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A RECENT NOTE ON THE ABSOLUTE RIESZ SUMMABILITY FACTOR OF INFINITE SERIES

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ABSTRACT

In this note we prove a new result concerning absolute summability factor of an infinite series via quasi - f -power increasing sequence, improving some conditions used by Bor [2] and Leindler [4] in recent results. In fact we are giving three improvements to the result of Leindler.

Key words: *Infinite series, absolute summability, summability factor.*

2000 (MSC): *40A05, 40D15, 40F05.*

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1. INTRODUCTION:

A positive sequence (b_n) is said to be almost increasing if exist a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \le b_n \le Bc_n$.

A positive sequence $a = (a_n)$ is said to be quasi $\beta-$power increasing if there exists a constant

$K = K(\beta, a) \ge 1$ such that

$$K\, n^{\beta}\, a_n \ge m^{\beta}\, a_m \tag{1.0}$$

holds for all $n \ge m$. If (1.0) stays with $\beta = 0$ then a is called a quasi increasing sequence . It should be noted that every almost increasing sequence is a quasi $\beta-$power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking $a_n = n^{-\beta}$.

A positive sequence $\alpha = (\alpha_n)$ is said to be a quasi$-f-$power increasing sequence, $f = (f_n)$, if there exits a constant $K = K(\alpha, f)$ such that

$$K\, f_n \alpha_n \ge f_m \alpha_m$$

holds for $n \ge m \ge 1$ (see [5]). Clearly if α is quasi-*f*-power increasing sequence, then $\alpha\, f$ is quasi increasing sequence.

By t_n we denote the nth (C, 1) mean of the sequence (na_n). The series $\sum a_n$ is said to be summable

$|C,1|_k$, $k \ge 1$, if (see[3])

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty.$$

A series $\sum a_n$ with partial sums s_n is said to be summable $\left|\overline{N}, p_n\right|_k, k \ge 1,$ if (see [1])

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$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_n - T_{n-1}\right|^k < \infty,$$

where (p_n) is a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty$$

and

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v .$$

The following results are proved

Theorem: 1.1 [2]. *Let* (X_n) *be a quasi* $\beta-$*power increasing sequence for some* $0 < \beta < 1,$ *and* (λ_n) *be a real sequence. If the conditions*

$$\sum_{n=1}^{m}\frac{1}{n}P_n = O(P_m) \tag{1.1}$$

$$\lambda_n X_n = O(1), \tag{1.2}$$

$$\sum_{n=1}^{m}\frac{1}{n}\left|t_n\right|^k = O(X_m), \tag{1.3}$$

$$\sum_{n=1}^{m}\frac{p_n}{P_n}\left|t_n\right|^k = O(X_m), \tag{1.4}$$

and

$$\sum_{n=1}^{\infty} n X_n \left|\Delta^2\lambda_n\right| < \infty, \quad \left(\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}\right) \tag{1.5}$$

are satisfied, then the series $\sum a_n\lambda_n$ *is summable* $\left|\overline{N}, p_n\right|_k,\ k \ge 1.$

Theorem: 1.2[4]. *If the sequence* (X_n) *is quasi* $\beta-$*power increasing for some* $0 \le \beta < 1,$ (λ_n) *satisfies the conditions*

$$\sum_{n=1}^{m}\lambda_n = o(m), \tag{1.6}$$

and

$$\sum_{n=1}^{m}\left|\Delta\lambda_n\right| = o(m), \tag{1.7}$$

further the conditions

$$\sum_{n=1}^{\infty} n X_n(\beta)\left|\Delta\left|\Delta\lambda_n\right|\right| < \infty, \tag{1.8}$$

(1.3) and (1.4) *holds, where* $X_n(\beta) = \max\left(n^{\beta}X_n, \log n\right),\ \Delta\lambda_n = \lambda_n - \lambda_{n+1}$ *, then the series* $\sum a_n\lambda_n$ *is summable* $\left|\overline{N}, p_n\right|_k,\ k \ge 1.$

