

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ABSTRACT

In the present paper we introduce generalized contraction mapping in fuzzy metric space and some fixed-point theorems for fuzzy metric space in the sense of George and Veeramani [George A, Veeramani P, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 1994;64:395-9] are given.

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1. INTRODUCTION AND PRELIMINARIES

After Zadeh pioneering paper [16], where the theory of fuzzy sets was introduced, many authors have introduced the concept of fuzzy metric in different ways. Kramosil and Michalek [8] generalized the concept of probabilistic metric space and studied an interesting notion of fuzzy metric space with the help of continuous t-norm. Later on, in order to construct a Hausdorff topology on the fuzzy metric space, George and Veeramani [4] modified the concept of fuzzy metric space introduced by Grabiec [5] and Kramosil and Michalek [8]. Fixed point theory has important applications in diverse disciplines of mathematics, statistics, engineering and economics in dealing with problems arising in: approximation theory, potential theory, game theory, mathematical economics, etc. Fixed point theory for contractiontype mappings in fuzzy metric space is closely related to the fixed-point theory for the same type of mappings in probabilistic metric space of Menger type (see [10, 15]). Many authors [3, 5, 7, 9, 13] have proved some fixed point theorems for various generalizations of contraction mappings in probabilistic and fuzzy metric spaces. In this paper we discuss generalized contraction mapping in fuzzy metric space in the sense George and Veeramani [4]. To define the fuzzy metric space we have to state several concepts as follows:

Definition 1.1 ([14])**.** A binary operation $* : [0,1] \times [0,1] \rightarrow [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:
(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

The t-norm minimum will be denoted by min, the usual product by $\cdot$. These two t-norms are themost commonly used in fuzzy logic.

Definition 1.2 ([4])**.** A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a nonemptyset, $*$ is a continuous t-norm and M is a fuzzy set of $X \times X \times (0,1)$ satisfying the following conditions, for all $x, y, z \in X, s, t > 0$:
(FM1) $M(x, y, t) > 0$;
(FM2) $M(x, y, t) = 1$ if and only if $x = y$;
(FM3) $M(x, y, t) = M(y, x, t)$;
(FM4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(FM5) $M(x, y, \cdot) : (0, +\infty) \rightarrow [0,1]$ is continuous.

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If, in the above definition, the triangular inequality (FM4) is replaced by

(NAF) $M(x,z,t) \geq M(x,y,t) * M(y,z,t), \forall x,y,z \in X, \forall t > 0$, then the triple $(X,M,*)$ is called a non-Archimedean fuzzy metric space.

George and Veeramani [4] proved that every fuzzy metric space $(X,M,*)$ on X generates a Hausdorff first countable topology τ_M on X which has as a base the family of sets of the form

$$\{B_M(x,\epsilon,t) : x \in X, 0 < \epsilon < 1, t > 0\},$$

where $B_M(x,r,t) = \{y \in X : M(x,y,t) > 1-r\}$ for all $r \in (0,1), t > 0$.

Remark 1.3 ([4]). In fuzzy metric space $(X,M,*)$, $M(x,y,\cdot)$ is non decreasing for all $x,y \in X$.

Definition 1.4 ([5]). A sequence x_n in X is said to be convergent to a point x in X (denoted by $x_n \to x$), if $M(x_n,x,t) \to 1$, for all $t > 0$.

Definition 1.5 ([4]).

(a) A sequence x_n in a fuzzy metric space $(X,M,*)$ is a Cauchy sequence if for each $\epsilon \in (0,1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n,x_m,t) > 1-\epsilon$ for all $n,m \geq n_0$.

(b) We say fuzzy metric space $(X,M,*)$ is complete if every Cauchy sequence is convergent withrespect to τ_M.

Remark 1.6 ([12]). Let $(X,M,*)$ be a fuzzy metric space then M is a continuous function on $X \times X \times (0,1)$.

Example 1.7. Let (X,d) be an ordinary metric space and $*$ be a t-norm.

(1) (See [4]) Let M_d be a fuzzy set on $X \times X \times (0,1)$ define as follow:

$$M_d(x,y,t) = \frac{ht^n}{ht^n + md(x,y)} \tag{1.1}$$

for all $t,h,m,n \in \mathbb{R}^+$. Then $(X,M_d,*)$ is a fuzzy metric space and called *induced fuzzy metricspace*. If in equation (1.1) we take $h = m = n = 1$ then we have

$$M_d(x,y,t) = \frac{t}{t + d(x,y)}.$$

This fuzzy metric space $(X,M_d,*)$ is called *standard fuzzy metric space*.

(2) (See [13]) It is immediate to show that (X,d) is a non-Archimedean metric space if and only if $(X,M_d,\min)$ is a non-Archimedean fuzzy metric space.

Further examples and results for fuzzy metric spaces may be found in [4-15].

