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COMMONFIXED POINT THEOREM IN BANACH SPACE

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ABSTRACT

In the present paper we establish a fixed point theorem in Banach space taking new rational expression, which satisfies the well-known results.

Key words: *Self mapping, Continuous mappings, Non contraction mappings, Banach Space.*

AMS subject classification: *47H10, 54H25,*

INTRODUCTION&PRELIMINARIES

The study of Non-Contraction mapping concerning the existence of fixed point draws attention of various authors in non linear analysis dealing with the study of Non –expansive mapping and the existence of fixed points.

It is well known that the differential and integral equations that arise in the physical problems are generally non linear, therefore the fixed point methods specially "Banach contraction Principle ([1],1922) provides a powerful tool for obtaining the solution of their equations which were very difficult to solve by any other methods.

It is also true that some qualitative properties of the solution of related equations are proved by functional analysis approach. Many authors have presented valuable results with non contraction mapping ([27], 2007) in Banach space.

Definition 2.1: (Banach space) A Banach space $(X, \|.\|)$is a normed vector space such that X is complete under the metric induced by the norm $\|.\|$

Example 2.1: The set of continuous functions on closed interval of real line with the norm $\|.\|$of function f given by

$$\|f\| = \sup_{x \in X} |f(x)|$$

is a Banach space ,where sup denotes the supermom.

Definition2.2: (Normed linear space) let $\|.\|$denotes a function from a linear space X into R that satisfies the following axioms

i) $\forall x \in X,\ \|x\| \geq 0, \|x\| = 0$ iff $x = 0$
ii) $\forall x, y \in X,\ \|x + y\| \leq \|x\| + \|y\|$
iii) $\forall x \in X,\ \propto \in R, \|\alpha x\| = |\alpha|\|x\|$

$\|x\|$is called the norm of x and $(X, \|..\|)$ is called a Normed linear space.

Example -2.2.1:$\langle R^n, \|.\|p\rangle, \forall x \in R^n, \|x\|_\infty = \max_{i=1}^{n} |x_i|$

Example -2.2.2: $\langle l_p, \|.\|p\rangle,\ 1 \leq p \leq \infty, \forall x \in l_p\ = \{x: x \in R^\infty, \sum_{i=1}^{\infty} |x|^p < \infty\},\ \|x\| = (\sum_1^\infty |x_i|^p)^{1/p}$

Definition 2.3: Asequence $\{x_n\}$ in a normed space is said to be a Cauchy sequence if$\|x_n - x_m\| \to 0$ as $m, n \to \infty$ $i.e.$ given $\varepsilon > 0$,there exist an integer N such that $\|x_n - x_m\| < \epsilon$,for all $m, n > N$

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3. MAIN RESULT

Theorem 3.1: Let f be a mapping of a Banach space X into itself. If f satisfies the following conditions;

f^2= I, where I is identity mapping. (3.1.1)

$$\|f(x) - f(y)\| \le \alpha\left[\frac{\|x-y\|\|x-f(x)\| + \|x-f(x)\|\|x-f(y)\| + \|x-f(y)\|\|y-f(x)\|}{\|x-y\|+\|y-f(y)\|}\right] + \gamma\,[\|x-f(x)\| + \|y-f(y)\|] + \delta\,[\|x-f(y)\| + \|y-f(x)\|] + \eta\,\|x-y\| \quad (3.1.2)$$

For every x, y Є X, where α, δ, η > 0 and 5α +4γ+2δ+η < 2, then f has a fixed point.

If α +2δ+η < 1 then f has an unique fixed point.

Proof: Suppose x is a point in the Banach space X.

Taking $y = \frac{1}{2}(f + I)(x)$, $z = f(y)$ and $u = 2y - z$ we have

$$\|z-x\| = \|f(y) - f^2(x)\| = \|f(y) - f(f(x))\|$$

$$\le \alpha\left[\frac{\|y-f(x)\|\|y-f(y)\| + \|y-f(y)\|\|y-f^2(x)\| + \|y-f^2(x)\|\|f(x)-f(y)\|}{\|y-f(x)\|+\|f(x)-f^2(x)\|}\right] + \gamma\,[\|y-f(y)\| + \|f(x)-f^2(x)\|] + \delta\,[\|y-f^2(x)\| + \|f(x)-f(y)\|] + \eta\,\|y-f(x)\|$$

$$\|z-x\| \le \alpha\left[\frac{\|y-f(x)\|\|y-f(y)\| + \|y-f(y)\|\|x-y\| + \|x-y\|\|f(x)-f(y)\|}{\|y-f(x)\|+\|x-f(x)\|}\right] + \gamma\,[\|y-f(y)\| + \|x-f(x)\|] + \delta\,[\|x-y\| + \|f(x)-f(y)\|] + \eta\,\|y-f(x)\|$$

$$\|z-x\| \le \alpha\left[\frac{\|y-f(y)\|(\|y-f(x)\|+\|x-y\|) + \|x-y\|\|f(x)-f(y)\|}{\|x-y\|}\right] + \gamma\,[\|y-f(y)\| + \|x-f(x)\|] + \delta\,[\|x-y\| + \|f(x)-f(y)\|] + \eta\,\|y-f(x)\|$$