2. LEMMAS:

Lemma: 2.1 Let (X_n) *be a quasi* $-\beta-$*power increasing sequence,* $0 < \beta < 1,$ *such that the conditions*

$$\lambda_n \to 0 \ \ as \ \ n \to \infty, \tag{2.1}$$

$$\sum_{n=1}^{\infty} n\, X_n(\beta) \left|\Delta\left|\Delta\lambda_n\right|\right| < \infty, \tag{2.2}$$

are satisfied. Then

$$n^{\beta+1} X_n \left|\Delta\lambda_n\right| = \mathrm{O}(1), \ \ as \ n \to \infty, \tag{2.3}$$

$$\sum_{n=1}^{\infty} n^{\beta} X_n \left|\Delta\lambda_n\right| < \infty, \tag{2.4}$$

and

$$n^{\beta} X_n \left|\lambda_n\right| = \mathrm{O}(1), \ \ as \ n \to \infty, \tag{2.5}$$

where $X_n(\beta) = n^{\beta} X_n$.

Proof: As $\Delta\lambda_n \to 0$, and $n^{\beta} X_n$ is non-decreasing, we have

$$\begin{aligned} n^{\beta+1} X_n \left|\Delta\lambda_n\right| &= n^{\beta+1} X_n \sum_{v=n}^{\infty} \Delta\left|\Delta\lambda_v\right| \\ &= \mathrm{O}(1) \sum_{v=n}^{\infty} v^{\beta+1} X_v \left|\Delta\left|\Delta\lambda_v\right|\right| \\ &= \mathrm{O}(1). \end{aligned}$$

This proves (2.3). To prove (2.4), we observe that

$$\begin{aligned} \sum_{n=1}^{\infty} n^{\beta} X_n \left|\Delta\lambda_n\right| &= \sum_{n=1}^{\infty} n^{\beta} X_n \sum_{v=n}^{\infty} \Delta\left|\Delta\lambda_v\right| \\ &= \mathrm{O}(1) \sum_{n=1}^{\infty} n^{\beta} X_n \sum_{v=n}^{\infty} \left|\Delta\left|\Delta\lambda_v\right|\right| \\ &= \mathrm{O}(1) \sum_{v=1}^{\infty} \left|\Delta\left|\Delta\lambda_v\right|\right| \sum_{n=1}^{v} n^{\beta} X_n \\ &= \mathrm{O}(1) \sum_{v=1}^{\infty} v^{\beta+1} X_v \left|\Delta\left|\Delta\lambda_v\right|\right| \\ &= \mathrm{O}(1). \end{aligned}$$

Finally

$$\begin{aligned} n^{\beta} X_n \left|\lambda_n\right| &= n^{\beta} X_n \sum_{v=n}^{\infty} \Delta\left|\lambda_v\right| \\ &\le \sum_{v=n}^{\infty} v^{\beta} X_v \left|\Delta\lambda_v\right| \\ &= \mathrm{O}(1), \ \ \text{by (2.4).} \end{aligned}$$

Lemma: 2.2[4]. *The conditions (1.6) and (1.7) implies (2.1).*

3. MAIN RESULT:

We state and prove the following new result

Theorem: 3.1. *If the sequences:* (X_n) *is quasi*$-\beta-$*power increasing,* $0<\beta<1$, (λ_n) *is a sequence of constants both satisfying conditions (1.1),(2.1), (2.2) and*

$$\sum_{n=1}^{m}\frac{1}{n\,(n^{\beta}X_n)^{k-1}}|t_n|^k = \mathrm{O}\left(m^{\beta}X_m\right), \tag{3.1}$$

$$\sum_{n=1}^{m}\frac{p_n}{P_n}\frac{1}{(n^{\beta}X_n)^{k-1}}|t_n|^k = \mathrm{O}\left(m^{\beta}X_m\right). \tag{3.2}$$

Then the series $\sum a_n\lambda_n$ *is summable* $\left|\overline{N},p_n\right|_k, k\geq 1.$.

Lemma: 3.2 Conditions (3.1) and (3.2) are weaker than conditions (1.3) and (1.4) respectively.