2. MAIN RESULTS

Gregori and Sepena introduced the notions of fuzzy contraction mapping and fuzzy contractive sequenceas follows:

Definition 2.1 ([7]). Let $(X,M,*)$ be a fuzzy metric space. We call the the mapping $T : X \to X$ is fuzzy contractive mapping, if there exists $\lambda \in (0,1)$ such that

$$\frac{1}{M(Tx,Ty,t)} - 1 \leq \lambda \left(\frac{1}{M(x,y,t)} - 1 \right)$$

for each $x,y \in X$ and $t > 0$, (λ is called the contractive constant of T).

Recall that a sequence x_n in a metric space (X,d) is said to be contractive if there exists $\lambda \in (0,1)$ such that $d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_{n+1})$, for all $n \in \mathbb{N}$.

Definition 2.2 ([7]). Let $(X,M,*)$ be a fuzzy metric space. A sequence x_n is called fuzzy contractive if there exists $\lambda \in (0,1)$ such that

$$\frac{1}{M(x_n, x_{n+1}, t)} - 1 \leq \lambda \left(\frac{1}{M(x_{n-1}, x_n, t)} - 1 \right)$$

for every $t > 0, n \in \mathbb{N}$.

Definition 2.3 ([7]). Let be $(X,M,*)$ a fuzzy metric space. A sequence is called x_n G-Cauchy iffor each $t > 0$ and $p \in \mathbb{N}$, $\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1$.

A fuzzy metric space in which every G-Cauchy sequence is convergent is called G-complete.

The following proposition is justified the above definitions.

Proposition 2.4 ([7]).

(a) The sequence x_n in the metric space (X,d) is contractive iff x_n is fuzzy contractive in the induced fuzzy metric space $(X,M_d,*)$.

(b) The standard fuzzy metric space $(X,M_d,*)$ is complete iff the metric space (X,d) is complete.

(c) If sequence x_n is fuzzy contractive in $(X,M_d,*)$ then it is G-Cauchy.

A continuous t-norm $*$ is of Hadžić-type if there exists a strictly increasing sequence $b_n \in (0,1)$ such that $b_n * b_n = b_n$ for all $n \in \mathbb{N}$, $* = \min$ is an example of such t-norm.

Lemma 2.5 ([9]). Each complete non-Archimedean fuzzy metric space $(X,M,*)$ with $*$ of Hadžić-typeis G-complete.

Let we recall the generalized contractions mapping in metric spaces due to Ćirić [2].

Theorem 2.6. Let (X,d) be a complete metric space and $T : X \to X$ be a self mapping on X such that for each $x,y \in X$.

$$d(Tx,Ty) \leq \alpha(x,y)d(x,y) + \beta(x,y)d(x,Tx) + \gamma(x,y)d(y,Ty) + \delta(x,y)\left[\frac{d(x,Ty) + d(y,Tx)}{2} \right] \quad (2.1)$$

where $\alpha, \beta, \gamma, \delta$ are functions from $X \times X$ into [0, 1) such that
$$\lambda = \sup\{\alpha(x,y) + \beta(x,y) + \gamma(x,y) + \delta(x,y); x,y \in X\} < 1.$$

Then T has a unique fixed point.

Mapping which satisfies (2.1)called generalized contractions. As observed in [1], a self mapping T on a metric space (X,d) is a generalized contraction if and only if T satisfies the following condition:

$$d(Tx,Ty) \leq \lambda \max \left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2} \right\} \quad (2.2)$$

where. Later on Ćirić take the term $\max\{d(x,Ty), d(y,Tx)\}$ instead of $[d(x,Ty) + d(y,Tx)]/2$ and by a new method of proof deduce theorem 2.6 for following which he called it quasi contraction mapping

$$d(Tx,Ty) \leq \lambda \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}. \quad (2.3)$$

The following is quasi-contraction theorem for non-Archimedean fuzzy metric spaces:

Theorem 2.7. Let be a complete non-Archimedean fuzzy metric space where the continuous t-norm $*$ is defined as min and $T : X \to X$ be a self mapping on X such that for each $x, y \in X$, $t > 0$.

$$\frac{1}{M(Tx,Ty,t)} - 1 \le \alpha \left(\frac{1}{M(x,y,t)} - 1 \right) + \beta \left(\frac{1}{M(x,Tx,t)} - 1 \right) + \gamma \left(\frac{1}{M(y,Ty,t)} - 1 \right)$$
$$+ \delta \left(\frac{1}{M(x,Ty,t)} - 1 \right) + \eta \left(\frac{1}{M(y,Tx,t)} - 1 \right), \tag{2.4}$$

where $\alpha, \beta, \gamma, \delta, \eta \in [0,1)$ and $\lambda = \alpha + \beta + \gamma + \delta + \eta < 1$, Then T has a unique fixed point.