$$\|z-x\| \le \alpha\left[\frac{\|y-f(y)\|\|x-f(x)\| + \|x-y\|\|f(x)-f(y)\|}{\|x-y\|}\right] + \gamma\,[\|y-f(y)\| + \|x-f(x)\|] + \delta\,[\|x-y\| + \|f(x)-f(y)\|] + \eta\,\|y-f(x)\|$$

$$\|z-x\| \le \alpha\left[\frac{\|y-f(y)\|\|x-f(x)\|}{\|x-y\|} + \|f(x)-f(y)\|\right] + \gamma\,[\|y-f(y)\| + \|x-f(x)\|] + \delta\,[\|x-y\| + \|f(x)-f(y)\|] + \eta\,\|y-f(x)\|$$

$$\|z-x\| \le \alpha\left[\frac{\|y-f(y)\|\|x-f(x)\|}{\left\|x-\frac{1}{2}(f+I)(x)\right\|} + \left\|f(x) - f(\tfrac{1}{2}(f+I)(x))\right\|\right] + \gamma\,[\|y-f(y)\| + \|x-f(x)\|] + \delta\left[\left\|x-\tfrac{1}{2}(f+I)(x)\right\| + \left\|f(x)-f(\tfrac{1}{2}(f+I)(x))\right\|\right] + \eta\left\|\tfrac{1}{2}(f+I)(x) - f(x)\right\|$$

$$\|z-x\| \le \alpha\left[\frac{\|y-f(y)\|\|x-f(x)\|}{\frac{1}{2}\|x-f(x)\|} + \frac{1}{2}\|x-f(x)\|\right] + \gamma\,[\|y-f(y)\| + \|x-f(x)\|] + \delta\left[\tfrac{1}{2}\|x-f(x)\| + \tfrac{1}{2}\|x-f(x)\|\right] + \eta\,\tfrac{1}{2}\|x-f(x)\|$$

$$\|z-x\| \le \alpha\left[2\|y-f(y)\| + \tfrac{1}{2}\|x-f(x)\|\right] + \beta\,\tfrac{1}{2}\|x-f(x)\| + \gamma\,[\|y-f(y)\| + \|x-f(x)\|] + \delta\,[\|x-f(x)\|] + \eta\,\tfrac{1}{2}\|x-f(x)\|$$

$$\|z-x\| \le \left(\frac{\alpha}{2} + +\gamma + \delta + \frac{\eta}{2}\right)\|x-f(x)\| + (2\alpha+\gamma)\,\|y-f(y)\| \quad (3.1.3)$$

Also

$$\|u-x\| = \|2y-z-x\| = \left\|\tfrac{1}{2}(f+I)(x) - z - x\right\| = \|f(x)-z\| = \|f(x)-f(y)\|$$

$\|u-x\| \le \alpha\left[\frac{\|x-y\|\|x-f(x)\|+\|x-f(x)\|\|x-f(y)\|+\|x-f(y)\|\|y-f(x)\|}{\|x-y\|+\|y-f(y)\|}\right]$
$+ \gamma\,[\|x-f(x)\|+\|y-f(y)\|] + \delta\,[\|x-f(y)\|+\|y-f(x)\|] + \eta\,\|x-y\|$

$\|u-x\| \le \alpha\left[\frac{\|x-f(x)\|(\|x-y\|+\|x-f(y)\|)+\|x-f(y)\|\|y-f(x)\|}{\|x-f(y)\|}\right]$
$+ \gamma\,[\|x-f(x)\|+\|y-f(y)\|] + \delta\,[\|x-f(y)\|+\|y-f(x)\|] + \eta\,\|x-y\|$

$\|u-x\| \le \alpha\left[\frac{\|x-f(x)\|\|y-f(y)\|}{\|x-f(y)\|}+\|y-f(x)\|\right]$
$+ \gamma\,[\|x-f(x)\|+\|y-f(y)\|] + \delta\,[\|x-f(y)\|+\|y-f(x)\|] + \eta\,\|x-y\|$

$\|u-x\| \le \alpha\left[\frac{\|x-f(x)\|\|y-f(y)\|}{\|x-f(y)\|}+\|y-f(x)\|\right] \gamma[\|x-f(x)\|+\|y-f(y)\|] + \delta\,[\|x-f(y)\|+\|y-f(x)\|] + \eta\,\|x-y\|$

$\|u-x\| \le \alpha\left[\frac{\|x-f(x)\|\|y-f(y)\|}{\left\|x-f\left(\frac{1}{2}(f+I)(x)\right)\right\|}+\left\|\frac{1}{2}(f+I)(x)-f(x)\right\|\right]$
$+ \gamma\,[\|x-f(x)\|+\|y-f(y)\|] + \delta\left[\left\|x-f\left(\frac{1}{2}(f+I)(x)\right)\right\|+\left\|\frac{1}{2}(f+I)(x)-f(x)\right\|\right]$
$+\eta\left\|x-\frac{1}{2}(f+I)(x)\right\| \|u-x\|$