Proof: If (1.3) holds, then we have

$$\sum_{n=1}^{m}\frac{|s_n|^k}{n(n^{\beta}X_n)^{k-1}} = \mathrm{O}\left(\frac{1}{X_1^{k-1}}\right)\sum_{n=1}^{m}\frac{1}{n}|s_n|^k = \mathrm{O}(X_m)\ ,$$

while if (3.1) is satisfied then,

$$\begin{aligned}\sum_{n=1}^{m}\frac{1}{n}|s_n|^k &= \sum_{n=1}^{m}\frac{1}{n(n^{\beta}X_n)^{k-1}}|s_n|^k(n^{\beta}X_n)^{k-1}\\
&= \sum_{n=1}^{m-1}\left(\sum_{v=1}^{n}\frac{|s_v|^k}{v(v^{\beta}X_v)^{k-1}}\right)\Delta(n^{\beta}X_n)^{k-1} + \left(\sum_{n=1}^{m}\frac{|s_n|^k}{n(n^{\beta}X_n)^{k-1}}\right)(m^{\beta}X_m)^{k-1}\\
&= \mathrm{O}(1)\sum_{n=1}^{m-1}n^{\beta}X_n\left|\Delta(n^{\beta}X_n)^{k-1}\right| + \mathrm{O}\left(m^{\beta}X_m\right)(m^{\beta}X_m)^{k-1}\\
&= \mathrm{O}\left((m-1)^{\beta}X_{m-1}\right)\sum_{n=1}^{m-1}\left(((n+1)^{\beta}X_{n+1})^{k-1}-(n^{\beta}X_n)^{k-1}\right) + \mathrm{O}(m^{\beta}X_m)^k\\
&= \mathrm{O}\left(m^{\beta}X_m\right)\left((m^{\beta}X_m)^{k-1}\right) + \mathrm{O}(m^{\beta}X_m)^k\\
&= \mathrm{O}(m^{\beta}X_m)^k\ .\end{aligned}$$

Therefore (1.3) implies (1.4) but not conversely.

The proof of the other part is similar.

Remark: Although the condition (1.1) has been added to the statement of theorem 3.1 but it may be mentioned that

Theorem: 3.1 give three improvements in comparing with Theorem 1.2 in the following sense:

1. (i) Conditions (3.1) and (3.2) are weaker than conditions (1.3) and (1.4) respectively (see lemma 3.2) .

(ii) The more advantage of our conditions is to obtain the desired result without any loss of powers through estimations. As an example the proof via conditions (1.3) and (1.4) impose to deal with

$|\lambda_n|^k$ *as* $|\lambda_n|^k = |\lambda_n|^k|\lambda_n| = \mathrm{O}\left(|\lambda_n|\right)$, loosing $|\lambda_n|^{k-1}$ as considered to be $\mathrm{O}(1)$. We have no such case via our conditions.

2. Condition (1.8) has been replaced by condition (2.2) which is weaker

3. Conditions (1.6) and (1.7) have replaced by condition (2.1) which is at least not stronger.

Proof of Theorem: 3.1. Let (T_n) denote the $(\bar{N}, p_n)$ mean of the series $\sum a_n \lambda_n$. Then

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r \lambda_r = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v \lambda_v .$$

Therefore, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} v a_v \left(\frac{1}{v} P_{v-1} \lambda_v \right),$$

and via Abel's transformation,

$$\begin{aligned} T_n - T_{n-1} &= \frac{n+1}{n} \frac{p_n}{P_n} t_n \lambda_n - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \lambda_v \frac{v+1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \Delta\lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \lambda_v \frac{1}{v} \\ &= T_{n1} + T_{n2} + T_{n3} + T_{n4} . \end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is enough to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{nj}|^k < \infty, \quad j = 1,2,3,4.$$