Proof. Let $x \in X$ and $t > 0$ be arbitrary and consider a sequence of Pickard iterations x_n, defined inductively by $x_0 = x$, $x_1 = x_0$, ..., $x_{n+1} = T(x_n)$ for each $n = \mathbb{N}$, we will show that x_n is fuzzy contractive. From (2.4) by replacing $x = x_{n-1}$, $y = x_n$ we get

$$\frac{1}{M(x_n, x_{n+1}, t)} - 1 = \frac{1}{M(Tx_{n-1}, Tx_n, t)} - 1$$
$$\le \alpha \left(\frac{1}{M(x_{n-1}, x_n, t)} - 1 \right) + \beta \left(\frac{1}{M(x_{n-1}, x_n, t)} - 1 \right) \tag{2.5}$$
$$+ \gamma \left(\frac{1}{M(x_n, x_{n+1}, t)} - 1 \right) + \delta \left(\frac{1}{M(x_{n-1}, x_{n+1}, t)} - 1 \right).$$

By our choice of t-norm $*$ and triangular inequality for the last parenthesis in (2.5) we have

$$\frac{1}{M(x_{n-1}, x_{n+1}, t)} - 1 \le \frac{1}{\min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}} - 1$$
$$= \max \left\{ \frac{1}{M(x_{n-1}, x_n, t)} - 1, \frac{1}{M(x_n, x_{n+1}, t)} - 1 \right\}, \tag{2.6}$$

from (2.5) we get

$$\frac{1}{M(x_n, x_{n+1}, t)} - 1 \le (\alpha + \beta + \gamma) \max \left\{ \frac{1}{M(x_{n-1}, x_n, t)} - 1, \frac{1}{M(x_n, x_{n+1}, t)} - 1 \right\}$$
$$+ \delta \max \left\{ \frac{1}{M(x_{n-1}, x_n, t)} - 1, \frac{1}{M(x_n, x_{n+1}, t)} - 1 \right\},$$

hence,

$$\frac{1}{M(x_n, x_{n+1}, t)} - 1 \le \lambda \max \left\{ \frac{1}{M(x_{n-1}, x_n, t)} - 1, \frac{1}{M(x_n, x_{n+1}, t)} - 1 \right\}$$

where $\lambda < 1$, this implies

$$\frac{1}{M(x_n, x_{n+1}, t)} - 1 \le \lambda \left(\frac{1}{M(x_{n-1}, x_n, t)} - 1 \right) \tag{2.7}$$

so sequence x_n is fuzzy contractive sequence. Since $(X, M, *)$ is a complete fuzzy metric space so by proposition 2.4 and lemma 2.5, sequence x_n converges to u for some $u \in X$. Now we show u isfixed point of T, from (2.4) we have

$$\frac{1}{M(Tu, Tx_n, t)} - 1 \leq \alpha \left(\frac{1}{M(u, x_n, t)} - 1 \right) + \beta \left(\frac{1}{M(u, Tu, t)} - 1 \right) + \gamma \left(\frac{1}{M(x_n, x_{n+1}, t)} - 1 \right)$$
$$+ \delta \left(\frac{1}{M(u, x_{n+1}, t)} - 1 \right) + \eta \left(\frac{1}{M(x_n, Tu, t)} - 1 \right), \tag{2.8}$$

taking the limit as $n \rightarrow +\infty$ we obtain

$$\frac{1}{M(Tu, u, t)} - 1 \leq \beta \left(\frac{1}{M(u, Tu, t)} - 1 \right) + \eta \left(\frac{1}{M(u, Tu, t)} - 1 \right)$$
$$\leq \lambda \left(\frac{1}{M(u, Tu, t)} - 1 \right).$$

Since $\lambda < 1$, we have $M(u, Tu, t) = 1$, thus, $Tu = u$, by (2.4) we find fixed point is unique.

Theorem 2.8. Let $(X, M, *)$ be a G-complete fuzzy metric space where the continuous t-norm $*$ is defined as min and $T : X \rightarrow X$ be self mapping on X such that for each $x, y \in X, t > 0$

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq \alpha \left(\frac{1}{M(x, y, t)} - 1 \right) + \beta \left(\frac{1}{M(x, Tx, t)} - 1 \right) + \gamma \left(\frac{1}{M(y, Ty, t)} - 1 \right)$$
$$+ \delta \left(\frac{1}{M(x, Ty, 2t)} + \frac{1}{M(y, Tx, 2t)} - 2 \right), \tag{2.9}$$

where $\alpha, \beta, \gamma, \delta \in [0, 1)$ and $\lambda = \alpha + \beta + \gamma + \delta < 1$, Then T has a unique fixed point.