$\le \alpha\left[\frac{\|x-f(x)\|\|y-f(y)\|}{\frac{1}{2}\|x-f(x)\|}+\frac{1}{2}\|x-f(x)\|\right]$
$+ \gamma\,[\|x-f(x)\|+\|y-f(y)\|] + \delta\left[\frac{1}{2}\|x-f(x)\|+\frac{1}{2}\|x-f(x)\|\right] + \eta\,\frac{1}{2}\|x-f(x)\|$

$\|u-x\| \le \alpha\,[2\|y-f(y)\|+\frac{1}{2}\|x-f(x)\|\ ] + \gamma\,[\|x-f(x)\|+\|y-f(y)\|] \ + \delta\|x-f(x)\| + \eta\frac{1}{2}\|x-f(x)\|$

$\|u-x\| \le (\frac{\alpha}{2}+\gamma+\delta+\frac{\eta}{2})\|x-f(x)\| + (2\alpha+\gamma)\,\|y-f(y)\|$(3.1.4)

Now,

$\|z-u\| = \|(z-x)-(x-u)\| \le (\|z-x\|+\|x-u\|)$

$\le [(\frac{\alpha}{2}+\gamma+\delta+\frac{\eta}{2})\|x-f(x)\| + (2\alpha+\gamma)\|y-f(y)\|] + [(\frac{\alpha}{2}\gamma+\delta+\frac{\eta}{2})\|x-f(x)\| + (2\alpha+\gamma)\,\|y-f(y)\|]$

$\|z-u\| \le (\alpha+2\gamma+2\delta+\eta)\|x-f(x)\| + (4\alpha+2\gamma)\|y-f(y)\|$

Also

$\|z-u\| = \|f(y)-(2y-z)\|$

$= \|f(y)-2y-f(y))\|$

$=2\|y-f(y)\|$(3.1.5)

From (3.1.5)

$2\|y-f(y)\| \le (\alpha+\beta+2\lambda+2\mu+2\gamma+2\delta+\eta)\|x-f(x)\| + (4\alpha+2\lambda+2\gamma)\,\|y-f(y)\|$
$[2-(4\alpha+2\gamma+2\lambda)]\,\|y-f(y)\|$

$\le (\alpha+\beta+2\lambda+2\mu+2\gamma+2\delta+\eta)\|x-f(x)\|$

$\|y-f(y)\| \le q\,\|x-f(x)\|$

Where $q = \frac{(\alpha+\beta+2\lambda+2\mu+2\gamma+2\delta+\eta)}{[2-(4\alpha+2\lambda+2\gamma)]} < 1$

Since $5\alpha+4\gamma+2\delta+\eta < 2$

Let $g = \frac{1}{2}(f + I)$ then for every $x \in X$

$$\|g^2(x) - g(x)\| = \|g(y) - y\|$$

$$= \left\|\frac{1}{2}(f + I)\, y - y\right\|$$

$$= \frac{1}{2}\|y - f(y)\|$$

$$\leq \frac{q}{2}\|x - f(x)\|$$

By the definition of q, we claim that $\{g^n(x)\}$ is a Cauchy sequence in X.

By the completeness, $\{g^n(x)\}$ converges to some element x_0 in X

i.e. $\lim_{n\to\infty} g^n(x) = x_0$

which implies that $g(x_0) = x_0$

hence $f(x_0) = x_0$

i.e. x_0 is fixed point of f

For the uniqueness:

If possible let y_0 ($\neq x_0$) be another fixed point of f then

$$\|x_0 - y_0\| = \|f(x_0) - f(y_0)\|$$

$$\leq \alpha\left[\frac{\|x_0-y_0\|\|x_0-f(x_0)\|+\|x_0-f(x_0)\|\|x_0-f(y_0)\|+\|x_0-f(y_0)\|\|y_0-f(x_0)\|}{\|x_0-y_0\|+\|y_0-f(y_0)\|}\right]$$
$$+\gamma\left[\|x_0 - f(x_0)\| + \|y_0 - f(y_0)\|\right] + \delta\left[\|x_0 - f(y_0)\| + \|y_0 - f(x_0)\|\right] + \eta\|x_0 - y_0\|$$

$$\|x_0 - y_0\| \leq \alpha\frac{\|x_0-y_0\|^2}{\|x_0-y_0\|} + 2\delta\|x_0 - y_0\| + \eta\|x_0 - y_0\|$$

$$\leq \alpha\|x_0 - y_0\| + 2\delta\|x_0 - y_0\| + \eta\|x_0 - y_0\|$$

$$\leq \alpha\|x_0 - y_0\| + 2\delta\|x_0 - y_0\| + \eta\|x_0 - y_0\|$$

$$\|x_0 - y_0\| = (\alpha + 2\delta + \eta)\|x_0 - y_0\|$$

Since $\alpha + 2\delta + \eta < 1$

$$\|x_0 - y_0\| = 0$$

$$x_0 = y_0$$

This complete the proof

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