Applying Holder's inequality, we have, in view of (2.4), (2.5) and (3.2),

$$\begin{aligned} \sum_{n=1}^{m} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n1}|^k &= O(1) \sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k |\lambda_n|^k \\ &= O(1) \sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{(n^\beta X_n)^{k-1}} \left(n^\beta X_n |\lambda_n| \right)^{k-1} |\lambda_n| \\ &= O(1) \sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{(n^\beta X_n)^{k-1}} |\lambda_n| \\ &= O(1) \sum_{n=1}^{m-1} \left(\sum_{v=1}^{n} \frac{p_v}{P_v} \frac{|t_v|^k}{(v^\beta X_v)^{k-1}} \right) |\Delta|\lambda_n|| + O(1) \left(\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{(n^\beta X_n)^{k-1}} \right) |\lambda_m| \\ &= O(1) \sum_{n=1}^{m-1} n^\beta X_n |\Delta\lambda_n| + O(1) m^\beta X_m |\lambda_m| \\ &= O(1), \end{aligned}$$

$$\begin{aligned} \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n2}|^k &= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v |t_v|^k |\lambda_v|^k \left(\sum_{v=1}^{n-1} \frac{p_v}{P_{n-1}} \right)^{k-1} \\ &= O(1) \sum_{v=1}^{m} p_v |t_v|^k |\lambda_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\ &= O(1) \sum_{v=1}^{m} \frac{p_v}{P_v} |t_v|^k |\lambda_v|^k \\ &= O(1), \text{ as in the case of } T_{n1} . \end{aligned}$$

In view of (2.2), (2.4) and (3.1), we have

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n3}|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\sum_{v=1}^{n-1}P_v^k\frac{|t_v|^k}{(v^\beta X_v)^{k-1}}|\Delta\lambda_v|\left(\sum_{v=1}^{n-1}v^\beta X_v|\Delta\lambda_v|\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}P_v^k\frac{|t_v|^k}{(v^\beta X_v)^{k-1}}|\Delta\lambda_v|\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\\
&= O(1)\sum_{v=1}^{m}\frac{1}{v}\frac{|t_v|^k}{(v^\beta X_v)^{k-1}}\left(v|\Delta\lambda_v|\right)\\
&= O(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}\frac{1}{r}\frac{|t_r|^k}{(r^\beta X_r)^{k-1}}\right)\Delta\left(v|\Delta\lambda_v|\right)+O(1)\left(\sum_{v=1}^{m}\frac{1}{v}\frac{|t_v|^k}{(v^\beta X_v)^{k-1}}\right)m|\Delta\lambda_m|\\
&= O(1)\sum_{v=1}^{m-1}v^\beta X_v\left(-|\Delta\lambda_v|+(v+1)|\Delta|\Delta\lambda_v||\right)+O(1)\,mX_m|\Delta\lambda_m|\\
&= O(1)\sum_{v=1}^{m}v^\beta X_v|\Delta\lambda_v|+O(1)\sum_{v=1}^{m}v^{\beta+1}X_v|\Delta|\Delta\lambda_v||+O(1)m^{\beta+1}X_m|\Delta\lambda_m|\\
&= O(1).\end{aligned}$$

In view of (2.5) and (3.1),

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n4}|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\sum_{v=1}^{n-1}\frac{P_v}{v}|t_v|^k|\lambda_v|^k\left(\sum_{v=1}^{n-1}\frac{P_v}{v}\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}\frac{P_v}{v}|t_v|^k|\lambda_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\frac{P_v}{v}|t_v|^k|\lambda_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\frac{1}{v}\frac{|t_v|^k}{(v^\beta X_v)^{k-1}}\left(v^\beta X_v|\lambda_v|\right)^{k-1}|\lambda_v|\\
&= O(1)\sum_{v=1}^{m}\frac{1}{v}\frac{|t_v|^k}{(v^\beta X_v)^{k-1}}|\lambda_v|\\
&= O(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}\frac{1}{r}\frac{|t_r|^k}{(r^\beta X_r)^{k-1}}\right)|\Delta|\lambda_v||+O(1)\left(\sum_{v=1}^{m}\frac{1}{v}\frac{|t_v|^k}{(v^\beta X_v)^{k-1}}\right)|\lambda_m|\\
&= O(1)\sum_{v=1}^{m}v^\beta X_v|\Delta\lambda_v|+O(1)\,m^\beta X_m|\lambda_m|\\
&= O(1).\end{aligned}$$

This completes the proof of the theorem.

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