Proof. The proof is very similar to the theorem 2.7. In stead of the equation (2.6) we have

$$\frac{1}{M(x_{n-1}, x_{n+1}, 2t)} - 1 \leq \frac{1}{\min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}} - 1$$
$$= \max \left\{ \frac{1}{M(x_{n-1}, x_n, t)} - 1, \frac{1}{M(x_n, x_{n+1}, t)} - 1 \right\}$$

proceed as the proof of the theorem 2.7 then we find sequence x_n is fuzzy contractive, thus, by proposition 2.4 is G-Cauchy. Since X is G-complete, x_n converges to u for some $u \in X$. Instead of(2.8) we find

$$\frac{1}{M(Tu, Tx_n, t)} - 1 \leq \alpha \left(\frac{1}{M(u, x_n, t)} - 1 \right) + \beta \left(\frac{1}{M(u, Tu, t)} - 1 \right) + \gamma \left(\frac{1}{M(x_n, x_{n+1}, t)} - 1 \right)$$
$$+ \delta \left(\frac{1}{M(u, x_{n+1}, 2t)} + \frac{1}{M(x_n, Tu, 2t)} - 2 \right)$$

taking the limit as $n \rightarrow +\infty$ we obtain

$$\frac{1}{M(Tu, u, t)} - 1 \leq \beta \left(\frac{1}{M(u, Tu, t)} - 1 \right) + \delta \left(\frac{1}{M(u, Tu, 2t)} - 1 \right) \leq \lambda \left(\frac{1}{M(u, Tu, t)} - 1 \right)$$

Since $\lambda < 1$, we have $M(u, Tu, t) = 1$, thus, $Tu = u$, by (2.9) we find fixed point is unique.

Remark 2.9. As observed in [1], by a similar proof we find the generalized contraction condition (2.4) and (2.9) are equivalent to following:

$$\frac{1}{M(Tx, Ty, t)} - 1 \le \lambda \max \left\{ \left(\frac{1}{M(x, y, t)} - 1 \right), \left(\frac{1}{M(x, Tx, t)} - 1 \right), \left(\frac{1}{M(y, Ty, t)} - 1 \right) \right.$$
$$\left. \left(\frac{1}{M(x, Ty, t)} - 1 \right), \left(\frac{1}{M(y, Tx, t)} - 1 \right) \right\}$$

and

$$\frac{1}{M(Tx, Ty, t)} - 1 \le \lambda \max \left\{ \left(\frac{1}{M(x, y, t)} - 1 \right), \left(\frac{1}{M(x, Tx, t)} - 1 \right), \left(\frac{1}{M(y, Ty, t)} - 1 \right) \right.$$
$$\left. \left(\frac{1}{M(x, Ty, 2t)} - 1 \right), \left(\frac{1}{M(y, Tx, 2t)} - 1 \right) \right\}$$

respectively, where $\lambda \in (0, 1)$.

Remark 2.10. For complete metric space (X, d), consider standard fuzzy metric space $(X, M_d, *)$ in example 1.7 then by the following equality

$$\frac{1}{M(x, y, t)} - 1 = \frac{d(x, y)}{t} \tag{2.10}$$

and abov remark we find contractions (2.2) and (2.4) are equivalent. Also by proposition 2.4-(a) and(b) and following the same line as Theorem 2.8, we automatically deduce fuzzy contractive sequence(2.7) is convergent, so theorem 2.6 is special case of theorem 2.8.In next theorem the assumption $* = \min$ in theorem 2.7 is not needed.

Theorem 2.11. Let $(X, M, *)$ be a G-complete fuzzy metric space and $T: X \to X$ be a self mapping on X such that for each $x, y \in X, t > 0$

$$\frac{1}{M(Tx, Ty, t)} - 1 \le \alpha \left(\frac{1}{M(x, y, t)} - 1 \right) + \beta \left(\frac{1}{M(x, Tx, t)} - 1 \right) + \gamma \left(\frac{1}{M(y, Ty, t)} - 1 \right), \tag{2.11}$$

where $\alpha, \beta, \gamma \in [0, 1)$ and $\lambda = \alpha + \beta + \gamma < 1$. Then T has a unique fixed point.

Proof. The assumption t-norm $* = \min$ in theorem 2.7 is applied only in equation (2.6), so the proof is very similar to theorem 2.7 and it is omitted.

Corollary 2.12. Let $(X, M_d, *)$ be a complete standard fuzzy metric space and $T: X \to X$ be a self mapping on X such that inequality (2.11) holds. Then T has a unique fixed point.

By above corollary and proposition 2.4-(b) and equation (2.10) we conclude the following theorem due to Reich [11].

Theorem 2.13. Let (X, d) be a complete metric space and $T: X \to X$ be a self mapping on X such that for each $x, y \in X$

$$d(Tx, Ty) \le \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty),$$

where $\alpha, \beta, \gamma \in [0, 1)$ and $\lambda = \alpha + \beta + \gamma < 1$. Then T has a unique fixed point.

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